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Electromagnetic Propagation in Homogeneous Media With Hermitian Permeability and Permittivity

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The problem of electromagnetic radiation traveling in a general homogeneous, but anisotropic and gyrotropic, medium has been solved. The plane wave representation is used to convert Maxwell's equations into a general eigenproblem, which allows for tensor permeability and permittivity and for electric and magnetic gyrotropy. This formulation can be applied directly to computer evaluation of wave velocities and polarizations for a given wave-normal direction. The reflection and refraction directions and amplitudes at an interface between two different, general anisotropic and gyrotropic media are solved. It is outlined how, with the aid of a digital computer, ray tracing through a general succession of anisotropic and gyrotropic, but locally homogeneous, media can be carried out.

I. INTRODUCTION

Traditionally, the theory of light propagation in transparent, anisotropic media has assumed that the permeability of the medium was the same as that of the vacuum, or at least that it was isotropic. Conversely, the theory of microwave propagation in ferrites and garnets has assumed that the permittivity was isotropic in spite of the permeability being anisotropic and possibly, gyrotropic. (References 1 and 2 are examples of the optics tradition and the microwave tradition, respectively.)

Both viewpoints are incomplete, particularly, when applied to infrared optics where anisotropy effects in both permeability and permittivity could be important. As a case in point, Yttrium Iron Garnet (YIG) is optically transparent at infrared wavelengths beyond 1 μm[3] and at microwave frequencies; it could be used at wavelengths where the effects of permeability and permittivity are comparable.

We develop the theory of electromagnetic wave propagation in a homogeneous medium where both the permeability and permittivity are anisotropic. Furthermore, by assuming that the permittivity and permeability tensors are Hermitian, not merely real and symmetric, we incorporate the more general effects of gyrotropy.

II. BASIC ASSUMPTIONS

We assume the general validity of Maxwell's equations without "source terms" (no charges or currents):

$$\nabla \times \mathbf{H} = \partial \mathbf{D}/\partial t \tag{1}$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \tag{2}$$

$$\nabla \cdot \mathbf{D} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

where **E**, **D**, **B**, **H** are complex vector fields which are functions of space and time ($\mathbf{r}$, t).

We shall restrict our considerations to plane-wave solutions where the above four vector fields can be cast in the form

$$\mathbf{F}(\mathbf{r}, t) = \mathbf{F}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \tag{5}$$

where $\mathbf{F}_0$ is a complex vector constant and ω is a real constant. The vector, $\mathbf{k}$, is usually considered to be real, but may be complex even in lossless media (for "evanescent waves").

To complete the specifications, we need to define material-dependent relationships between the four vector fields of Maxwell's equations. We do so by means of the electromagnetic energy density:

$$U = \tfrac{1}{2}\mathbf{E}^* \cdot \mathbf{D} + \tfrac{1}{2}\mathbf{H}^* \cdot \mathbf{B}. \tag{6}$$

For a plane wave, we can substitute (5) into (6) to show that U is independent of time and position (assuming $\mathbf{k}$ is real):

$$U = \tfrac{1}{2}\mathbf{E}_0^* \cdot \mathbf{D}_0 + \tfrac{1}{2}\mathbf{H}_0^* \cdot \mathbf{B}_0. \tag{7}$$

By assuming that U is a purely quadratic function of the components of $\mathbf{E}_0^*$, we obtain

$$\mathbf{D}_0 = \overleftrightarrow{\epsilon}\mathbf{E}_0. \tag{8}$$

The Hermiticity of $\overleftrightarrow{\epsilon}$, the permittivity or dielectric tensor, follows from the reality of U.

Similarly, by assuming that U is a purely quadratic function of the components of $\mathbf{H}_0$, we conclude that

$$\mathbf{B}_0 = \overleftrightarrow{\mu}\mathbf{H}_0, \tag{9}$$

where the permeability tensor, $\overleftrightarrow{\mu}$, is Hermitian.

Our assumptions are consistent with magneto-optic effects where, conventionally, a Polder tensor[4] is defined, e.g.,

$$\overleftrightarrow{\mu} = \begin{pmatrix} \mu & i\mu_a & 0 \\ -i\mu_a & \mu & 0 \\ 0 & 0 & \mu' \end{pmatrix}. \tag{10}$$

Using the plane-wave representation (5) in Maxwell's first pair of equations, (1) and (2), we obtain

$$\mathbf{k} \times \mathbf{H} = -\omega\mathbf{D}. \tag{11}$$

$$\mathbf{k} \times \mathbf{E} = \omega\mathbf{B}. \tag{12}$$

Similar substitutions into Maxwell's last pair of equations, (3) and (4),

$$\mathbf{k} \cdot \mathbf{D} = 0 \tag{13}$$

$$\mathbf{k} \cdot \mathbf{B} = 0, \tag{14}$$

which, in fact, follow from (11) and (12) when $\omega \neq 0$.

