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A Traffic Overflow System With a Large Primary Queue

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We analyze a traffic overflow system that consists of two groups of trunks, with waiting spaces for each group, and some overflow capability from the primary to the secondary group. We consider the case in which the number of waiting spaces in the primary queue is large compared to the corresponding number in the secondary queue and to the number of trunks in the secondary group. The case of an infinite number of waiting spaces in the primary queue is also allowed. We contrast the approach presented with some previous approaches that are suitable when the number of waiting spaces in the primary queue is not comparatively large. As with previous approaches, the aim is to reduce the dimensions of the system of equations to be solved in order to calculate various steady-state quantities of interest. Our results include expressions for the loss probabilities, the probability of overflow from the primary to the secondary group, and the average waiting times in the queues. We also obtain the stability condition under which the results are valid when the number of waiting spaces in the primary queue is infinite.

I. INTRODUCTION

In this paper, a particular traffic overflow system with queueing is analyzed. The same system has been investigated previously where techniques were developed for reducing the dimensions of the system of equations to be solved in order to calculate various steady-state quantities of interest.[1,2] This analysis is a considerable improvement over the earlier study for the case when the number of waiting spaces in the primary queue is large compared to the corresponding number

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in the secondary queue and to the number of secondary trunks.[1] A suitable analysis has already been made for the case when the number of waiting spaces in the secondary queue is large compared to the corresponding number in the primary queue and to the number of trunks in the two groups.[2]

In cases of interest, the sparse system of linear equations for determining the steady-state probabilities of the number of demands in the two groups may have very large dimensions. The reduced system of equations then has considerably fewer dimensions, but the system will be dense. Numerical results based on the solution of the original sparse system of equations by successive over-relaxation techniques, and on the solution of the reduced system of equations obtained earlier by Morrison,[1] were presented by Kaufman, Seery, and Morrison.[3] The numerical values of the various steady-state quantities of interest obtained by the two procedures agree to many significant figures. Other procedures for solving the original system of equations are discussed in Refs. 1 and 4.

The dimensions of the reduced system of equations obtained in this paper are independent of the number of waiting spaces in the primary queue. On the other hand, the dimensions of the reduced system of equations derived in Ref. 1 continue to increase as the number of waiting spaces in the primary queue increases. This is even more drastically the case for the original sparse system of equations. Consequently, the reduced system of equations that we derive is advantageous when the number of waiting spaces in the primary queue is large, and even more so when the number is infinite.

The traffic overflow system considered consists of two groups, a primary and a secondary, with $n_k > 0$ servers and q_k waiting spaces, which receive demands from independent Poisson sources S_k with arrival rates $\lambda_k > 0$, $k = 1$ and 2, respectively, as shown in Fig. 1. The service times of the demands are independent and exponentially distributed with mean service rate $\mu > 0$. If all n_2 servers in the secondary group are busy when a demand from S_2 arrives, the demand is queued if one of the q_2 waiting spaces is available; otherwise it is lost, that is, blocked and cleared from the system. Demands waiting in the secondary queue enter service, in some prescribed order, as servers in the secondary group become free.

If all n_1 servers in the primary group are busy when a demand from S_1 arrives, and there is a free server in the secondary group and no demands waiting in the secondary queue, the demand is served in the secondary. If there are no free servers, then the demand is queued in the primary queue, if one of the q_1 waiting spaces is available; otherwise it is lost. Previously, two different cases were considered for the treatment of demands waiting in the primary queue.[1,2] In case I, a

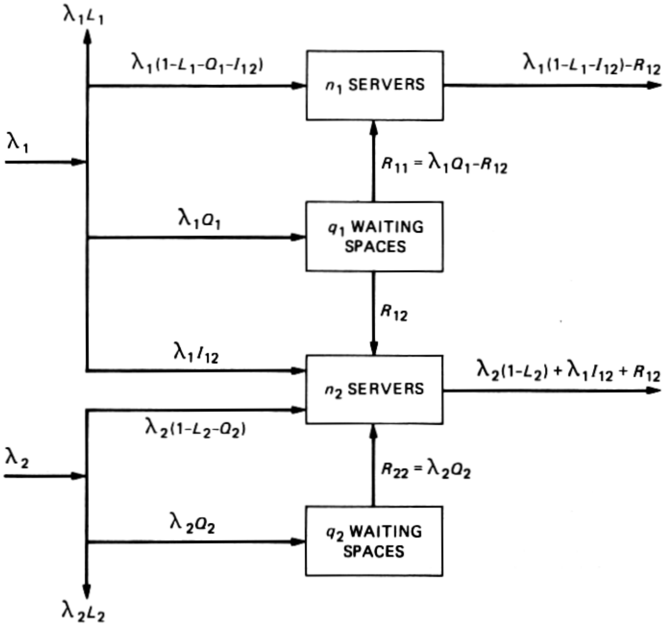

Fig. 1—Mean flow rates for an overflow system with queueing.

demand waiting in the primary queue may enter service either in the primary group, when a server becomes free, or in the secondary group, if a server becomes free and there are no demands waiting in the secondary queue. In case II, no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary group to become free.

We consider only case I and use a different approach to analyze the overflow system. The new approach is preferable to the earlier approach[1] when q_1 is large compared to q_2 and n_2, or even infinite. The alternate earlier approach[2] is preferable when q_2 is large compared to q_1, n_1, and n_2, or even infinite.

Let p_{ij} denote the steady-state probability that there are i demands in the primary group and j demands in the secondary group, either in service or waiting. These probabilities satisfy a set of generalized birth-and-death equations, which take the form of partial difference equations connecting nearest neighboring states. The basic technique is to separate variables in regions away from certain boundaries of the state space, the elements of which are (i, j). These regions are shown in Fig. 2a. The analogous regions corresponding to the earlier analyses[1,2] are shown in Figs. 2b and 2c. The separation of variables leads to two sets of eigenvalue problems for the separation constant. The eigenvalues are roots of polynomial equations. The probabilities p_{ij} are then represented in terms of the corresponding eigenfunctions.

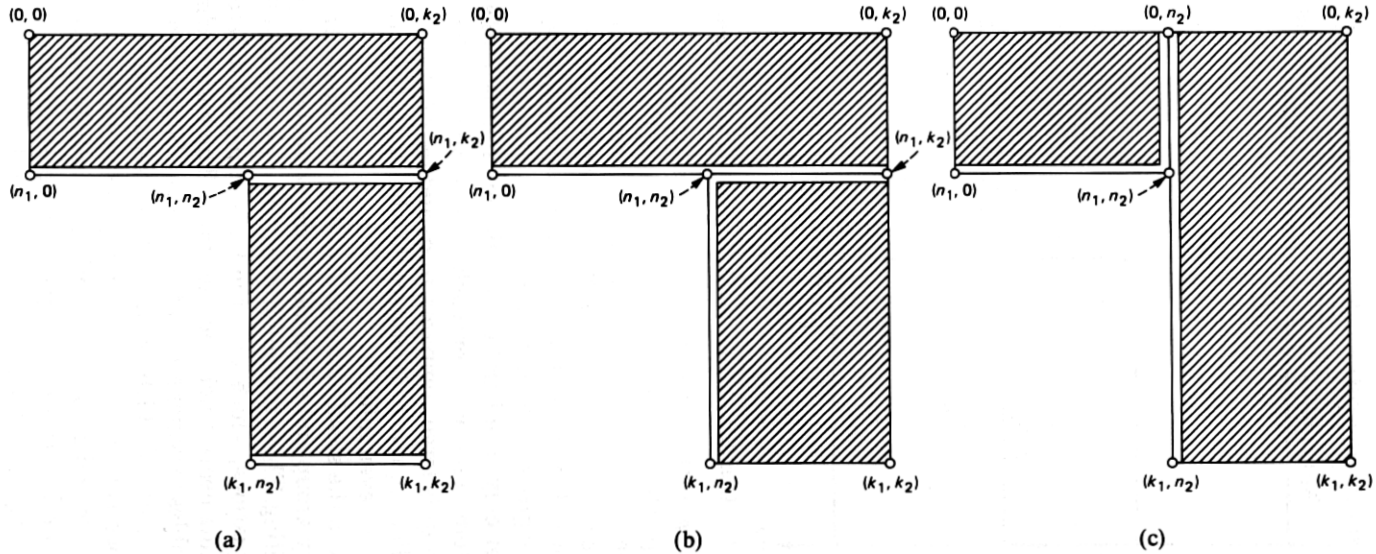

Fig. 2—Boundaries of regions in state space for the analysis of (a) this paper, and (b) and (c) the earlier papers.

The constant coefficients in these representations are determined from the boundary conditions and the normalization condition that the sum of the probabilities is unity. There are two sets of constants, corresponding to the representations of the probabilities p_{ij} in the two shaded regions in Fig. 2a. In general, these constants have to be evaluated numerically. However, we do show that the set of constants corresponding to the representation in the primary queueing region may be expressed explicitly in terms of the other set of constants. This further reduction leads to a problem of fewer dimensions than the one obtained in the first analysis,[1] regardless of the size of $q_1 > 0$.

Various steady-state quantities of interest may be expressed in terms of the probabilities p_{ij}. Among these are the loss (or blocking) probabilities, the probability of overflow from the primary to the secondary group, the probabilities that a demand is queued, and the average waiting times in the queues. These quantities may be expressed directly in terms of the constant coefficients occurring in the representations of the probabilities p_{ij}, averting the need to calculate the p_{ij}. This is a key advantage of the reduction in the dimensions of the problem.

The solution is built up in stages. In Section II, we consider solutions to the birth-and-death equations in the region $0 \le i \le n_1 - 1$, $0 \le j \le n_2 + q_2$ of state space. This corresponds to the case $q_1 = 0$, and was analyzed earlier as a special case.[1] Its inclusion here is for the sake of completeness. In Section III, we examine the solution in the region of state space corresponding to queueing in the primary, and give a heuristic derivation of the stability condition when $q_1 = \infty$. Section IV discusses the boundary and normalization conditions, while Section V is devoted to the calculation of various steady-state quantities of interest. In Section VI, we show how to achieve a further reduction in the dimensions of the problem by the introduction of a generating function. This reduction is obtained in another way in Section VII. Properties of the eigenfunctions that occur in the representations of the probabilities p_{ij} are given in Appendix A, the eigenvalues corresponding to the primary queueing region are discussed in Appendix B, and results pertaining to the generating function are derived in Appendix C.

