

## Memoryless Nonlinearities With Gaussian Inputs: Elementary Results

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*The distortion-to-signal power ratio at the output of a memoryless nonlinearity is determined by simple calculations. This is done by application of Bussgang's theorem, which may also be obtained as a special case of Price's theorem. Specific results are given for hard and soft instantaneous and envelope limiters.*

### I. INTRODUCTION

We present elementary derivations of some useful results for memoryless nonlinearities driven by Gaussian noise. The present relations may be obtained as special cases of more general results,<sup>1</sup> but the present methods are elementary and give physical understanding.

Let a nonlinearity have input  $x$ , output  $z$ , and nonlinear characteristic  $h(x)$ ;

$$z = h(x). \quad (1)$$

Let  $x(t)$  be a stationary, Gaussian random noise. Then it is well-known (Bussgang's theorem) that the cross-correlation between input and output has the same shape as the autocorrelation of the input,

$$\langle x(t + \tau)z(t) \rangle = \alpha \cdot \langle x(t + \tau)x(t) \rangle. \quad (2)$$

This may be seen as a special case of Price's theorem,<sup>2,3</sup> or directly as outlined in the appendix.

From (2) we can write the output  $z(t)$  of any instantaneous nonlinearity with Gaussian input  $x(t)$  as

$$z(t) = \alpha \cdot x(t) + y(t), \quad (3)$$

where  $y(t)$  will be uncorrelated with  $x(t)$ ;

$$\langle x(t + \tau)y(t) \rangle = 0. \quad (4)$$

The constant  $\alpha$  in eq. (3) is given as follows:

$$\begin{aligned}\alpha &= \frac{1}{\sqrt{2\pi\phi(0)}} \int_{-\infty}^{\infty} \frac{x}{\phi(0)} e^{-\frac{x^2}{2\phi(0)}} h(x) dx \\ &= \frac{1}{\sqrt{2\pi\phi(0)}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\phi(0)}} h'(x) dx,\end{aligned}\quad (5)$$

where

$$\langle x(t) \rangle = 0, \quad \phi(\tau) = \langle x(t + \tau)x(t) \rangle. \quad (6)$$

## II. NARROW-BAND INPUT

If  $x(t)$  is narrow-band, it is useful to write

$$\begin{aligned}x(t) &= R(t)\cos[\omega_c t + \theta(t)] \\ &= \frac{1}{2} R(t)e^{-j\theta(t)} e^{-j\omega_c t} + \frac{1}{2} R(t)e^{j\theta(t)} e^{j\omega_c t},\end{aligned}\quad (7)$$

where  $R(t)$  and  $\theta(t)$ , the envelope and angle, slowly varying compared to  $\omega_c t$ , are defined in terms of  $x(t)$  and its Hilbert transform  $\hat{x}(t)$  in the usual way:

$$R(t)e^{j\theta(t)} e^{j\omega_c t} = x(t) + j\hat{x}(t). \quad (8)$$

The output of the nonlinearity may then be written<sup>4</sup>

$$z(t) = \sum_{n=0}^{\infty} A_n(R)\cos(n\omega_c t + n\theta), \quad (9)$$

where  $R$  and  $\theta$  are understood to be functions of  $t$ . The different terms of (9) occupy separate narrow bands, centered around their respective midband frequencies  $n\omega_c$ . The output envelopes  $A_n(R)$  are nonlinear functions of the input envelope  $R$ .<sup>4</sup>

We regard the first term of (3) as the undistorted, or signal, component of the nonlinearity output, and the second term of (3) as distortion. Thus, the signal component of (9) resides exclusively in the  $n = 1$  term, to which we subsequently restrict our attention. We redefine  $z(t)$  to be the response to (9) of a zonal filter centered on  $\omega_c$ , and drop the subscript 1 on the coefficient of the  $n = 1$  term of (9), to yield

$$z(t) = A(R)\cos(\omega_c t + \theta) \quad (10)$$

as the output to be investigated. In eq. (10),  $A(R)$  is given by<sup>5</sup>

$$A(R) = \frac{2}{\pi} \int_0^{\pi} h(R \cos \beta) \cos \beta d\beta, \quad (11)$$

in terms of the instantaneous nonlinear characteristic  $h(\ )$  of (1).