III. THE WAVE VELOCITY

We rewrite the plane-wave representation (5) in a suggestive form

$$\mathbf{F} = \mathbf{F}_0 \exp[-(\mathrm{Im}\mathbf{k}) \cdot \mathbf{r}] \cdot \exp[i|\mathrm{Re}\ \mathbf{k}|(\hat{n} \cdot \mathbf{r} - \mathrm{v}_w t)], \tag{15}$$

where

$$\hat{n} = \mathrm{Re}\ \mathbf{k}/|\mathrm{Re}\ \mathbf{k}| \tag{16}$$

is a unit vector that is normal to the wave fronts, and the wave velocity

$$\mathrm{v}_w = \omega/|\mathrm{Re}\ \mathbf{k}|. \tag{17}$$

Thus, v_w represents the rate at which the wave fronts appear to be advancing in the $\hat{n}$ direction.

We usually assume that $\mathrm{Im}\ \mathbf{k} = 0$. However, for $\mathrm{Im}\ \mathbf{k} \neq 0$ we have an "evanescent wave"; a wave whose intensity diminishes in the Im $\mathbf{k}$ direction. It will be shown later, under the discussion of Poynting's vector, that Im $\mathbf{k}$ is perpendicular to Poynting's vector in nondissipative media.

Let us assume now that $\mathrm{Im}\ \mathbf{k} = 0$ and that, consequently, $\hat{n}$ is parallel to $\mathbf{k}$. We can restate (25) and (26) as follows:

$$\hat{n} \times \mathbf{H} = -\mathrm{v}_w\mathbf{D} = -\mathrm{v}_w\overleftrightarrow{\epsilon}\mathbf{E} \tag{18}$$

$$\hat{n} \times \mathbf{E} = v_w \mathbf{B} = v_w \overleftrightarrow{\mu} \mathbf{H}. \tag{19}$$

If we go to a six-dimensional representation, we may combine (18) and (19) into a single-matrix equation:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \hat{n}_z & -\hat{n}_y \\ 0 & 0 & 0 & -\hat{n}_z & 0 & \hat{n}_x \\ 0 & 0 & 0 & \hat{n}_y & -\hat{n}_x & 0 \\ 0 & -\hat{n}_z & \hat{n}_y & 0 & 0 & 0 \\ \hat{n}_z & 0 & -\hat{n}_x & 0 & 0 & 0 \\ -\hat{n}_y & \hat{n}_x & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix}$$

$$= v_w \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} & 0 & 0 & 0 \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} & 0 & 0 & 0 \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_{xx} & \mu_{xy} & \mu_{xz} \\ 0 & 0 & 0 & \mu_{yx} & \mu_{yy} & \mu_{yz} \\ 0 & 0 & 0 & \mu_{zx} & \mu_{zy} & \mu_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \\ H_x \\ H_y \\ H_z \end{pmatrix}. \tag{20}$$

For convenience, we shall compress (20) by defining a more compact notation

$$\begin{pmatrix} 0 & -\overleftrightarrow{N} \\ \overleftrightarrow{N} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = v_w \begin{pmatrix} \overleftrightarrow{\epsilon} & 0 \\ 0 & \overleftrightarrow{\mu} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \tag{21}$$

or

$$\overleftrightarrow{\alpha} \boldsymbol{\nu} = v_y \overleftrightarrow{\beta} \boldsymbol{\nu}. \tag{22}$$

Note that (22) is clearly a general eigenproblem. The tensor, $\overleftrightarrow{\beta}$, is determined by the permeability and permittivity of the medium. If $\mathbf{k}$ is real, then $\overleftrightarrow{\alpha}$ is determined completely by $\hat{n}$.

Both $\overleftrightarrow{\alpha}$ and $\overleftrightarrow{\beta}$ are usually Hermitian. Tensor, $\overleftrightarrow{\alpha}$, is symmetric. It is real if $\mathbf{k}$ was real. We can restate (6) as

$$2U = \boldsymbol{\nu}^* \cdot \overleftrightarrow{\beta} \boldsymbol{\nu}. \tag{23}$$

Away from dramatic resonances, we may assume that U is positive definite as well as real. If so, $\overleftrightarrow{\beta}$ is not only Hermitian but positive definite.