II. REPRESENTATION OUTSIDE THE PRIMARY QUEUEING REGION

For convenience, we define

$$k_1 = n_1 + q_1, \qquad k_2 = n_2 + q_2. \tag{1}$$

Let p_{ij} $(0 \le i \le k_1, 0 \le j \le k_2)$ denote the steady-state probability that there are i demands in the primary, and j demands in the secondary. Define also the traffic intensities

$$a_1 = \lambda_1/\mu, \qquad a_2 = \lambda_2/\mu. \tag{2}$$

As usual, we let δ_{ij} denote the Kronecker delta, i.e.,

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{3}$$

Finally, we introduce the function

$$\chi_l = \begin{cases} 1, & l \geq 0, \\ 0, & l < 0. \end{cases} \tag{4}$$

The probabilities p_{ij} satisfy a set of generalized birth-and-death equations. For case I, it may be shown[1] that

$$[a_1(1 - \delta_{ik_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)]p_{ij}$$

$$= (1 - \chi_{i-n_1-1}\chi_{n_2-j-1})[a_1(1 - \delta_{i0})p_{i-1,j}$$

$$+ (1 - \delta_{jk_2})\min(j + 1, n_2)p_{i,j+1}] \tag{5}$$

$$+ (1 - \delta_{j0})[a_1\delta_{in_1}\chi_{n_2-j} + a_2(1 - \chi_{i-n_1-1}\chi_{n_2-j})]p_{i,j-1}$$

$$+ (1 - \delta_{ik_1})[(1 - \chi_{i-n_1}\chi_{n_2-j-1})\min(i + 1, n_1) + n_2\chi_{i-n_1}\delta_{jn_2}]p_{i+1,j},$$

for $0 \leq i \leq k_1$, $0 \leq j \leq k_2$. These equations were constructed to imply that

$$p_{ij} = 0, n_1 + 1 \leq i \leq k_1, \qquad 0 \leq j \leq n_2 - 1, \tag{6}$$

since there can be no queued demands in the primary when there is a free secondary server.

The normalization condition is

$$\sum_{i=0}^{n_1}\sum_{j=0}^{k_2} p_{ij} + \sum_{i=n_1+1}^{k_1}\sum_{j=n_2}^{k_2} p_{ij} = 1. \tag{7}$$

For $0 \leq i \leq n_1 - 1$ and $0 \leq j \leq k_2$, the variables in (5) may be separated,[1] and there are solutions of the form $\alpha_i\beta_j$, where

$$(a_1 + i + \rho)\alpha_i = a_1(1 - \delta_{i0})\alpha_{i-1} + (i + 1)\alpha_{i+1}, \tag{8}$$

for $0 \leq i \leq n_1 - 1$, and

$$[a_2(1 - \delta_{jk_2}) + \min(j, n_2) - \rho]\beta_j$$

$$= a_2(1 - \delta_{j0})\beta_{j-1} + (1 - \delta_{jk_2})\min(j + 1, n_2)\beta_{j+1}, \tag{9}$$

for $0 \leq j \leq k_2$, and ρ is a separation constant. The solution of (8) may be expressed in terms of Poisson-Charlier polynomials.[5,6] We denote here the solution of (8) for which $\alpha_0 = 1$ by $s_i(\rho, a_1)$. The properties of $s_i(\lambda, a)$ which we will need in the analysis are contained in Appendix A.

For $0 \leq j \leq n_2 - 1$, (9) becomes

$$(a_2 + j - \rho)\beta_j = a_2(1 - \delta_{j0})\beta_{j-1} + (j + 1)\beta_{j+1}, \tag{10}$$

and we see that there are solutions β_j proportional to $s_j(-\rho, a_2)$ for $0 \le j \le n_2$. For $n_2 \le j \le k_2$, we find that

$$[a_2(1-\delta_{jk_2}) + n_2 - \rho]\beta_j = a_2\beta_{j-1} + n_2(1-\delta_{jk_2})\beta_{j+1}. \tag{11}$$

To find a representation for solutions to (11), we define[1]

$$\Psi_l(\rho) = \left(\frac{n_2}{a_2}\right)^{l/2} U_l\left(\frac{a_2 + n_2 - \rho}{2\sqrt{a_2 n_2}}\right) \tag{12}$$

and

$$\phi_j(\rho) = \Psi_{k_2-j}(\rho) - \Psi_{k_2-j-1}(\rho), \tag{13}$$

where $U_l(x)$ denotes the lth Chebyshev polynomial of the second kind.[7] The properties of these functions are discussed in Appendix A. However, we note here that $U_0(x) \equiv 1$, $U_{-1}(x) \equiv 0$, and hence $\phi_{k_2}(\rho) \equiv 1$. From (11), and (134), it follows that β_j is proportional to $\phi_j(\rho)$ for $n_2 - 1 \le j \le k_2$.

Thus, as a basic solution to (9), we take

$$\beta_j = \begin{cases} s_j(-\rho, a_2)\phi_{n_2}(\rho), & 0 \le j \le n_2, \\ s_{n_2}(-\rho, a_2)\phi_j(\rho), & n_2 - 1 \le j \le k_2, \end{cases} \tag{14}$$

where

$$s_{n_2-1}(-\rho, a_2)\phi_{n_2}(\rho) = s_{n_2}(-\rho, a_2)\phi_{n_2-1}(\rho). \tag{15}$$

With the help of (13), (128), (129), and (133), (15) may be expressed in the form,

$$\rho[s_{n_2}(1-\rho, a_2)\Psi_{q_2}(\rho) - s_{n_2-1}(1-\rho, a_2)\Psi_{q_2-1}(\rho)] = 0. \tag{16}$$

The expression in the square brackets is a polynomial in ρ of degree k_2. It was shown[1] that its zeros are positive and distinct, and we denote them by ρ_m, $m = 1, 2, \cdots, k_2$. We adopt also the convention $\rho_0 = 0$. The calculation of the $\{\rho_m\}$ for $m > 0$ is related to an eigenvalue problem for a symmetric tridiagonal matrix.[1]

For the moment, we consider $q_1 = 0$, so that $k_1 = n_1$. This corresponds to the case of queueing in the secondary only, which has been analyzed as a special case.[1] In terms of the separated solutions to (5) for $0 \le i \le n_1 - 1$, a representation of the solution for $0 \le i \le n_1$, $0 \le j \le k_2$ is

$$p_{ij} = \begin{cases} \sum_{m=0}^{k_2} d_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2)\phi_{n_2}(\rho_m), & 0 \le j \le n_2, \\ \sum_{m=0}^{k_2} d_m s_i(\rho_m, a_1) s_{n_2}(-\rho_m, a_2)\phi_j(\rho_m), & n_2 \le j \le k_2. \end{cases} \tag{17}$$

The constants $\{d_m\}$ are determined only to within a multiplicative factor by the boundary conditions at $i = n_1$, $0 \le j \le k_2$, since the boundary condition at $j = n_2$ may be shown to be redundant.[1] The multiplicative factor is determined by the normalization condition (7). In general, the constants $\{d_m\}$ have to be calculated numerically.

For $q_1 > 0$, the representation (17) may still be used for $0 \le i \le n_1$, $0 \le j \le k_2$, but a different representation must be found for $n_1 + 1 \le i \le k_1$, $n_2 \le j \le k_2$. The earlier approach[1] was to find separated solutions to (5) in the region $n_1 + 1 \le i \le k_1$, $n_2 + 1 \le j \le k_2$, i.e., away from the boundary at $j = n_2$. This led to a representation of p_{ij} for $n_1 \le i \le k_1$, $n_2 \le j \le k_2$. Solutions were found that match those in (17) for $i = n_1$, $n_2 \le j \le k_2$, and an additional q_1 solutions were found that vanish at $i = n_1$. By the superposition of a suitable linear combination of these q_1 solutions, the boundary conditions at $j = n_2$, $n_1 + 1 \le i \le k_1$ could be satisfied, without disturbing the matching of the representations at $i = n_1$, $n_2 \le j \le k_2$. The constants $\{d_m\}$ are still available to satisfy the boundary conditions at $i = n_1$, $0 \le j \le k_2$, and again the condition at $j = n_2$ is redundant.

Since the number of additional constants to be determined in the above approach is q_1, this procedure is not suitable if q_1 is large. In the next section, we investigate an alternate approach that is suitable if q_1 is large or even infinite.

III. REPRESENTATION IN THE PRIMARY QUEUEING REGION

We now assume that $q_1 \ge 2$ and $q_2 \ge 1$, although these restrictions may be relaxed. For $n_1 + 1 \le i \le k_1 - 1$ and $n_2 \le j \le k_2$, the variables in (5) may be separated and there are solutions to the partial difference equations of the form

$$\sigma^{i-n_1} f_j, \qquad n_1 \le i \le k_1, \qquad n_2 \le j \le k_2,$$

where

$$\left[\left(n_1 - \frac{a_1}{\sigma}\right)(1-\sigma) + a_2(1-\delta_{jk_2}) + n_2(1-\delta_{jn_2}\sigma)\right] f_j = a_2(1-\delta_{jn_2}) f_{j-1} + n_2(1-\delta_{jk_2}) f_{j+1}, \qquad n_2 \le j \le k_2, \tag{18}$$

and σ is a separation constant. Since $a_1 \ne 0$, and we are interested in nontrivial solutions, $\sigma \ne 0$. In particular, for $n_2 + 1 \le j \le k_2$, we have

$$[a_2(1-\delta_{jk_2}) + n_2 - \omega] f_j = a_2 f_{j-1} + n_2(1-\delta_{jk_2}) f_{j+1}, \tag{19}$$

where

$$\omega = \left(\frac{a_1}{\sigma} - n_1\right)(1-\sigma). \tag{20}$$

We see from (134) that a solution which satisfies the boundary condition at $j = k_2$ is

$$f_j = f_{k_2}\phi_j(\omega), \qquad n_2 \le j \le k_2, \tag{21}$$

with ϕ_j defined as in (13).

It remains to satisfy the boundary condition at $j = n_2$, namely,

$$[a_2 + n_2(1 - \sigma) - \omega]f_{n_2} = n_2 f_{n_2+1}. \tag{22}$$

With the help of (21) and (13), we obtain

$$n_2(1 - \sigma)\phi_{n_2}(\omega) + (a_2 - \omega)[\Psi_{q_2}(\omega) - \Psi_{q_2-1}(\omega)] - n_2[\Psi_{q_2-1}(\omega) - \Psi_{q_2-2}(\omega)] = 0. \tag{23}$$

This reduces with the help of (20) and (133) to

$$(1 - \sigma)H(\sigma) = 0, \tag{24}$$

where

$$H(\sigma) = \left(n_1 - \frac{a_1}{\sigma}\right)\Psi_{q_2}(\omega) + n_2\phi_{n_2}(\omega), \tag{25}$$

with ω defined as in (20). Hence, $\sigma = 1$ or $H(\sigma) = 0$.

Now $\Psi_l(\omega)$ is a polynomial of degree l in ω. Hence, $\sigma^{q_2+1}H(\sigma)$ is a polynomial of degree $2q_2 + 1$ in σ, so that $H(\sigma) = 0$ has $2q_2 + 1$ roots. In Appendix B, we show that these roots are positive and distinct, and that at least q_2 of them lie between 0 and 1, and at least q_2 of them are greater than 1. More precisely, define

$$A_1 = n_1 + n_2\left[1 - \frac{\Psi_{q_2-1}(0)}{\Psi_{q_2}(0)}\right]. \tag{26}$$

Then, if $a_1 < A_1$, $q_2 + 1$ roots lie between 0 and 1, and q_2 roots exceed 1. If $a_1 > A_1$, then q_2 roots lie between 0 and 1, and $q_2 + 1$ roots exceed 1. Further, if $a_1 = A_1$, then $H(1) = 0$. We denote the roots of $H(\sigma) = 0$ by σ_r, $r = 1, \cdots, 2q_2 + 1$, with $0 < \sigma_1 < \cdots < \sigma_{2q_2+1}$. For convenience, we define $\sigma_0 = 1$. Then $\sigma_{q_2} < 1$ and $\sigma_{q_2+2} > 1$, and $\sigma_{q_2+1} < 1$ if $a_1 < A_1$, and $\sigma_{q_2+1} > 1$ if $a_1 > A_1$. Note that σ_r, $r = 0, \cdots, 2q_2 + 1$ are distinct if $a_1 \ne A_1$. On the other hand, $\sigma_0 = \sigma_{q_2+1}$ if $a_1 = A_1$.