The output [eqs. (10) and (11)] contains both signal and distortion components. For Gaussian input  $x(t)$ ,  $R(t)$  is Rayleigh, and  $\alpha$  of (5) is given in terms of envelope by

$$\begin{aligned}\alpha &= \frac{(1/2)\langle R \cdot A(R) \rangle}{\phi(0)} \\ &= \frac{1}{2\phi(0)} \int_0^\infty RA(R) \cdot \frac{R}{\phi(0)} e^{-\frac{R^2}{2\phi(0)}} dR \\ &= \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{2\phi(0)}} \int_0^\infty \left[ \operatorname{erfc} \frac{R}{\sqrt{2\phi(0)}} + \frac{2}{\sqrt{\pi}} \frac{R}{\sqrt{2\phi(0)}} e^{-\frac{R^2}{2\phi(0)}} \right] A'(R) dR, \quad (12)\end{aligned}$$

where

$$\operatorname{erfc} u = \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-t^2} dt. \quad (13)$$

The output signal power  $S$  is given by

$$S = \alpha^2 \cdot \phi(0). \quad (14)$$

The constant  $\alpha$  in eq. (14) may be calculated from either (5) or (12), whichever is more convenient. The second line of (12) is equivalent to the  $m = 1$  case of eq. (9) of Ref. 1.

In eq. (10),  $z(t)$  contains both the signal component

$$\alpha \cdot x(t) = \alpha \cdot R(t) \cos[\omega_c + \theta(t)] \quad (15)$$

and a distortion component, of power  $D$ . To find this, we determine the total power  $S + D$  in  $z(t)$  of (10), and subtract  $S$ . We have

$$S + D = \frac{1}{2} \langle A^2(R) \rangle = \frac{1}{2} \int_0^\infty A^2(R) \cdot \frac{R}{\phi(0)} e^{-\frac{R^2}{2\phi(0)}} dR. \quad (16)$$

Since  $S \leq S + D$ , we must have from (12), (14), and (16)

$$\langle R \cdot A(R) \rangle^2 \leq 2\phi(0) \cdot \langle A^2(R) \rangle. \quad (17)$$

Noting that

$$2\phi(0) = \langle R^2 \rangle, \quad (18)$$

(17) is simply the Schwarz inequality.

### III. HARD (INSTANTANEOUS OR ENVELOPE) LIMITER

The instantaneous nonlinear characteristic of (1) is:

$$h(x) = \operatorname{sgn} x = \begin{cases} 1, & x > 0. \\ -1, & x < 0. \end{cases} \quad (19)$$

The corresponding envelope characteristic of (11) is:

$$A(R) = \frac{4}{\pi}, \quad R > 0. \quad (20)$$

The signal gain  $\alpha$  of (3) is, by the second relation of either (5) or (12):

$$\alpha = \sqrt{\frac{2}{\pi\phi(0)}} \quad (21)$$

The output signal power  $S$  is by (14):

$$S = \frac{2}{\pi}. \quad (22)$$

Finally, the total power in the first zone is given by (16):

$$S + D = \frac{8}{\pi^2}. \quad (23)$$

Therefore,

$$\frac{D}{S + D} = 1 - \frac{\pi}{4}. \quad (24)$$

#### IV. SOFT INSTANTANEOUS LIMITER

The instantaneous nonlinear characteristic of (1) is:

$$h(x) = \begin{cases} x, & |x| \leq X. \\ 1, & |x| \geq X. \end{cases} \quad (25)$$

This signal gain  $\alpha$  is from the second relation of (5):

$$\alpha = \frac{2}{\sqrt{\pi}} \cdot \operatorname{erf} \frac{X}{\sqrt{2\phi(0)}}, \quad (26)$$

where

$$\operatorname{erf} u = \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt = 1 - \operatorname{erfc} u. \quad (27)$$

#### V. SOFT ENVELOPE LIMITER

The envelope nonlinear characteristic of (11) is:

$$A(R) = \begin{cases} R, & 0 \leq R \leq A. \\ A, & A \leq R \leq \infty. \end{cases} \quad (28)$$

The signal gain  $\alpha$  is, by the second relation of (12):

$$\alpha = \frac{\sqrt{\pi}}{2} \frac{A}{\sqrt{2\phi(0)}} \cdot \operatorname{erfc} \frac{A}{\sqrt{2\phi(0)}} + 1 - e^{-\frac{A^2}{2\phi(0)}}. \quad (29)$$

By (14), the output signal power is:

$$S = \phi(0) \cdot \left[ \frac{\sqrt{\pi}}{2} \frac{A}{\sqrt{2\phi(0)}} \cdot \operatorname{erfc} \frac{A}{\sqrt{2\phi(0)}} + 1 - e^{-\frac{A^2}{2\phi(0)}} \right]^2. \quad (30)$$