When $\overleftrightarrow{\beta}$ is both Hermitian and positive definite, we can find its inverse and its square root (all of which are Hermitian and positive definite as well). Thus, we are able to convert (8) to an ordinary eigenproblem of a Hermitian matrix:

$$(\overleftrightarrow{\beta}^{-1/2} \overleftrightarrow{\alpha} \overleftrightarrow{\beta}^{-1/2})(\overleftrightarrow{\beta}^{1/2} \nu) = v_w (\overleftrightarrow{\beta}^{1/2} \nu). \tag{24}$$

The Hermiticity of $(\overleftrightarrow{\beta}^{-1/2} \overleftrightarrow{\alpha} \overleftrightarrow{\beta}^{-1/2})$ guarantees that there are six eigen-solutions, each with a real v_w.

We can use the Hermiticity of $\overleftrightarrow{\alpha}$ and $\overleftrightarrow{\beta}$ to prove orthogonality:

$$\boldsymbol{\nu}_1^* \cdot \overleftrightarrow{\alpha}\, \boldsymbol{\nu}_2 = \mathrm{v}_{w1}^* \boldsymbol{\nu}_1^* \cdot \overleftrightarrow{\beta}\, \boldsymbol{\nu}_2 = \mathrm{v}_{w2} \boldsymbol{\nu}_1^* \cdot \overleftrightarrow{\beta}\, \boldsymbol{\nu}_2. \tag{25}$$

Clearly, when v_{w1}^* does not equal v_{w2}, the corresponding eigenvectors satisfy the orthogonality relations:

$$\boldsymbol{\nu}_1^* \cdot \overleftrightarrow{\alpha}\, \boldsymbol{\nu}_2 = 0 = \boldsymbol{\nu}_1^* \cdot \overleftrightarrow{\beta}\, \boldsymbol{\nu}_2. \tag{26}$$

Additional orthogonality relationships can be proved if $\mathrm{v}_{w1}^n \neq \mathrm{v}_{w2}^n$ $(n = 1, 2, 3 \cdots)$:

$$\boldsymbol{\nu}_1^* \cdot \overleftrightarrow{\beta} (\overleftrightarrow{\beta}^{-1} \overleftrightarrow{\alpha})^n \boldsymbol{\nu}_2 = 0. \tag{27}$$

Two eigensolutions can be provided automatically:

$$\boldsymbol{\nu}_5 = \begin{pmatrix} \hat{n} \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\nu}_6 = \begin{pmatrix} 0 \\ \hat{n} \end{pmatrix}$$

$$\mathrm{v}_{w5} = 0 \quad \text{and} \quad \mathrm{v}_{w6} = 0. \tag{28}$$

Thus, we have to find four more eigensolutions to this six-dimensional eigenproblem.

The reality of $\hat{n}$ and the special form of $\overleftrightarrow{\beta}$ enable us to simplify the problem still further. Suppose we have a solution to (21). It follows then that

$$\begin{pmatrix} 0 & -\overleftrightarrow{N} \\ \overleftrightarrow{N} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ -\mathbf{H} \end{pmatrix} = -\mathrm{v}_w \begin{pmatrix} \overleftrightarrow{\epsilon} & 0 \\ 0 & \overleftrightarrow{\mu} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ -\mathbf{H} \end{pmatrix}. \tag{29}$$

Thus, for each eigensolution with wave velocity v_w, there is an analogous solution that travels in the opposite direction.

In summary, of the six solutions to the eigenproblem (20) for a specified, real $\hat{n}$, two are trivial and have zero wave velocity, two are positive and correspond to two polarized solutions propagating in the $+\hat{n}$ direction, and the last two are negative, and, consequently, travel in the $-\hat{n}$ direction.

It is worthwhile noting that we have included nonreciprocal cases, since Faraday rotation, the traditional nonreciprocal effect, is manifested by complex, yet Hermitian, permittivity and/or permeability tensors. These effects do not seem to cause the wave velocities in the $-\hat{n}$ direction to differ from those in the $+\hat{n}$ direction. The nonreciprocal effects must be manifest in the differing polarizations between the forward and the backward directions.