For $q_1 < \infty$, we assume for simplicity that $a_1 \ne A_1$. We then have $2q_2 + 2$ distinct roots, and it follows that we may represent the probabilities in the form

$$p_{ij} = \sum_{r=0}^{2q_2+1} e_r\sigma_r^{i-n_1}\phi_j(\omega_r), \qquad n_1 \le i \le k_1, \qquad n_2 \le j \le k_2, \tag{27}$$

where

$$\omega_r = \left(\frac{a_1}{\sigma_r} - n_1\right)(1 - \sigma_r)$$

and the constants $\{e_r\}$ are to be determined. Consistency of the representations (27) and (17) at $i = n_1$ requires that

$$\sum_{r=0}^{2q_2+1} e_r \phi_j(\omega_r) = \sum_{m=0}^{k_2} d_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \phi_j(\rho_m), \tag{28}$$

for $n_2 \le j \le k_2$.

For $q_1 = \infty$, we assume that $a_1 < A_1$; the reason for this will be apparent shortly. We use the representation (27), but the normalization condition (7) implies that $e_0 = 0$ and $e_r = 0$, $r = q_2 + 2, \cdots, 2q_2 + 1$ since $\sigma_0 = 1$ and $\sigma_r > 1$, $r = q_2 + 2, \cdots, 2q_2 + 1$. We thus have $q_2 + 1$ conditions to replace the $q_2 + 1$ boundary conditions lost on removal of the boundary at $i = k_1$. If $a_1 > A_1$, then $\sigma_{q_2+1} > 1$, and we would have to take $e_{q_2+1} = 0$ also. However, this would leave only q_2 constants e_r, $r = 1, \cdots, q_2$ to satisfy the $q_2 + 1$ conditions (28), which is insufficient.

The stability condition $a_1 < A_1$, for $q_1 = \infty$, may be derived heuristically as follows. Consider the situation when there are demands waiting in the primary queue. It follows that all n_2 servers in the secondary are busy. As the point of instability is reached, the primary queue will never be empty, and hence acts as an infinite source for the secondary servers. Demands from this source enter the secondary only when a server becomes available and there are no demands waiting in the secondary queue. Since the secondary system is a loss system, it is ergodic.

Let $P_j, j = n_2, \cdots, k_2$, be the steady-state probability that there are j demands in the secondary. Then the rate at which demands in the primary queue are served in the secondary is $n_2 \mu P_{n_2}$, and the rate at which they are served in the primary is $n_1 \mu$. Since $\lambda_1 = a_1 \mu$, it follows that the stability condition is

$$a_1 < n_1 + n_2 P_{n_2}. \tag{29}$$

But, as for the system without the infinite source,[8]

$$P_j = \left(\frac{a_2}{n_2}\right)^{j-n_2} P_{n_2}, \qquad n_2 \le j \le k_2. \tag{30}$$

With the infinite source present, $P_j = 0, j = 0, \cdots, n_2 - 1$, and P_{n_2} is determined by

$$\sum_{j=n_2}^{k_2} P_j = 1. \tag{31}$$

From (30), (31), and (136), we obtain

$$P_{n_2} = \left[1 - \frac{\Psi_{q_2-1}(0)}{\Psi_{q_2}(0)}\right]. \tag{32}$$

From (26), (29), and (32), it follows that the stability condition for $q_1 = \infty$ is $a_1 < A_1$.

IV. BOUNDARY CONDITIONS

The constants $\{d_m\}$ and $\{e_r\}$ introduced in (17) and (27) must be chosen so as to satisfy the boundary conditions at $i = n_1$ and $i = k_1$. The boundary conditions at $i = k_1$ imply that

$$[a_2(1 - \delta_{jk_2}) + n_1 + n_2]p_{k_1,j} = a_1 p_{k_1-1,j} + a_2(1 - \delta_{jn_2})p_{k_1,j-1} + n_2(1 - \delta_{jk_2})p_{k_1,j+1} \tag{33}$$

for $n_2 \le j \le k_2$. At $i = n_1$, we have

$$(a_1 + a_2 + n_1 + j)p_{n_1,j} = a_1 p_{n_1-1,j} + (a_1 + a_2)(1 - \delta_{j0})p_{n_1,j-1} + (j + 1)p_{n_1,j+1} \tag{34}$$

for $0 \le j \le n_2 - 1$,

$$(a_1 + a_2 + n_1 + n_2)p_{n_1,n_2} = a_1 p_{n_1-1,n_2} + (a_1 + a_2)p_{n_1,n_2-1} + n_2 p_{n_1,n_2+1} + (n_1 + n_2)p_{n_1+1,n_2}, \tag{35}$$

and

$$[a_1 + a_2(1 - \delta_{jk_2}) + n_1 + n_2]p_{n_1,j} = a_1 p_{n_1-1,j} + a_2 p_{n_1,j-1} + n_1 p_{n_1+1,j} + n_2(1 - \delta_{jk_2})p_{n_1,j+1}, \tag{36}$$

for $n_2 + 1 \le j \le k_2$.

If we substitute (17) in (34), we find, on reduction with the help of (125), (128), and (129), that

$$\sum_{m=0}^{k_2} d_m \phi_{n_2}(\rho_m)[\rho_m s_{n_1}(1 + \rho_m, a_1)s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1)s_j(-1 - \rho_m, a_2)] = 0, \tag{37}$$

for $0 \le j \le n_2 - 1$. Similarly, if we substitute from (17) and (27), (36) implies, after reduction, that

$$\sum_{m=1}^{k_2} d_m \rho_m s_{n_1}(1 + \rho_m, a_1)s_{n_2}(-\rho_m, a_2)\phi_j(\rho_m) - \sum_{r=0}^{2q_2+1} e_r(n_1\sigma_r - a_1)\phi_j(\omega_r) = 0, \tag{38}$$

for $n_2 + 1 \le j \le k_2$, and (33) implies, with the help of the fact that $f_j = \phi_j(\omega_r)$ satisfies (18) for $0 \le r \le 2q_2 + 1$, that

$$\sum_{r=0}^{2q_2+1} e_r \sigma_r^{q_1} [(n_1 + n_2 \delta_{jn_2})\sigma_r - a_1]\phi_j(\omega_r) = 0, \tag{39}$$

for $n_2 \le j \le k_2$. Finally, (17), (27), (15), (125), (128), and (129) in (35) imply that

$$\sum_{m=0}^{k_2} d_m \phi_{n_2}(\rho_m)[\rho_m s_{n_1}(1 + \rho_m, a_1)s_{n_2}(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1)s_{n_2}(-1 - \rho_m, a_2)] - (n_1 + n_2) \sum_{r=0}^{2q_2+1} e_r \sigma_r \phi_{n_2}(\omega_r) = 0. \tag{40}$$

We will now show that the boundary condition at $i = n_1, j = n_2$ is redundant and may hence be dropped. Summing (39) from $j = n_2$ to k_2, we find that

$$\sum_{r=0}^{2q_2+1} e_r \sigma_r^{q_1} [(n_1 \sigma_r - a_1)\Psi_{q_2}(\omega_r) + n_2 \sigma_r \phi_{n_2}(\omega_r)] = 0. \tag{41}$$

For $r = 1, 2, \cdots, 2q_2 + 1$, it is true that

$$H(\sigma_r) \equiv \left(n_1 - \frac{a_1}{\sigma_r}\right)\Psi_{q_2}(\omega_r) + n_2 \phi_{n_2}(\omega_r) = 0, \tag{42}$$

and $\sigma_0 = 1$ and $\omega_0 = 0$, so that (41) and (26) imply that

$$e_0(A_1 - a_1) = 0,$$

that is,

$$e_0 = 0, \tag{43}$$

since $a_1 \neq A_1$ by assumption. The redundancy of the boundary condition (40) may then be verified by summing (37) from $j = 0$ to $n_2 - 1$, (38) from $j = n_2 + 1$ to k_2, and (28) from $j = n_2$ to k_2, and by using the facts that ρ_m satisfies (16) for $m = 0, 1, 2, \cdots, k_2$, and $e_0 = 0$, together with the recurrence relations found in Appendix A.

Consider (39) for $j = n_2$. We note that $\phi_{n_2}(\omega_r)$ *is* nonzero; otherwise (22) with $f_j = \phi_j(\omega_r)$ would imply that $\phi_{n_2+1}(\omega_r) = 0$, and consequently that $\phi_j(\omega_r) \equiv 0$, in contradiction to the fact that $\phi_{k_2}(\omega_r) \equiv 1$. Further, since the roots of $H(\sigma)$ are distinct, it follows that not all of the square-bracketed terms are zero. Hence, the equation is nonvacuous. Since summation of (39) from $j = n_2$ to k_2 implies $e_0 = 0$, if we set $e_0 = 0$ in (38) and (39), then (39) for $j = n_2$ is redundant and accordingly may be dropped. We remark that even for $a_1 = A_1$, we are still free to take $e_0 = 0$ since $\sigma_{q_2+1} = 1$ in that case. As before, the boundary conditions at $j = n_2$ ($i = n_1$ and $i = k_1$) are redundant. Therefore, it follows that the constants $\{d_m\}$ and $\{e_r\}$ are determined by the boundary conditions at $i = n_1$ for $0 \le j \le n_2 - 1$ and $n_2 + 1 \le j \le k_2$, the consistency relationships at $i = n_1$, $n_2 \le j \le k_2$, the boundary conditions at $i = k_1$, $n_2 + 1 \le j \le k_2$ if q_1 is finite, and the normalization condition (7).

We summarize the results of the last three sections as follows:

$$p_{ij} = \begin{cases} \sum_{m=0}^{k_2} d_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2) \phi_{n_2}(\rho_m), & 0 \le j \le n_2, \\ \sum_{m=0}^{k_2} d_m s_i(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \phi_j(\rho_m), & n_2 \le j \le k_2, \end{cases} \tag{44}$$

for $0 \le i \le n_1$ and,

$$p_{ij} = \sum_{r=1}^{2q_2+1} e_r \sigma_r^{i-n_1} \phi_j(\omega_r), \qquad n_2 \le j \le k_2, \tag{45}$$

for $n_1 \le i \le k_1$, $(q_1 < \infty)$, where

$$\rho_m[s_{n_2}(1 - \rho_m, a_2)\Psi_{q_2}(\rho_m) - s_{n_2-1}(1 - \rho_m, a_2)\Psi_{q_2-1}(\rho_m)] = 0, \tag{46}$$

for $m = 0, 1, 2, \cdots, k_2$ and

$$H(\sigma_r) = 0, \qquad r = 1, 2, 3, \cdots, 2q_2 + 1 \tag{47}$$

with $\omega_r \equiv \left(\frac{a_1}{\sigma_r} - n_1\right)(1 - \sigma_r)$.