This is equivalent to eq. (24a) of Ref. 1. The total power in the first zone is by (16):

$$S + D = \phi(0) \cdot \left[ 1 - e^{-\frac{A^2}{2\phi(0)}} \right]. \quad (31)$$

Therefore, for the soft envelope limiter (28):

$$\frac{D}{S + D} =$$

$$1 - \left[ \frac{\sqrt{\pi}}{2} \frac{A}{\sqrt{2\phi(0)}} \operatorname{erfc} \frac{A}{\sqrt{2\phi(0)}} + 1 - e^{-\frac{A^2}{2\phi(0)}} \right]^2 / \left[ 1 - e^{-\frac{A^2}{2\phi(0)}} \right]. \quad (32)$$

Asymptotically:

$$\frac{D}{S + D} \sim \frac{1}{2} e^{-\frac{A^2}{2\phi(0)}} / \left[ \frac{A^2}{2\phi(0)} \right], \quad \frac{A^2}{2\phi(0)} \gg 1. \quad (33)$$

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## APPENDIX

### Derivation of Eq. (2)

A special case of Price's theorem<sup>2</sup> may be stated as follows.<sup>3</sup> Let  $x_1$  and  $x_2$  be jointly Gaussian random variables with first and second moments given as follows:

$$\begin{aligned} \langle x_1 \rangle &= \langle x_2 \rangle = 0. \\ \langle x_1^2 \rangle &= \langle x_2^2 \rangle = R_{11}. \\ \langle x_1 x_2 \rangle &= R_{12}. \end{aligned} \quad (34)$$

The normalized covariance is

$$\rho_{12} = \frac{R_{12}}{R_{11}}. \quad (35)$$

Let  $x_1$  and  $x_2$  be passed through two different instantaneous nonlinearities with characteristics  $h_1$  and  $h_2$ , yielding outputs  $h_1(x_1)$  and  $h_2(x_2)$  with output cross-variance

$$\psi = \langle h_1(x_1) h_2(x_2) \rangle. \quad (36)$$

Then

$$\frac{\partial \psi}{\partial \rho_{12}} = \langle h'_1(x_1) h'_2(x_2) \rangle, \quad (37)$$

where

$$h'_1(x_1) = \left. \frac{dh_1(x)}{dx} \right|_{x=x_1}, \quad h'_2(x_2) = \left. \frac{dh_2(x)}{dx} \right|_{x=x_2}. \quad (38)$$

To derive Bussgang's theorem set:

$$h_1(x) = h(x).$$

$$h_2(x) = x.$$

$$x_1 = x(t).$$

$$x_2 = x(t + \tau). \quad (39)$$

$x(t)$  = stationary Gaussian noise.

$$\phi(\tau) = \langle x(t + \tau)x(t) \rangle; \quad \rho(t) = \frac{\phi(\tau)}{\phi(0)}.$$

Substituting (39) into (34) to (38),

$$\frac{\partial \psi}{\partial \phi} = \frac{1}{\phi(0)} \langle h'(x) \rangle = \text{constant}. \quad (40)$$

Integrating,

$$\psi(\tau) = \langle x(t + \tau)h(x) \rangle = \text{constant} \cdot \phi(\tau), \quad (41)$$

the result given in (2).

The following alternative derivation, without using Price's theorem, was given by Jack Salz. From (2)

$$\alpha = \frac{\langle x(t + \tau)z(t) \rangle}{\langle x(t + \tau)x(t) \rangle}. \quad (42)$$

Express the nonlinear characteristic of (1) as the contour integral of its transform;<sup>4</sup>

$$z = h(x) = \frac{1}{2\pi} \int_C F(ju) e^{jxu} du. \quad (43)$$

Write

$$x = \left. \frac{1}{j} \frac{d}{dv} e^{jvx} \right|_{v=0}. \quad (44)$$

Then,

$$\begin{aligned}\langle x(t + \tau)z(t) \rangle &= \frac{1}{2\pi j} \lim_{v \rightarrow 0} \frac{d}{dv} \int_C F(ju) \langle e^{j(ux(t) + vx(t+\tau))} \rangle du \\ &= \phi(\tau) \cdot \frac{j}{2\pi} \int_C F(ju) u e^{-u^2 \phi(0)/2} du,\end{aligned}\quad (45)$$

the same result as (41). Substituting in (42),

$$\alpha = \frac{j}{2\pi} \int_C F(ju) u e^{-u^2 \phi(0)/2} du.\quad (46)$$

Equation (46) is readily shown to be equivalent to (5), by substituting the transform relation (43) into (5) and interchanging the order of integration.

## REFERENCES

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