Polarization is specified usually by the orientation of the electromagnetic field components normal to the relevant velocity direction (here $\hat{n}$). Thus, from eqs. (13) and (14) we should concentrate upon **D**, when **k** is real. In general, the components of **D**, say, are complex; then the polarization should be considered elliptical. The special case of $\mathbf{D} \propto \mathbf{D}^*$ is called "linear polarization." Since **D** is confined to the plane perpendicular to **k**, its description for a specified $\hat{n}$ can be reduced always to two complex vector components; this two-dimensional representation is known as the Jones vector.[5,6]

Nonreciprocity is the inequivalence of wave velocity (or attenuation, which we ignore here) for a given polarization for the $\pm\hat{n}$ directions. The reason that Faraday rotation is nonreciprocal is that the polarization for a given magnitude of wave velocity in the $\pm\hat{n}$ directions are not the same; the Jones vector representation for the $\pm\hat{n}$ directions associate right-handed elliptical polarization to left-hand elliptical polarization in the opposite direction. Two polarizations in the $\pm\hat{n}$ directions would be considered the same if the corresponding **D** fields obeyed the proportionality

$$\mathbf{D}_1 \propto \mathbf{D}_2^*. \tag{30}$$

Thus, nonreciprocity for solutions consistent with (20) could occur only when $\mathbf{D}^*$ is not proportional to **D**; no nonreciprocity occurs for modes that are linearly polarized. It also appears to be a consequence of the form of (20) that no polarization-independent nonreciprocity can be produced in homogeneous media. However, if the medium is bianisotropic, then polarization-independent nonreciprocity is possible (see Ref. 7).

The orthogonality relations (26) and (27) derived earlier, can be used to show that the two polarizations associated with the two different wave velocities in a given $\hat{n}$-direction are "perpendicular." Let us call these solutions, "fast" and "slow." By assuming that $\omega \neq 0$, we can derive from the orthogonality relation

$$\mathbf{E}_{\text{fast}}^* \cdot \mathbf{D}_{\text{slow}} + \mathbf{H}_{\text{fast}}^* \cdot \mathbf{B}_{\text{slow}} = 0. \tag{31}$$

Since $\mathbf{E}_{\text{fast}}, -\mathbf{H}_{\text{fast}}$ is again a solution with yet another v_w, we can derive from orthogonality

$$\mathbf{E}_{\text{fast}}^* \cdot \mathbf{D}_{\text{slow}} - \mathbf{H}_{\text{fast}}^* \cdot \mathbf{B}_{\text{slow}} = 0. \tag{32}$$

Thus, we can conclude from (31) and (32) that

$$\mathbf{E}_{\text{fast}}^* \cdot \mathbf{D}_{\text{slow}} = 0 \tag{33}$$

$$\mathbf{H}_{\text{fast}}^* \cdot \mathbf{B}_{\text{slow}} = 0. \tag{34}$$

We see from (31) also that there is no interference in the energy density between the fast and slow solutions for a given direction, $\hat{n}$. Suppose at some location

$$\boldsymbol{\nu} = a\boldsymbol{\nu}_{\text{fast}} + b\boldsymbol{\nu}_{\text{slow}}, \tag{35}$$

then

$$U = \tfrac{1}{2}\boldsymbol{\nu} \cdot \vec{\beta}\boldsymbol{\nu} = \tfrac{1}{2}|a|^2 \boldsymbol{\nu}_{\text{fast}}^* \cdot \vec{\beta}\boldsymbol{\nu}_{\text{fast}} + \tfrac{1}{2}|b|^2 \boldsymbol{\nu}_{\text{slow}}^* \cdot \vec{\beta}\boldsymbol{\nu}_{\text{slow}}. \tag{36}$$

Usually, the literature considers the more restrictive cases,[8] where either $\overleftrightarrow{\mu}$ or $\overleftrightarrow{\epsilon}$ is isotropic and conclude

$$\mathbf{D}^*_{\text{fast}} \cdot \mathbf{D}_{\text{slow}} = 0(?),$$

which is not always true for more general cases such as are considered here.

IV. THE POYNTING VECTOR

If we consider the time dependence of U, we derive:

$$\frac{dU}{dt} = \frac{1}{2} \frac{d\boldsymbol{\nu}^* \overleftrightarrow{\beta}}{dt} \boldsymbol{\nu} + \frac{1}{2} \boldsymbol{\nu}^* \cdot \frac{d\overleftrightarrow{\beta} \boldsymbol{\nu}}{dt} = \text{Re}\left(\boldsymbol{\nu}^* \cdot \frac{d\overleftrightarrow{\beta} \boldsymbol{\nu}}{dt} \right). \tag{37}$$

By using Maxwell's eqs. (1) and (2), we obtain further:

$$dU/dt = \text{Re}(\mathbf{E}^* \cdot \nabla \times \mathbf{H} - \mathbf{H}^* \cdot \nabla \times \mathbf{E}) = \nabla \cdot [\text{Re}(\mathbf{E}^* \times \mathbf{H})]. \tag{38}$$