The $k_2 + 2q_2 + 2$ constants $\{d_m\}$ and $\{e_r\}$ are determined, with $e_0 = 0$, by (28) and (37) to (39) only to within a multiplicative constant, which is determined by the normalization condition. From (44), (46), (13), (130), it follows that

$$\sum_{j=0}^{k_2} p_{ij} = d_0 s_i(0, a_1)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)], \tag{48}$$

for $0 \le i \le n_1$, so that (7), (13), (45), and (130) imply that

$$d_0 s_{n_1}(1, a_1)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)] + \sum_{r=1}^{2q_2+1} e_r \Gamma_{q_1}(\sigma_r)\Psi_{q_2}(\omega_r) = 1, \tag{49}$$

where

$$\Gamma_q(\xi) = \sum_{l=0}^{q-1} \xi^{q-l} = \begin{cases} \xi(1 - \xi^q)/(1 - \xi), & \xi \ne 1, \\ q, & \xi = 1. \end{cases} \tag{50}$$

We point out that the special case $q_1 \ge 1$, $q_2 = 0$ has previously been analyzed.[1] In the primary queue region, solutions of the form

$$p_{i,n_2} = \left(\frac{a_1}{n_1 + n_2}\right)^{i-n_1} p_{n_1,n_2}, \qquad n_1 \le i \le k_1, \tag{51}$$

where

$$p_{n_1,n_2} = \sum_{m=0}^{k_2} d_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \phi_{n_2}(\rho_m), \tag{52}$$

were found. We note that since $k_2 = n_2$ for $q_2 = 0$, $\phi_{n_2}(\rho) = \phi_{k_2}(\rho) \equiv 1$, above. For $q_1 = \infty$, the stability condition was found to be $a_1 < n_1 + n_2$. It is readily verified that the representations (44) and (45), and the stability condition $a_1 < A_1$, produce the same result.

For $q_1 = \infty$ and $q_2 \geq 1$, we must set $e_r = 0$ for $r = q_2 + 2, \cdots, 2q_2 + 1$, as well as $e_0 = 0$, as discussed in Section III. The remaining $k_2 + q_2 + 2$ constants $\{d_m\}$ and $\{e_r\}$ are then determined by (28), (37), and (38) to within a multiplicative constant, which is determined by the normalization condition. But, from (50),

$$\Gamma_\infty(\xi) \equiv \lim_{q_1 \to \infty} \Gamma_{q_1}(\xi) = \frac{\xi}{(1-\xi)}, \qquad 0 \leq \xi < 1. \tag{53}$$

Hence, for $q_1 = \infty$, the normalization condition (49) is

$$d_0 s_{n_1}(1, a_1)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)] + \sum_{r=0}^{q_2+1} \frac{e_r \sigma_r}{(1-\sigma_r)} \Psi_{q_2}(\omega_r) = 1. \tag{54}$$

V. SOME STEADY-STATE QUANTITIES

We now proceed with the calculation of various steady-state quantities of interest. These quantities are shown in Fig. 1, which depicts mean flow rates. The loss probabilities are, for the primary and secondary, respectively,

$$L_1 = \sum_{j=n_2}^{k_2} p_{k_1,j}, \qquad L_2 = \sum_{i=0}^{k_1} p_{i,k_2}, \tag{55}$$

while the probabilities that a demand from the primary, or secondary, source is queued on arrival are

$$Q_1 = \sum_{i=n_1}^{k_1-1} \sum_{j=n_2}^{k_2} p_{ij}, \ (q_1 \geq 1), \qquad Q_2 = \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2-1} p_{ij}, \ (q_2 \geq 1). \tag{56}$$

The probability that a demand arriving from the primary source overflows immediately is

$$I_{12} = \sum_{j=0}^{n_2-1} p_{n_1,j}. \tag{57}$$

Since the mean service rate is μ for each server, the mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1 \mu \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}, \qquad (q_1 \geq 1), \tag{58}$$

while the mean departure rate from the secondary queue is

$$R_{22} = n_2\mu \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} p_{ij}, \qquad (q_2 \geq 1). \tag{59}$$

The mean rate at which queued demands in the primary overflow to the secondary servers is

$$R_{12} = n_2\mu \sum_{i=n_1+1}^{k_1} p_{i,n_2}, \qquad (q_1 \geq 1). \tag{60}$$

The average queue populations are, for the primary and secondary, respectively,

$$V_1 = \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} (i - n_1)p_{ij}, \qquad (q_1 \geq 1), \tag{61}$$

and

$$V_2 = \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} (j - n_2)p_{ij}, \qquad (q_2 \geq 1). \tag{62}$$

Also, the average number of demands in service in the two groups are

$$X_1 = \sum_{i=0}^{n_1} \sum_{j=0}^{k_2} ip_{ij} + n_1(1 - \delta_{q_1,0}) \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}, \tag{63}$$

and

$$X_2 = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} jp_{ij} + n_2 \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}. \tag{64}$$

An application of Little's theorem[8] to the primary and secondary queues shows that the average waiting times of the queued demands in the primary and secondary are given by

$$W_1 = \frac{V_1}{\lambda_1 Q_1}, \ (q_1 \geq 1), \qquad W_2 = \frac{V_2}{\lambda_2 Q_2}, \ (q_2 \geq 1), \tag{65}$$

respectively, independently of the service order within each queue. Also, if we apply Little's theorem to the primary and secondary groups of servers, we obtain

$$\lambda_1(1 - L_1 - I_{12}) - R_{12} = \mu X_1, \quad \lambda_2(1 - L_2) + \lambda_1 I_{12} + R_{12} = \mu X_2, \tag{66}$$

since the mean service rate is μ.

The steady-state quantities of interest may be expressed directly in terms of the constants $\{d_m\}$ and $\{e_r\}$, with the help of the representations (44) and (45). From (55), with the help of (13), we find that

$$L_1 = \sum_{r=1}^{2q_2+1} e_r \sigma_r^{q_1} \Psi_{q_2}(\omega_r), \tag{67}$$

and with the help of (130) and the fact that $\phi_{k_2}(\rho) \equiv 1$, we find that

$$L_2 = \sum_{m=0}^{k_2} d_m s_{n_1}(1 + \rho_m, a_1) s_{n_2}(-\rho_m, a_2) + \sum_{r=1}^{2q_2+1} e_r \Gamma_{q_1}(\sigma_r), \tag{68}$$

where $\Gamma_q(\xi)$ is as defined in (50). Similarly, it follows from (57) and (130) that

$$I_{12} = \sum_{m=0}^{k_2} d_m s_{n_1}(\rho_m, a_1) s_{n_2-1}(1 - \rho_m, a_2) \phi_{n_2}(\rho_m). \tag{69}$$

Next, we find from (56), (50), and (13) that

$$Q_1 = \sum_{r=1}^{2q_2+1} \left(\frac{e_r}{\sigma_r}\right) \Gamma_{q_1}(\sigma_r) \Psi_{q_2}(\omega_r). \tag{70}$$

Also, (58), (50), and (13) imply that

$$R_{11} = n_1 \mu \sum_{r=1}^{2q_2+1} e_r \Gamma_{q_1}(\sigma_r) \Psi_{q_2}(\omega_r), \tag{71}$$

while from (60) and (50) it follows that

$$R_{12} = n_2 \mu \sum_{r=1}^{2q_2+1} e_r \Gamma_{q_1}(\sigma_r) \phi_{n_2}(\omega_r). \tag{72}$$

It may be shown directly from (5) that

$$R_{11} + R_{12} = \lambda_1 Q_1, \tag{73}$$

and, using (42), it is readily verified that this is consistent with the representations (70) to (72).

At this point, we define

$$r_j = \sum_{i=0}^{k_1} p_{ij}, \qquad n_2 \le j \le k_2. \tag{74}$$

If we sum on i in (5), we obtain

$$[a_2(1 - \delta_{jk_2}) + n_2] r_j = a_2 r_{j-1} + n_2(1 - \delta_{jk_2}) r_{j+1}, \tag{75}$$

for $n_2 + 1 \le j \le k_2$. It follows that

$$n_2 r_j = a_2 r_{j-1}, \qquad n_2 + 1 \le j \le k_2. \tag{76}$$

Hence, since $L_2 = r_{k_2}$, from (55) and (74), we find that

$$r_j = \left(\frac{n_2}{a_2}\right)^{k_2 - j} L_2, \qquad n_2 \le j \le k_2. \tag{77}$$

Then, it follows from (50) and (59) that

$$R_{22} = \lambda_2 \Gamma_{q_2}\left(\frac{n_2}{a_2}\right) L_2. \tag{78}$$

Also, from (56) and (77), we deduce that

$$Q_2 = \Gamma_{q_2}\left(\frac{n_2}{a_2}\right)L_2. \tag{79}$$

These representations satisfy $R_{22} = \lambda_2 \dot{Q}_2$. We remark that this latter result, as well as (73), hold since, in the steady state, the average arrival and departure rates are equal, for both the primary and secondary queues.

From (61) and (13), we find that

$$V_1 = \sum_{r=1}^{2q_2+1} e_r \Lambda_{q_1}(\sigma_r)\Psi_{q_2}(\omega_r), \tag{80}$$

where

$$\Lambda_q(\xi) = \sum_{l=1}^{q} l\xi^l = \begin{cases} \xi[1-(q+1)\xi^q + q\xi^{q+1}]/(1-\xi)^2, & \xi \neq 1, \\ \tfrac{1}{2}q(q+1), & \xi = 1. \end{cases} \tag{81}$$

We note in passing that

$$\Gamma_q(\xi) = q\xi^{q+1} + (1-\xi)\Lambda_q(\xi), \tag{82}$$

a formula that may be numerically useful. From (62), (74), (77), and (81), it follows that

$$V_2 = \left(\frac{n_2}{a_2}\right)^{q_2} \Lambda_{q_2}\left(\frac{a_2}{n_2}\right)L_2. \tag{83}$$

Finally, from (63), with the help of (7), (48), and (131), we find that

$$X_1 = n_1 - d_0 s_{n_1-1}(2, a_1)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)], \tag{84}$$

and, from (64), with the help of (7), (17), (130), and (131),

$$X_2 = n_2 - \sum_{m=0}^{k_2} d_m s_{n_1}(1+\rho_m, a_1)s_{n_2-1}(2-\rho_m, a_2)\phi_{n_2}(\rho_m). \tag{85}$$

We remark that in view of (67) to (69), (72), (84), and (85), the relations presented in (66) provide a useful numerical check.