As we have identified dU/dt with the divergence of a "flux," this flux should be identified with Poynting's vector. With this in mind, we make two definitions

$$\mathbf{G} \equiv \mathbf{E}^* \times \mathbf{H} \tag{39}$$

and

$$\mathbf{S} \equiv \text{Re}\, \mathbf{G}. \tag{40}$$

For a plane wave of real $\mathbf{k}$, $\mathbf{G}$ is independent of time and space. However, it may be complex. The real part of $\mathbf{G}$, namely, $\mathbf{S}$, is Poynting's vector and is always real, obviously.

The vectors $\mathbf{G}$ and $\mathbf{G}^*$ are useful because they are "dual" to $\mathbf{k}$. Consider

$$\begin{aligned} \mathbf{G} \times \mathbf{D} &= \mathbf{H}(\mathbf{E}^* \cdot \mathbf{D}) - \mathbf{E}^*(\mathbf{H} \cdot \mathbf{D}) \\ &= \mathbf{H}(\mathbf{E}^* \cdot \mathbf{D}) + \omega^{-1}\mathbf{E}^*(\mathbf{H} \cdot \mathbf{k} \times \mathbf{H}) = U\mathbf{H}. \end{aligned}$$

Consequently,

$$\mathbf{G} \times \mathbf{D} = U\mathbf{H}. \tag{41}$$

A similar derivation shows

$$\mathbf{G}^* \times \mathbf{B} = -U\mathbf{E}. \tag{42}$$

These two equations are analogous to (12) and (11) and suggest a series of steps that are usable for calculating the ray velocity.

The ray velocity vector, $\mathbf{v}_r$, which is parallel to the Poynting vector, $\mathbf{S}$, is defined as

$$\mathbf{v}_r \equiv \mathbf{S}/U \tag{43a}$$

$$v_r \equiv |\mathbf{v}_r|. \tag{43b}$$

The Appendix proves that the Poynting vector is indeed the ray direction.

Unlike the theory for **k**, we will *not* be able to assume that **G** is real. Nevertheless, we define

$$\hat{g} \equiv \mathbf{G}/|\mathrm{Re}\,\mathbf{G}| \tag{44}$$

and

$$\hat{s} \equiv \mathrm{Re}\,\hat{g} = \mathbf{S}/|\mathbf{S}|. \tag{45}$$

With these notational conventions, we restate (5) and (6)

$$\hat{g}^* \times \mathbf{B} = -\mathrm{v}_r^{-1}\mathbf{E} = -\mathrm{v}_r^{-1}\overleftrightarrow{\epsilon}^{-1}\mathbf{D} \tag{46}$$

$$\hat{g} \times \mathbf{D} = \mathrm{v}_r^{-1}\mathbf{H} = \mathrm{v}_r^{-1}\overleftrightarrow{\mu}^{-1}\mathbf{B}. \tag{47}$$

These two relations can be rewritten as an eigencondition:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \hat{g}_z^* & -\hat{g}_y^* \\ 0 & 0 & 0 & -\hat{g}_z^* & 0 & \hat{g}_x^* \\ 0 & 0 & 0 & \hat{g}_y^* & -\hat{g}_x^* & 0 \\ 0 & -\hat{g}_z & \hat{g}_y & 0 & 0 & 0 \\ \hat{g}_z & 0 & -\hat{g}_x & 0 & 0 & 0 \\ -\hat{g}_y & \hat{g}_x & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \end{pmatrix} = \mathrm{v}_r^{-1}\overleftrightarrow{\beta}^{-1} \begin{pmatrix} D_x \\ D_y \\ D_z \\ B_x \\ B_y \\ B_z \end{pmatrix}. \tag{48}$$

And a terser notation, analogous to (22), could be

$$\overleftrightarrow{\Gamma}\boldsymbol{v} = \mathrm{v}_r^{-1}\overleftrightarrow{\beta}^{-1}\boldsymbol{v}, \tag{49}$$

where $\overleftrightarrow{\Gamma}$ is Hermitian; $\overleftrightarrow{\beta}$ is positive definite and Hermitian because

$$2U = \boldsymbol{v}^* \cdot \overleftrightarrow{\beta}^{-1}\boldsymbol{v} \tag{50}$$

in analogy to (23).