We point out that the results of this section up to this point are valid for $q_1 < \infty$. We recall that for $q_1 = \infty$, $e_r = 0$ for $r = q_2 + 2, \cdots, 2q_2 + 1$, since $\sigma_r > 1$ for these values of r. In addition, the stability condition $a_1 < A_1$, with A_1 defined in (26), is necessary. Under these assumptions, since $0 < \sigma_r < 1$ for $r = 1, \cdots, q_2 + 1$, we find, analogously to (67), that

$$L_1|_{q_1=\infty} = 0, \tag{86}$$

as expected. Also, with the help of (53), we obtain, analogously to (68) and (70) to (72),

$$L_2|_{q_1=\infty} = \sum_{m=0}^{k_2} d_m s_{n_1}(1+\rho_m, a_1) s_{n_2}(-\rho_m, a_2) + \sum_{r=1}^{q_2+1} \frac{e_r\sigma_r}{(1-\sigma_r)}, \tag{87}$$

and

$$Q_1|_{q_1=\infty} = \sum_{r=1}^{q_2+1} \frac{e_r}{(1-\sigma_r)} \Psi_{q_2}(\omega_r), \tag{88}$$

$$R_{11}|_{q_1=\infty} = n_1\mu \sum_{r=1}^{q_2+1} \frac{e_r\sigma_r}{(1-\sigma_r)} \Psi_{q_2}(\omega_r), \tag{89}$$

and

$$R_{12}|_{q_1=\infty} = n_2\mu \sum_{r=1}^{q_2+1} \frac{e_r\sigma_r}{(1-\sigma_r)} \phi_{n_2}(\omega_r). \tag{90}$$

Similarly, from (81), we have

$$\Lambda_\infty(\xi) = \lim_{q_1\to\infty} \Lambda_{q_1}(\xi) = \frac{\xi}{(1-\xi)^2}, \qquad 0<\xi<1. \tag{91}$$

Hence, corresponding to (80), we obtain

$$V_1|_{q_1=\infty} = \sum_{r=1}^{q_2+1} \frac{e_r\sigma_r}{(1-\sigma_r)^2} \Psi_{q_2}(\omega_r). \tag{92}$$

The expressions for I_{12}, R_{22}, Q_2, V_2, X_1, and X_2, given by (69), (78), (79), and (83) to (85), still hold. Of course, the constants $\{d_m\}$ and $\{e_m\}$, which occur in these expressions, (87) to (90) and (92), are those pertaining to $q_1 = \infty$, and are determined in the manner discussed at the end of Section IV.

VI. ELIMINATION OF ONE SET OF CONSTANTS

We will now show how to determine the constants $\{e_r\}$ in the representation (27) of the probabilities p_{ij} for $n_1 \le i \le k_1$, $n_2 \le j \le k_2$, in terms of the constants $\{d_m\}$ in the representation (17) of the probabilities p_{ij} for $0 \le i \le n_1$, $0 \le j \le k_2$. For $n_1 \le i < k_1$, $n_2 \le j \le k_2$ we have, from (5),

$$\begin{aligned}[a_1 + a_2(1-\delta_{jk_2}) + n_1 + n_2]p_{ij} \\ = a_1 p_{i-1,j} + [(a_1+a_2)\delta_{in_1}\delta_{jn_2} + a_2(1-\delta_{jn_2})]p_{i,j-1} \\ + (n_1 + n_2\delta_{jn_2})p_{i+1,j} + n_2(1-\delta_{jk_2})p_{i,j+1}.\end{aligned} \tag{93}$$

We consider this equation for $n_1 \le i < \infty$, and introduce the generating function

$$G_j(z) = \sum_{i=n_1}^{\infty} p_{ij} z^{i-n_1}. \tag{94}$$

The series will be convergent for sufficiently small $|z|$.

We consider $z \neq 0$ and define

$$\zeta = (1 - z)\left(\frac{n_1}{z} - a_1\right). \tag{95}$$

Then, from (93) to (95), it follows that

$$\left[a_2(1 - \delta_{jk_2}) + n_2\left(1 - \frac{1}{z}\,\delta_{jn_2}\right) - \zeta\right]G_j(z) - a_2(1 - \delta_{jn_2})G_{j-1}(z) - n_2(1 - \delta_{jk_2})G_{j+1}(z) = a_1 p_{n_1-1,j} + (a_1 + a_2)\delta_{jn_2}p_{n_1,j-1} - \frac{1}{z}\,(n_1 + n_2\delta_{jn_2})p_{n_1,j}, \tag{96}$$

for $n_2 \le j \le k_2$. We now define $G_j^{(l)}(z)$ for $n_2 \le l \le k_2$ as the solutions of the equations

$$\left[a_2(1 - \delta_{jk_2}) + n_2\left(1 - \frac{1}{z}\,\delta_{jn_2}\right) - \zeta\right]G_j^{(l)}(z) - a_2(1 - \delta_{jn_2})G_{j-1}^{(l)}(z) - n_2(1 - \delta_{jk_2})G_{j+1}^{(l)}(z) = -\frac{1}{z}\,\delta_{jl}, \tag{97}$$

for $n_2 \le j \le k_2$. Then, from (96) and (97), we have

$$G_j(z) = \sum_{l=n_2}^{k_2} (n_1 p_{n_1,l} - a_1 z p_{n_1-1,l})G_j^{(l)}(z) + [n_2 p_{n_1,n_2} - (a_1 + a_2)z p_{n_1,n_2-1}]G_j^{(n_2)}(z). \tag{98}$$

We define

$$\Upsilon_j(z) = \left(\frac{a_2}{n_2}\right)^{j-n_2}\left[\Psi_{j-n_2}(\zeta) - \frac{n_2}{a_2 z}\,\Psi_{j-n_2-1}(\zeta)\right], \tag{99}$$

where $\Psi_l(\rho)$ is as defined in (12). It is shown in Appendix C that the solution of (97) is

$$G_j^{(l)}(z) = \begin{cases} \left(\dfrac{n_2}{a_2}\right)^{l-n_2} \dfrac{\phi_l(\zeta)\Upsilon_j(z)}{(1 - z)H(1/z)}, & n_2 \le j \le l, \\[2ex] \left(\dfrac{n_2}{a_2}\right)^{l-n_2} \dfrac{\Upsilon_l(z)\phi_j(\zeta)}{(1 - z)H(1/z)}, & l \le j \le k_2, \end{cases} \tag{100}$$

for $n_2 \le l \le k_2$, where $\phi_j(\rho)$ and $H(\sigma)$ are as defined in (13) and (25), respectively, with ω as defined in (20). We know that $H(\sigma_r) = 0$, $r = 1, \cdots, 2q_2 + 1$, with $0 < \sigma_1 < \cdots < \sigma_{2q_2+1}$. As discussed in Section III, $\sigma_{q_2} < 1$, $\sigma_{q_2+2} > 1$, and $\sigma_{q_2+1} < 1$ if $a_1 < A_1$, and $\sigma_{q_2+1} > 1$ if $a_1 > A_1$, where A_1 is given by (26). We assume that $a_1 \neq A_1$, so that $\sigma_{q_2+1} \neq 1$.

Now, from (12), (13), (20), (25), (95), and (99), $z^{q_2}H(1/z)$, $z^{k_2-j}\phi_j(\zeta)$ and $z^{j-n_2}\Upsilon_j(z)$ are polynomials in z of degrees $2q_2 + 1$, $2(k_2 - j)$ and

$2(j - n_2)$, respectively. It follows from (100) that $zG_j^{(l)}(z) \to 0$ as $z \to \infty$, for $n_2 \le j \le k_2$, $n_2 \le l \le k_2$. Hence, from (98), we have $G_j(z) \to 0$ as $z \to \infty$, for $n_2 \le j \le k_2$. It follows that we may expand $G_j(z)$ in partial fractions in the form

$$G_j(z) = \sum_{r=0}^{2q_2+1} \frac{g_{jr}}{(z - 1/\sigma_r)}, \qquad n_2 \le j \le k_2, \tag{101}$$

where $\sigma_0 = 1$, and

$$g_{jr} = \lim_{z \to 1/\sigma_r} [(z - 1/\sigma_r)G_j(z)]. \tag{102}$$

From (93 , for $n_1 \le i < k_1$, (94), and (101), we deduce that

$$p_{ij} = -\sum_{r=0}^{2q_2+1} \sigma_r^{i+1-n_1} g_{jr}, \qquad n_1 \le i \le k_1, \qquad n_2 \le j \le k_2. \tag{103}$$

We proceed with the explicit calculation of g_{jr}. Let

$$g_{jr}^{(l)} = \lim_{z \to 1/\sigma_r} [(z - 1/\sigma_r)G_j^{(l)}(z)]. \tag{104}$$

Then, from (98), (102), and (104), we obtain

$$\sigma_r g_{jr} = \sum_{l=n_2}^{k_2} (n_1 \sigma_r p_{n_1,l} - a_1 p_{n_1-1,l}) g_{jr}^{(l)} + [n_2 \sigma_r p_{n_1,n_2} - (a_1 + a_2) p_{n_1,n_2-1}] g_{jr}^{(n_2)}. \tag{105}$$

As we noted in Section IV, $\phi_{n_2}(\omega_r) \neq 0$. It is shown in Appendix C that

$$\Upsilon_j(1/\sigma_r) = \frac{\phi_j(\omega_r)}{\phi_{n_2}(\omega_r)}, \qquad n_2 \le j \le k_2, \tag{106}$$

with ω_r given by (47). Hence, from (100), (104), and (106), since $\sigma_0 = 1$, we obtain

$$g_{j0}^{(l)} = -\left(\frac{n_2}{a_2}\right)^{l-n_2} \frac{\phi_l(0)\phi_j(0)}{\phi_{n_2}(0)H(1)} = -\frac{\phi_j(0)}{H(1)} \tag{107}$$

from (137). We note that $H(1) \neq 0$, since we have assumed that $a_1 \neq A_1$. We also obtain

$$g_{jr}^{(l)} = \left(\frac{n_2}{a_2}\right)^{l-n_2} \frac{\phi_l(\omega_r)\phi_j(\omega_r)}{\sigma_r(1 - \sigma_r)H'(\sigma_r)\phi_{n_2}(\omega_r)}, \qquad r = 1, \cdots, 2q_2 + 1, \tag{108}$$

where the prime denotes derivative.

From (27), (103), (105), and (107), since $\sigma_0 = 1$, it follows that

$$H(1)e_0 = \sum_{l=n_2}^{k_2} (n_1 p_{n_1,l} - a_1 p_{n_1-1,l}) + n_2 p_{n_1,n_2} - (a_1 + a_2) p_{n_1,n_2-1}. \tag{109}$$

If we substitute for $p_{n_1,l}$ and $p_{n_1-1,l}$, $n_2 \le l \le k_2$, and p_{n_1,n_2-1} from (17), and sum on l, and use (13) and (128), we obtain

$$H(1)e_0 = \sum_{m=0}^{k_2} d_m[\rho_m s_{n_1-1}(1+\rho_m, a_1)s_{n_2}(-\rho_m, a_2)\Psi_{q_2}(\rho_m)$$

$$+ n_2 s_{n_1}(\rho_m, a_1)s_{n_2}(-\rho_m, a_2)\phi_{n_2}(\rho_m)$$

$$- (a_1+a_2)s_{n_1}(\rho_m, a_1)s_{n_2-1}(-\rho_m, a_2)\phi_{n_2}(\rho_m)]. \quad (110)$$

After reduction with the help of (46) and some recurrence relations, we deduce from (110) that

$$H(1)e_0 = -\sum_{m=0}^{k_2} d_m\phi_{n_2}(\rho_m)[\rho_m s_{n_1}(1+\rho_m, a_1)s_{n_2-1}(1-\rho_m, a_2)$$

$$+ a_1 s_{n_1}(\rho_m, a_1)s_{n_2-1}(-\rho_m, a_2)]. \quad (111)$$

But, summation of (37) from $j = 0$ to $n_2 - 1$ shows that the term on the right-hand side of (111) is zero, so that

$$H(1)e_0 = 0. \quad (112)$$

Since $H(1) \neq 0$ for $a_1 \neq A_1$, it follows that $e_0 = 0$, as was shown in Section IV.

Next, from (27), (103), (105), and (108), we deduce that

$$\sigma_r(1-\sigma_r)H'(\sigma_r)\phi_{n_2}(\omega_r)e_r$$

$$= \sum_{l=n_2}^{k_2} (a_1 p_{n_1-1,l} - n_1\sigma_r p_{n_1,l})\left(\frac{n_2}{a_2}\right)^{l-n_2}\phi_l(\omega_r)$$

$$+ [(a_1+a_2)p_{n_1,n_2-1} - n_2\sigma_r p_{n_1,n_2}]\phi_{n_2}(\omega_r), \quad (113)$$

for $r = 1, \cdots, 2q_2 + 1$. If we substitute for $p_{n_1,l}$ and $p_{n_1-1,l}$, $n_2 \le l \le k_2$, and p_{n_1,n_2-1} from (17), we obtain

$$\sigma_r(1-\sigma_r)H'(\sigma_r)e_r = \sum_{m=0}^{k_2} d_m[a_1 s_{n_1-1}(\rho_m, a_1) - n_1\sigma_r s_{n_1}(\rho_m, a_1)]$$

$$\cdot s_{n_2}(-\rho_m, a_2)\sum_{l=n_2}^{k_2}\left(\frac{n_2}{a_2}\right)^{l-n_2}\phi_l(\rho_m)\phi_l(\omega_r)/\phi_{n_2}(\omega_r) + \sum_{m=0}^{k_2} d_m s_{n_1}(\rho_m, a_1)$$

$$\cdot[(a_1+a_2)s_{n_2-1}(-\rho_m, a_2) - n_2\sigma_r s_{n_2}(-\rho_m, a_2)]\phi_{n_2}(\rho_m), \quad (114)$$

for $r = 1, \cdots, 2q_2 + 1$. The sums on l are given explicitly by (144), or by (145) if $\rho_m = \omega_r$.

If we set $\sigma_r = 1$ in the right-hand side of (114), so that $\omega_r = 0$ from (47), it is found after reduction that the expression reduces to $-H(1)e_0$, as given by (111), and hence is zero, from (112). It follows that $(1 - \sigma_r)$ may be factored from both sides of (114). Consequently, we may consider the limiting case $a_1 \to A_1$, and obtain a finite expression for e_{q_2+1}, even though $\sigma_{q_2+1} \to 1$ as $a_1 \to A_1$.

Now that we have determined the constants $\{e_r\}$ in terms of the constants $\{d_m\}$, it remains to determine the latter. These are determined from (37) for $0 \le j \le n_2 - 1$, (39) for $n_2 + 1 \le j \le k_2$ and (49). The redundancy of (39) for $j = n_2$ was discussed in Section IV, and follows from the fact that $H(1)e_0 = 0$. Thus, the constants $\{d_m\}$ are determined from the boundary conditions at $i = n_1$, $0 \le j \le n_2 - 1$ and at $i = k_1$, $n_2 + 1 \le j \le k_2$, and from the normalization condition, the boundary condition at $i = k_1$, $j = n_2$ being redundant. The constants $\{d_m\}$ have to be calculated numerically.

VII. AN ALTERNATE DERIVATION

In this final section, we will give another derivation of the expressions for e_0 and e_r, $r = 1, \cdots, 2q_2 + 1$, given in (109) and (113). This derivation is more direct, but somewhat obscure, and was not evident until the results had been established with the help of the generating function approach.

We make use of the representation (27) for the probabilities p_{ij}, which holds for $n_1 \le i \le k_1$ and $n_2 \le j \le k_2$, and the boundary conditions at $i = n_1$, $n_2 \le j \le k_2$, as given by (35) and (36). In particular, from (27) we have

$$p_{n_1,j} = \sum_{r=0}^{2q_2+1} e_r \phi_j(\omega_r), \qquad n_2 \le j \le k_2. \tag{115}$$

Next, from (27) and (36), with the help of (47) and (134), we obtain

$$p_{n_1-1,j} = \sum_{r=0}^{2q_2+1} \frac{e_r}{\sigma_r} \phi_j(\omega_r), \qquad n_2 + 1 \le j \le k_2. \tag{116}$$

Also, from (27) and (35), with the help of (47) and (140), we find that

$$a_1 p_{n_1-1,n_2} + (a_1 + a_2) p_{n_1,n_2-1} = a_1 \sum_{r=0}^{2q_2+1} \frac{e_r}{\sigma_r} \phi_{n_2}(\omega_r). \tag{117}$$

From (115) to (117), it follows that

$$\begin{aligned}\sum_{j=n_2}^{k_2} &(a_1 p_{n_1-1,j} - n_1 \sigma_m p_{n_1,j}) \left(\frac{n_2}{a_2}\right)^{j-n_2} \phi_j(\omega_m) \\ &+ [(a_1 + a_2) p_{n_1,n_2-1} - n_2 \sigma_m p_{n_1,n_2}] \phi_{n_2}(\omega_m) \\ &= \sigma_m \sum_{r=0}^{2q_2+1} e_r \left[\left(\frac{a_1}{\sigma_r \sigma_m} - n_1 \right) \sum_{j=n_2}^{k_2} \left(\frac{n_2}{a_2}\right)^{j-n_2} \phi_j(\omega_m) \phi_j(\omega_r) \right. \\ &\qquad \left. - n_2 \phi_{n_2}(\omega_m) \phi_{n_2}(\omega_r) \right]. \end{aligned} \tag{118}$$

Now, from (47),

$$(\omega_r - \omega_m) = (\sigma_m - \sigma_r)\left(\frac{a_1}{\sigma_r\sigma_m} - n_1\right). \tag{119}$$

But, for $a_1 \neq A_1$, we know that $\sigma_r \neq \sigma_m$ for $r \neq m$. Hence, if we set $\rho = \omega_m$ in (143), and make use of (119) and (142), we obtain

$$\left(\frac{a_1}{\sigma_r\sigma_m} - n_1\right) \sum_{j=n_2}^{k_2} \left(\frac{n_2}{a_2}\right)^{j-n_2} \phi_j(\omega_m)\phi_j(\omega_r)$$
$$= n_2\phi_{n_2}(\omega_m)\phi_{n_2}(\omega_r), \qquad r \neq m, \qquad r, m = 0, \cdots, 2q_2 + 1. \tag{120}$$

This implies that all the terms on the right-hand side of equation (118), except the ones corresponding to $r = m$, are zero.

Next, from (13), (20), (25), and (133), it follows that

$$(1 - \sigma)H(\sigma) = a_2\phi_{n_2-1}(\omega) - n_2\sigma\phi_{n_2}(\omega), \tag{121}$$

and

$$\frac{d}{d\sigma}[(1 - \sigma)H(\sigma)]$$
$$= \left(n_1 - \frac{a_1}{\sigma^2}\right)[a_2\phi'_{n_2-1}(\omega) - n_2\sigma\phi'_{n_2}(\omega)] - n_2\phi_{n_2}(\omega). \tag{122}$$

Hence, from (145) we obtain

$$\left(\frac{a_1}{\sigma_r^2} - n_1\right) \sum_{j=n_2}^{k_2} \left(\frac{n_2}{a_2}\right)^{j-n_2} [\phi_j(\omega_r)]^2 - n_2[\phi_{n_2}(\omega_r)]^2$$
$$= \phi_{n_2}(\omega_r)\left\{\frac{d}{d\sigma}[(1 - \sigma)H(\sigma)]\right\}_{\sigma=\sigma_r}, \qquad r = 0, \cdots, 2q_2 + 1. \tag{123}$$

But, $\sigma_0 = 1$ and $\omega_0 = 0$, and $H(\sigma_r) = 0$, $r = 1, \cdots, 2q_2 + 1$, so that

$$\left\{\frac{d}{d\sigma}[(1-\sigma)H(\sigma)]\right\}_{\sigma=\sigma_r} = \begin{cases} -H(1), & r=0 \\ (1-\sigma_r)H'(\sigma_r), & r=1, \cdots, 2q_2+1. \end{cases} \tag{124}$$

The expressions in (109) and (113) for e_0 and e_r, $r = 1, \cdots, 2q_2 + 1$, respectively, follow directly from (118), if we set $m = 0$ and $m = 1, \cdots, 2q_2 + 1$, respectively, and make use of (120), (123), (124), and (137).

APPENDIX A

Properties of the Eigenfunctions

We define $s_i(\lambda, a)$ by the recurrence relation

$$(a + i + \lambda)s_i(\lambda, a)$$
$$= a(1 - \delta_{i0})s_{i-1}(\lambda, a) + (i + 1)s_{i+1}(\lambda, a); \; s_0(\lambda, a) = 1, \tag{125}$$

for $i = 0, 1, \cdots$. Thus, $s_n(\lambda, a)$ is a polynomial of degree n in both λ and a, and it may be related to a Poisson-Charlier polynomial.[5,6] However, we will give here the properties of $s_n(\lambda, a)$ that we will need. An explicit formula is[1]

$$s_i(\lambda, a) = \sum_{r=0}^{i} \frac{(\lambda)_r a^{i-r}}{r!(i-r)!}, \tag{126}$$

where

$$(\lambda)_0 = 1, \qquad (\lambda)_r = \lambda(\lambda+1)\cdots(\lambda+r-1), \qquad r = 1, 2, \cdots. \tag{127}$$

It was also shown[1] that

$$(i+1)s_{i+1}(\lambda, a) = a s_i(\lambda, a) + \lambda s_i(\lambda+1, a), \tag{128}$$

and

$$s_i(\lambda, a) = s_i(\lambda+1, a) - (1-\delta_{i0})s_{i-1}(\lambda+1, a). \tag{129}$$

From (129), it follows that

$$\sum_{i=0}^{n} s_i(\lambda, a) = s_n(\lambda+1, a), \tag{130}$$

and, from (128) and (130), we deduce that

$$\sum_{i=0}^{n} (n-i)s_i(\lambda, a) = (1-\delta_{n0})s_{n-1}(\lambda+2, a). \tag{131}$$

We now turn our attention to the Chebyshev polynomials of the second kind,[7] $U_l(x)$. They may be defined by the recurrence relation

$$2xU_l(x) = U_{l+1}(x) + U_{l-1}(x); \qquad U_{-1}(x) \equiv 0, \qquad U_0(x) \equiv 1, \tag{132}$$

for $l = 0, 1, \cdots$. From (12) and (132), it follows that

$$(a_2 + n_2 - \rho)\Psi_l(\rho) = a_2\Psi_{l+1}(\rho) + n_2\Psi_{l-1}(\rho);$$
$$\Psi_{-1}(\rho) \equiv 0, \qquad \Psi_0(\rho) \equiv 1. \tag{133}$$