In accordance with the analogous discussion of the solutions (22), the six real eigenvalue solutions to (49) will consist of two zero v_r^{-1}, two positive v_r^{-1}, and their two negatives. Similar orthogonality relations can also be generated.

The duality[9] would be complete if $\hat{g}$ were equal to $\hat{s}$. If **k** and $\overleftrightarrow{\beta}$ were real, we could expect that **G** = **S**. However, we have demonstrated by computer that a real **k** does not always lead to a real **G**.

Thus, we must conclude that to solve for v_r starting with a given $\hat{s}$, we must find the appropriate values of Im $\hat{g}$ that produce a real **k** in a test such as

$$U\mathbf{k} = \omega(\mathbf{D}^* \times \mathbf{B}). \tag{51}$$

We shall show now that if $\mathbf{k}$ is complex, then its imaginary part is perpendicular to $\mathbf{S}$. From (11) and (12), we can generate

$$-\mathbf{E}^* \cdot \mathbf{k} \times \mathbf{H} = \omega \mathbf{E}^* \cdot \mathbf{D} \tag{52}$$

$$\mathbf{H}^* \cdot \mathbf{k} \times \mathbf{E} = \omega \mathbf{H}^* \cdot \mathbf{B}. \tag{53}$$

Adding (52) and (53), we obtain

$$\mathbf{k} \cdot (\mathbf{E}^* \times \mathbf{H} + \mathbf{E} \times \mathbf{H}^*) = \omega(\mathbf{E}^* \cdot \mathbf{D} + \mathbf{H}^* \cdot \mathbf{B}). \tag{54}$$

Using the definitions of Poynting's vector and energy density, we can restate (54) as

$$\mathbf{k} \cdot \mathbf{S} = \omega U. \tag{55}$$

Because ω, U, and $\mathbf{S}$ are always assumed to be real, we can conclude that

$$\mathbf{S} \cdot \mathrm{Im}\, \mathbf{k} = 0. \tag{56}$$

V. REFLECTION AND REFRACTION

We can find the reflection and refraction of a plane wave at a flat interface between two different, homogeneous media of the general type we have been considering.

Figure 1 indicates what we should expect at the interface between two media. The $\mathbf{k}_0$ is the propagation vector of the incident wave. Usually, there will be two reflected waves, represented by $\mathbf{k}_1$ and $\mathbf{k}_2$, and two refracted (transmitted) waves, represented by $\mathbf{k}_3$ and $\mathbf{k}_4$. Details of the four "scattered" waves are determined by boundary conditions.

To determine $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and $\mathbf{k}_4$ for a particular $\mathbf{k}_0$, the boundary conditions we need to consider are the equality of the $\omega(\mathbf{k})$ associated with all five $\mathbf{k}$ vectors. In addition, because the geometry possesses two-dimensional translational symmetry in the plane of the interface, we must match the components in the plane of the interface for all the $\mathbf{k}$ vectors. These constraints are sufficient to determine the four "scattered" $\mathbf{k}$ vectors for a given $\mathbf{k}_0$.

Finding the scattered $\mathbf{k}$ vectors amounts to specifying the components of $\gamma \equiv \mathbf{k}/\omega$ that are in the plane of the interface, $\gamma_{\parallel}$, and solving an eigenproblem that provides

$$\gamma_{\perp}^{-1} \equiv \omega / |\mathbf{k} \cdot \hat{x}_{\perp}|. \tag{57}$$

We define $\hat{x}_{\perp}$, for notational convenience, as a unit vector perpendicular to the interface and directed in the same sense as the incident radiation.

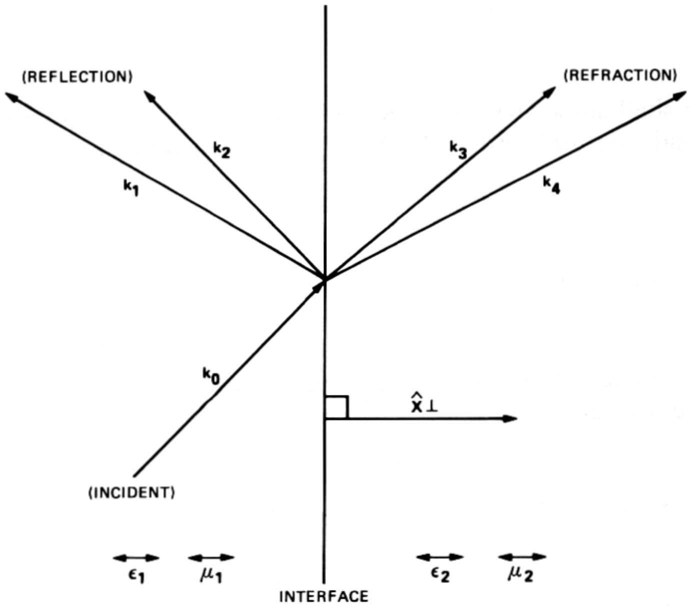

Fig. 1—Scattering at an interface between two anisotropic media.

From (11) and (12), we have

$$-\gamma \times \mathbf{H} = \overleftrightarrow{\epsilon}\mathbf{E} \tag{58}$$

$$\gamma \times \mathbf{E} = \overleftrightarrow{\mu}\mathbf{H}. \tag{59}$$

Rearranging, we have

$$-\gamma_{\perp}\hat{x}_{\perp} \times \mathbf{H} = \overleftrightarrow{\epsilon}\mathbf{E} + \gamma_{\|} \times \mathbf{H} \tag{60}$$

$$\gamma_{\perp}\hat{x}_{\perp} \times \mathbf{E} = \overleftrightarrow{\mu}\mathbf{H} - \gamma_{\|} \times \mathbf{E}. \tag{61}$$

Thus, we have a general eigenproblem again:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \hat{x}_{\perp z} & -\hat{x}_{\perp y} \\ 0 & 0 & 0 & -\hat{x}_{\perp z} & 0 & \hat{x}_{\perp x} \\ 0 & 0 & 0 & \hat{x}_{\perp y} & -\hat{x}_{\perp x} & 0 \\ 0 & -\hat{x}_{\perp z} & \hat{x}_{\perp y} & 0 & 0 & 0 \\ \hat{x}_{\perp z} & 0 & -\hat{x}_{\perp x} & 0 & 0 & 0 \\ -\hat{x}_{\perp y} & \hat{x}_{\perp x} & 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{\nu}$$

$$= \gamma_{\perp}^{-1} \left[\overleftrightarrow{\beta} + \begin{pmatrix} 0 & 0 & 0 & 0 & -\gamma_{\|x} & \gamma_{\|z} \\ 0 & 0 & 0 & \gamma_{\|z} & 0 & -\gamma_{\|x} \\ 0 & 0 & 0 & -\gamma_{\|y} & \gamma_{\|x} & 0 \\ 0 & \gamma_{\|z} & -\gamma_{\|y} & 0 & 0 & 0 \\ -\gamma_{\|z} & 0 & \gamma_{\|x} & 0 & 0 & 0 \\ \gamma_{\|y} & -\gamma_{\|x} & 0 & 0 & 0 & 0 \end{pmatrix} \right] \boldsymbol{\nu}$$

$$\equiv \overleftrightarrow{\alpha}_{\perp} \boldsymbol{\nu} = \gamma_{\perp}^{-1} \overleftrightarrow{\beta}_{\perp} \boldsymbol{\nu}. \tag{62}$$

This eigenproblem has a lot of similarity to (20) in that the matrices on each side of the equation are Hermitian. Further, we expect two extraneous solutions for which the eigenvalues $\gamma_\perp^{-1} = 0$ and the eigenvectors

$$\boldsymbol{\nu}_5 = \begin{pmatrix} \hat{x}_\perp \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\nu}_6 = \begin{pmatrix} 0 \\ \hat{x}_\perp \end{pmatrix}, \tag{63}$$

are analogous to (28). We expect two "reflected" solutions with negative

$$\hat{s}_\perp \equiv \hat{x}_\perp \cdot \hat{s}. \tag{64}$$

The sign of $\gamma_\perp$ is not so relevant because **k** and **S** are rarely parallel in anisotropic media.

We expect a maximum of two "refracted" solutions when we use $\overleftrightarrow{\epsilon}_2$ and $\overleftrightarrow{\mu}_2$ in (62) and look for solutions with positive $\hat{s}_\perp$. We shall have less than the maximum number of refracted solutions when $\gamma_\parallel$ is getting large enough so that $\overleftrightarrow{\beta}_\perp$ ceases to be positive definite.

If $\overleftrightarrow{\beta}_\perp$ is neither positive definite nor negative definite, a derivation similar to (24) does not exist and some of the solutions for $\gamma_\perp^{-1}$ may be complex. If $\gamma_\perp$ is complex, then the corresponding **E** and **H** fields satisfy from (26):

$$\boldsymbol{\nu}^* \cdot \overleftrightarrow{\alpha}_\perp \boldsymbol{\nu} = 0. \tag{65}$$

Therefore, we have an evanescent solution:

$$\hat{s}_\perp = 0. \tag{66}$$

Note that the sign of $\gamma_\perp$ is *not* the criterion for acceptable solutions to (62), rather the sign of $\hat{s}_\perp$ is the criterion.