From (13) and (133), we deduce that

$$[a_2(1-\delta_{jk_2}) + n_2 - \rho]\phi_j(\rho) = a_2\phi_{j-1}(\rho) + n_2(1-\delta_{jk_2})\phi_{j+1}(\rho), \tag{134}$$

for $j \le k_2$. Since[7]

$$U_l\left[\frac{1}{2}\left(\xi + \frac{1}{\xi}\right)\right] = \sum_{r=0}^{l} \xi^{2r-l}, \tag{135}$$

it follows that

$$\Psi_l(0) = \left(\frac{n_2}{a_2}\right)^{l/2} U_l\left(\frac{a_2+n_2}{2\sqrt{a_2 n_2}}\right) = \sum_{r=0}^{l}\left(\frac{n_2}{a_2}\right)^r. \tag{136}$$

Hence, from (13), we have

$$\phi_j(0) = \left(\frac{n_2}{a_2}\right)^{k_2-j}. \tag{137}$$

If we set $\rho = \omega_r$ in (134), and also use (134) as it stands, we obtain

$$(\omega_r - \rho)\phi_j(\rho)\phi_j(\omega_r) = a_2[\phi_{j-1}(\rho)\phi_j(\omega_r) - \phi_j(\rho)\phi_{j-1}(\omega_r)] - n_2(1 - \delta_{jk_2})[\phi_j(\rho)\phi_{j+1}(\omega_r) - \phi_{j+1}(\rho)\phi_j(\omega_r)]. \tag{138}$$

If we now multiply (138) by $(n_2/a_2)^{j-n_2}$, and sum on j, we deduce that

$$(\omega_r - \rho) \sum_{j=n_2}^{k_2} \left(\frac{n_2}{a_2}\right)^{j-n_2} \phi_j(\rho)\phi_j(\omega_r) = a_2[\phi_{n_2-1}(\rho)\phi_{n_2}(\omega_r) - \phi_{n_2}(\rho)\phi_{n_2-1}(\omega_r)]. \tag{139}$$

But, from (21) and (22),

$$[a_2 + n_2(1 - \sigma_r) - \omega_r]\phi_{n_2}(\omega_r) = n_2\phi_{n_2+1}(\omega_r), \tag{140}$$

for $r = 0, \cdots, 2q_2 + 1$. Also, from (134), since $q_2 \geq 1$,

$$(a_2 + n_2 - \omega_r)\phi_{n_2}(\omega_r) = a_2\phi_{n_2-1}(\omega_r) + n_2\phi_{n_2+1}(\omega_r). \tag{141}$$

From (140) and (141), it follows that

$$a_2\phi_{n_2-1}(\omega_r) = n_2\sigma_r\phi_{n_2}(\omega_r), \qquad r = 0, \cdots, 2q_2 + 1. \tag{142}$$

Hence, from (139) and (142), we have

$$(\omega_r - \rho) \sum_{j=n_2}^{k_2} \left(\frac{n_2}{a_2}\right)^{j-n_2} \phi_j(\rho)\phi_j(\omega_r) = \phi_{n_2}(\omega_r)[a_2\phi_{n_2-1}(\rho) - n_2\sigma_r\phi_{n_2}(\rho)], \qquad r = 0, \cdots, 2q_2 + 1. \tag{143}$$

If we set $\rho = \rho_m$ in (143) and make use of the fact that ρ_m, $m = 0, \cdots, k_2$, are the roots of equation (15), we deduce that

$$(\omega_r - \rho_m)s_{n_2}(-\rho_m, a_2) \sum_{j=n_2}^{k_2} \left(\frac{n_2}{a_2}\right)^{j-n_2} \phi_j(\rho_m)\phi_j(\omega_r) = \phi_{n_2}(\rho_m)\phi_{n_2}(\omega_r)[a_2 s_{n_2-1}(-\rho_m, a_2) - n_2\sigma_r s_{n_2}(-\rho_m, a_2)]. \tag{144}$$

Also, if we divide (143) by $(\omega_r - \rho)$ and let $\rho \to \omega_r$, and use (142) and L'Hospital's rule, we find that

$$\sum_{j=n_2}^{k_2} \left(\frac{n_2}{a_2}\right)^{j-n_2} [\phi_j(\omega_r)]^2 = \phi_{n_2}(\omega_r)[n_2\sigma_r\phi'_{n_2}(\omega_r) - a_2\phi'_{n_2-1}(\omega_r)], \qquad r = 0, \cdots, 2q_2 + 1, \tag{145}$$

where the prime denotes derivative.

APPENDIX B

Eigenvalues Corresponding to the Primary Queueing Region

We consider here the zeros of

$$H(\sigma) \equiv \left(n_1 - \frac{a_1}{\sigma}\right)\Psi_{q_2}(\omega) + n_2\phi_{n_2}(\omega), \tag{146}$$

where

$$\omega = \left(\frac{a_1}{\sigma} - n_1\right)(1 - \sigma). \tag{147}$$

As may be seen from (12) and (13), $\Psi_{q_2}(\omega)$ and $\phi_{n_2}(\omega)$ are both polynomials in ω of degree q_2. It follows that $\sigma^{q_2+1}H(\sigma)$ is a polynomial in σ of degree $2q_2 + 1$, so that $H(\sigma) = 0$ has $2q_2 + 1$ roots. Now[7]

$$U_l(\cos\theta) = \frac{\sin(l+1)\theta}{\sin\theta}. \tag{148}$$

We define

$$\alpha_m = a_2 + n_2 - 2\sqrt{a_2 n_2}\cos\left(\frac{m\pi}{q_2+1}\right), \qquad m = 1, \cdots, q_2, \tag{149}$$

and we note that $0 < \alpha_1 < \cdots < \alpha_{q_2}$. From (12), it follows that

$$\Psi_{q_2}(\alpha_m) = 0, \qquad m = 1, \cdots, q_2. \tag{150}$$

Next, we consider

$$\phi_{n_2}(\omega) \equiv \Psi_{q_2}(\omega) - \Psi_{q_2-1}(\omega), \tag{151}$$

and we note from (137) that $\phi_{n_2}(0) > 0$. If $q_2 = 1$, then $\phi_{n_2}(\alpha_1) = -1$, and the unique zero β_1 of $\phi_{n_2}(\omega)$ satisfies $0 < \beta_1 < \alpha_1$. For $q_2 \geq 2$, we define

$$\zeta_s = a_2 + n_2 - 2\sqrt{a_2 n_2}\cos\left(\frac{s\pi}{q_2}\right), \qquad s = 1, \cdots, q_2 - 1, \tag{152}$$

and we note that

$$0 < \alpha_1 < \zeta_1 < \cdots < \alpha_{q_2-1} < \zeta_{q_2-1} < \alpha_{q_2}. \tag{153}$$

From (12) and (148), it follows that

$$\Psi_{q_2-1}(\zeta_s) = 0, \qquad s = 1, \cdots, q_2 - 1. \tag{154}$$

Then, from (150), (151), (153), and (154), we deduce that

$$\phi_{n_2}(\alpha_1) = -\Psi_{q_2-1}(\alpha_1) < 0, \qquad \phi_{n_2}(\zeta_1) = \Psi_{q_2}(\zeta_1) < 0,$$
$$\phi_{n_2}(\alpha_2) = -\Psi_{q_2-1}(\alpha_2) > 0, \cdots. \tag{155}$$

We let

$$\phi_{n_2}(\beta_m) = 0, \qquad m = 1, \cdots, q_2, \tag{156}$$

with $\beta_1 < \beta_2 < \cdots < \beta_{q_2}$. Then, it follows that

$$0 < \beta_1 < \alpha_1 < \zeta_1 < \beta_2 < \alpha_2 < \cdots < \zeta_{q_2-1} < \beta_{q_2} < \alpha_{q_2}. \qquad (157)$$

Now, from (147), it is seen that ω decreases from $+\infty$ to 0 as σ increases from 0+ to $\min(1, a_1/n_1)$, and ω increases from 0 to $+\infty$ as σ increases from $\max(1, a_1/n_1)$ to $+\infty$. Hence γ_m and δ_m, $m = 1, \cdots, q_2$, are determined uniquely by the relations

$$\left(\frac{a_1}{\gamma_m} - n_1\right)(1 - \gamma_m) = \alpha_m, \qquad 0 < \gamma_m < \min\left(1, \frac{a_1}{n_1}\right),$$

$$\left(\frac{a_1}{\delta_m} - n_1\right)(1 - \delta_m) = \beta_m, \qquad 0 < \delta_m < \min\left(1, \frac{a_1}{n_1}\right). \qquad (158)$$

It follows from (157) that

$$0 < \gamma_{q_2} < \delta_{q_2} < \cdots < \gamma_1 < \delta_1 < \min(1, a_1/n_1). \qquad (159)$$

Now, from (146) and (147), since $\phi_{n_2}(0) > 0$, we have $H(a_1/n_1) > 0$. But, from (150) and (156) to (158),

$$H(\delta_1) = \left(n_1 - \frac{a_1}{\delta_1}\right)\Psi_{q_2}(\beta_1) < 0, \qquad H(\gamma_1) = n_2\phi_{n_2}(\alpha_1) < 0. \qquad (160)$$

Also, for $q_2 \geq 2$,

$$H(\delta_2) = \left(n_1 - \frac{a_1}{\delta_2}\right)\Psi_{q_2}(\beta_2) > 0, \qquad H(\gamma_2) = n_2\phi_{n_2}(\alpha_2) > 0, \qquad (161)$$

and so on. Finally, consideration of the sign of $\Psi_{q_2}(\omega)$ for $\omega \to +\infty$ leads to

$$H(0+) = \begin{cases} -\infty, & \text{if } q_2 \text{ is even,} \\ +\infty, & \text{if } q_2 \text{ is odd.} \end{cases} \qquad (162)$$

From the above results, it follows that there are roots of $H(\sigma) = 0$ in the intervals $(0, \gamma_{q_2})$, $(\delta_{q_2}, \gamma_{q_2-1})$, $\cdots$, (δ_2, γ_1) and $(\delta_1, a_1/n_1)$. Since $\delta_1 < 1$, there are at least q_2 roots in the interval (0, 1).

Next, from (159), we have

$$\max\left(1, \frac{a_1}{n_1}\right) < \frac{a_1}{n_1\delta_1} < \frac{a_1}{n_1\gamma_1} < \cdots < \frac{a_1}{n_1\delta_{q_2}} < \frac{a_1}{n_1\gamma_{q_2}}. \qquad (163)$$

Also, from (146), (147), (150), and (156) to (158), we obtain

$$H\left(\frac{a_1}{n_1\delta_1}\right) = n_1(1 - \delta_1)\Psi_{q_2}(\beta_1) > 0, \quad H\left(\frac{a_1}{n_1\gamma_1}\right) = n_2\phi_{n_2}(\alpha_1) < 0, \qquad (164)$$

and, for $q_2 \geq 2$,

$$H\left(\frac{a_1}{n_1\delta_2}\right) = n_1(1 - \delta_2)\Psi_{q_2}(\beta_2) < 0, \quad H\left(\frac{a_1}{n_1\gamma_2}\right) = n_2\phi_{n_2}(\alpha_2) > 0, \qquad (165)$$

and so on. Hence there are q_2 roots of $H(\sigma) = 0$ in the intervals

$$\left(\frac{a_1}{n_1\delta_1}, \frac{a_1}{n_1\gamma_1}\right), \cdots, \left(\frac{a_1}{n_1\delta_{q_2}}, \frac{a_1}{n_1\gamma_{q_2}}\right).$$

These q_2 roots are greater than 1, and we have seen that at least q_2 of the remaining $q_2 + 1$ roots lie in the interval (0, 1). It is evident that the $2q_2 + 1$ roots of $H(\sigma) = 0$ are positive and distinct.

It remains to consider the root that lies in the interval $(\delta_1, a_1/n_1)$. But, from (146) and (147), with the help of (13), we obtain

$$H(1) = \Psi_{q_2}(0)(A_1 - a_1), \tag{166}$$

where A_1 is as defined in (26). Also, we have shown that $H(\delta_1) < 0$ and $H(a_1/n_1) > 0$. Hence, if $a_1 < A_1$ this root lies in the interval $(\delta_1, 1)$, and there are $q_2 + 1$ roots in the interval (0, 1), and q_2 roots greater than 1. If $a_1 > A_1$, then $a_1/n_1 > 1$ and the root lies in the interval $(1, a_1/n_1)$, and there are q_2 roots in the interval (0, 1), and $q_2 + 1$ roots greater than 1. If $a_1 = A_1$, then one root is unity, and there are q_2 roots in the interval (0, 1), and q_2 roots greater than 1.

APPENDIX C

Results Pertaining to the Generating Function

We derive here the solutions of (97), and we first consider $l = n_2$. Then,

$$[a_2(1 - \delta_{jk_2}) + n_2 - \zeta]G_j^{(n_2)}(z) = a_2 G_{j-1}^{(n_2)}(z) + n_2(1 - \delta_{jk_2})G_{j+1}^{(n_2)}(z), \tag{167}$$

for $n_2 + 1 \le j \le k_2$. It follows from (134) that

$$G_j^{(n_2)}(z) = G_{k_2}^{(n_2)}(z)\phi_j(\zeta), \qquad n_2 \le j \le k_2, \tag{168}$$

since $\phi_{k_2}(\zeta) \equiv 1$. If we set $j = n_2$ in (97), and substitute from (168), we obtain

$$\left\{\left[a_2 + n_2\left(1 - \frac{1}{z}\right) - \zeta\right]\phi_{n_2}(\zeta) - n_2\phi_{n_2+1}(\zeta)\right\}G_{k_2}^{(n_2)}(z) = -\frac{1}{z}. \tag{169}$$

With the help of (13), (95), and (133), we deduce that

$$\left[a_2 + n_2\left(1 - \frac{1}{z}\right) - \zeta\right]\phi_{n_2}(\zeta) - n_2\phi_{n_2+1}(\zeta) = \left(1 - \frac{1}{z}\right)[(n_1 - a_1 z)\Psi_{q_2}(\zeta) + n_2\phi_{n_2}(\zeta)] = \left(1 - \frac{1}{z}\right)H(1/z), \tag{170}$$

from (20) and (25). Hence,

$$(1 - z)H(1/z)G_{k_2}^{(n_2)}(z) = 1. \tag{171}$$

Since $\Upsilon_{n_2}(z) \equiv 1$, from (99), we have established (100) for $l = n_2$.

Next, we consider $l = k_2$. Then, from (97)

$$\left[a_2 + n_2\left(1 - \frac{\delta_{jn_2}}{z}\right) - \zeta\right]G_j^{(k_2)}(z) = a_2(1 - \delta_{jn_2})G_{j-1}^{(k_2)}(z) + n_2G_{j+1}^{(k_2)}(z), \quad (172)$$

for $n_2 \leq j \leq k_2 - 1$. But, it may be verified from (95), (99), and (133), that

$$\left[a_2 + n_2\left(1 - \frac{\delta_{jn_2}}{z}\right) - \zeta\right]\Upsilon_j(z) = a_2(1 - \delta_{jn_2})\Upsilon_{j-1}(z) + n_2\Upsilon_{j+1}(z), \quad (173)$$

for $j \geq n_2$. It follows that

$$G_j^{(k_2)}(z) = G_{n_2}^{(k_2)}(z)\Upsilon_j(z), \qquad n_2 \leq j \leq k_2, \quad (174)$$

since $\Upsilon_{n_2}(z) \equiv 1$. If we set $j = k_2$ in (97), and substitute from (174), we obtain

$$[(n_2 - \zeta)\Upsilon_{k_2}(z) - a_2\Upsilon_{k_2-1}(z)]G_{n_2}^{(k_2)}(z) = -\frac{1}{z}. \quad (175)$$

With the help of (13), (95), (99), and (133), this may be written in the form

$$\left(\frac{a_2}{n_2}\right)^{q_2}(1 - z)[(n_1 - a_1z)\Psi_{q_2}(\zeta) + n_2\phi_{n_2}(\zeta)]G_{n_2}^{(k_2)}(z) = 1. \quad (176)$$

It follows from (170) that

$$\left(\frac{a_2}{n_2}\right)^{q_2}(1 - z)H(1/z)G_{n_2}^{(k_2)}(z) = 1. \quad (177)$$

Since $\phi_{k_2}(\zeta) \equiv 1$, we have established (100) for $l = k_2$.

Finally, we consider $n_2 < l < k_2$. Then, from (97),

$$\left[a_2 + n_2\left(1 - \frac{\delta_{jn_2}}{z}\right) - \zeta\right]G_j^{(l)}(z) = a_2(1 - \delta_{jn_2})G_{j-1}^{(l)}(z) + n_2G_{j+1}^{(l)}(z), \quad (178)$$

for $n_2 \leq j \leq l - 1$, and

$$[a_2(1 - \delta_{jk_2}) + n_2 - \zeta]G_j^{(l)}(z) = a_2G_{j-1}^{(l)}(z) + n_2(1 - \delta_{jk_2})G_{j+1}^{(l)}(z), \quad (179)$$

for $l + 1 \leq j \leq k_2$. It follows from (134) and (173) that

$$G_j^{(l)}(z) = \begin{cases} G_{n_2}^{(l)}(z)\Upsilon_j(z), & n_2 \leq j \leq l, \\ G_{k_2}^{(l)}(z)\phi_j(\zeta), & l \leq j \leq k_2, \end{cases} \quad (180)$$

since $\Upsilon_{n_2}(z) \equiv 1$ and $\phi_{k_2}(\zeta) \equiv 1$. The consistency of the representations for $j = l$ requires that

$$G_{n_2}^{(l)}(z)\Upsilon_l(z) = G_{k_2}^{(l)}(z)\phi_l(\zeta). \tag{181}$$

Also, if we set $j = l$ in (97), and substitute from (180), we obtain

$$(a_2 + n_2 - \zeta)G_{k_2}^{(l)}(z)\phi_l(\zeta) - a_2 G_{n_2}^{(l)}(z)\Upsilon_{l-1}(z) - n_2 G_{k_2}^{(l)}(z)\phi_{l+1}(\zeta) = -\frac{1}{z}. \tag{182}$$

It remains to solve (181) and (182) for $G_{n_2}^{(l)}(z)$ and $G_{k_2}^{(l)}(z)$.

Now, since $l < k_2$, it follows from (134) and (182) that

$$a_2[G_{k_2}^{(l)}(z)\phi_{l-1}(\zeta) - G_{n_2}^{(l)}(z)\Upsilon_{l-1}(z)] = -\frac{1}{z}. \tag{183}$$

If we eliminate $G_{n_2}^{(l)}(z)$ with the help of (181), we find that

$$a_2 z G_{k_2}^{(l)}(z)[\Upsilon_l(z)\phi_{l-1}(\zeta) - \phi_l(\zeta)\Upsilon_{l-1}(z)] = -\Upsilon_l(z). \tag{184}$$

But, from (134) and (173), for $n_2 < j < k_2$,

$$(a_2 + n_2 - \zeta)\phi_j(\zeta) = a_2\phi_{j-1}(\zeta) + n_2\phi_{j+1}(\zeta), \tag{185}$$

and

$$(a_2 + n_2 - \zeta)\Upsilon_j(z) = a_2\Upsilon_{j-1}(z) + n_2\Upsilon_{j+1}(z). \tag{186}$$

Hence,

$$a_2[\Upsilon_j(z)\phi_{j-1}(\zeta) - \phi_j(\zeta)\Upsilon_{j-1}(z)] = n_2[\Upsilon_{j+1}(z)\phi_j(\zeta) - \phi_{j+1}(\zeta)\Upsilon_j(z)], \tag{187}$$

for $n_2 < j < k_2$. Since $n_2 < l < k_2$, it follows that

$$\Upsilon_l(z)\phi_{l-1}(\zeta) - \phi_l(\zeta)\Upsilon_{l-1}(z) = \left(\frac{a_2}{n_2}\right)^{l-n_2-1}[\Upsilon_{n_2+1}(z)\phi_{n_2}(\zeta) - \phi_{n_2+1}(\zeta)\Upsilon_{n_2}(z)]. \tag{188}$$

But, from (99) and (133),

$$n_2[\Upsilon_{n_2+1}(z)\phi_{n_2}(\zeta) - \phi_{n_2+1}(\zeta)\Upsilon_{n_2}(z)] = \left[a_2 + n_2\left(1 - \frac{1}{z}\right) - \zeta\right]\phi_{n_2}(\zeta) - n_2\phi_{n_2+1}(\zeta). \tag{189}$$

Hence, from (170), (188), and (189), we obtain

$$a_2[\Upsilon_l(z)\phi_{l-1}(\zeta) - \phi_l(\zeta)\Upsilon_{l-1}(z)] = \left(\frac{a_2}{n_2}\right)^{l-n_2}\left(1 - \frac{1}{z}\right)H\left(\frac{1}{z}\right). \tag{190}$$

It follows from (184) that

$$\left(\frac{a_2}{n_2}\right)^{l-n_2}(1-z)H\left(\frac{1}{z}\right)G_{k_2}^{(l)}(z) = \Upsilon_l(z). \tag{191}$$

In view of (180), (181), and (191), we have established (100) for $n_2 < l < k_2$.

We will now establish the relationship (106). From (21) and (22), it follows that

$$[a_2 + n_2(1-\sigma_r) - \omega_r]\phi_{n_2}(\omega_r) = n_2\phi_{n_2+1}(\omega_r). \tag{192}$$

Hence, from (47), (95), and (189), since $\Upsilon_{n_2}(z) \equiv 1$, we have

$$\Upsilon_{n_2+1}(1/\sigma_r)\phi_{n_2}(\omega_r) = \phi_{n_2+1}(\omega_r). \tag{193}$$

As we noted in Section IV, $\phi_{n_2}(\omega_r) \neq 0$. Since $\Upsilon_{n_2}(1/\sigma_r) \equiv 1$, we see that (106) holds for both $j = n_2$ and $j = n_2 + 1$. But, from (47), (95), (185), and (186), $\phi_j(\omega_r)$ and $\Upsilon_j(1/\sigma_r)$ are both solutions of

$$(a_2 + n_2 - \omega_r)h_j = a_2h_{j-1} + n_2h_{j+1}, \qquad n_2 < j < k_2. \tag{194}$$

It follows that (106) holds for $n_2 \leq j \leq k_2$.

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