VI. SCATTERING AMPLITUDES

The solution of (62) has provided the values of $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and $\mathbf{k}_4$ for a given $\mathbf{k}_0$. Let us label the corresponding eigenvectors: $\boldsymbol{\nu}_1$, $\boldsymbol{\nu}_2$, $\boldsymbol{\nu}_3$, $\boldsymbol{\nu}_4$, and $\boldsymbol{\nu}_0$. We shall assume no particular normalization for the eigenvectors, but suppose for a given $\boldsymbol{\nu}_0$ there will be particular amplitudes $a_i\boldsymbol{\nu}_i$ for each scattered wave. Naturally, the solutions for the a_i are dependent upon the normalizations of the $\boldsymbol{\nu}_i$.

From Maxwell's eqs. (1) and (2), we must satisfy continuity of the components of **E** and **H** in the plane of the interface. These conditions may be restated as

$$\overleftrightarrow{\alpha}_\perp \boldsymbol{\nu}_0 = \overleftrightarrow{\alpha}_\perp(-a_1\boldsymbol{\nu}_1 - a_2\boldsymbol{\nu}_2 + a_3\boldsymbol{\nu}_3 + a_4\boldsymbol{\nu}_4). \tag{67}$$

Equation (67) can be restated as four simultaneous equations by suitable definitions. In the following, assume the index, i, takes on the integer values 1 through 4. Let

$$b_i \equiv \boldsymbol{\nu}_i^* \cdot \vec{\vec{\alpha}}_\perp \boldsymbol{\nu}_0 \tag{68}$$

$$c_{ij} \equiv \boldsymbol{\nu}_i^* \cdot \vec{\vec{\alpha}}_\perp \boldsymbol{\nu}_j . \tag{69}$$

Then it follows that

$$b_i = \sum_{j=1}^{4} c_{ij} a_j . \tag{70}$$

The form of the Hermitian, 4 × 4 matrix, (c_{ij}), is simplified somewhat by orthogonality relations:

$$c_{12} = c_{21} = c_{34} = c_{43} = 0. \tag{71}$$

Thus, by solving (70), we obtain the four coupling coefficients, a_i.

VII. RAY TRACING

We envision that our results will find general applicability in problems where "ray-tracing" is to be performed through anisotropic/gyrotropic media and where the relative attenuation of alternative paths is to be explored. We shall indicate how to use the preceding results to this end.

As indicated in the discussion following equation (50), there are difficulties in working directly with the Poynting vector, **S**. Even so it is the more macroscopic variable as it is the ray direction, as opposed to **k** which depends upon the wavefront normal. Most of the difficulty is resolved if we make the assumption (which does not seem too restrictive in practice) that the initial medium of the ray is to be isotropic (such as air, glass, index matching fluid, etc.).

For isotropic media, the **k** direction is identical with the ray direction. It is relatively straightforward to calculate $\gamma_\parallel$ at the first interface when the initial medium is isotropic. By solving (62) and, subsequently, calculating the corresponding $\mathbf{S}_i$ and, if desired, the amplitudes, a_i, we can decide which of the scattered ray(s) we may wish to trace further.

After we have chosen which scattered wave to follow, we use the Poynting vector to trace the ray to the next interface. We know the **k** of this ray and are not faced with the awkward task of reconstructing it from other information. If we are interested in exact phase relationships, these are easy to obtain:

$$\boldsymbol{\nu}(\text{second interface}) = \exp[i(\mathbf{k}\cdot\Delta\mathbf{r})]\boldsymbol{\nu}(\text{first interface}). \tag{72}$$

The spatial displacement from the first to the second interface has been designated $\Delta\mathbf{r}$.

We can calculate the $\gamma_\parallel$ at the second interface and proceed to solve (62). We calculate the new $\mathbf{S}_i$ of the scattered waves, etc., and choose which one we will follow to the next interface. This process can be iterated as often as necessary to trace the ray of interest through an arbitrary optical system made up of homogeneous media and interfaces.

VIII. SUMMARY AND CONCLUSIONS

We have developed a general formalism suitable for computation in the analysis of optical propagation through a succession of anisotropic, gyrotropic, homogeneous media. Such problems arise in a large variety of optical devices that are made of materials such as YIG, $LiNbO_3$, rutile, and calcite.

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APPENDIX

Equivalence of Ray Direction and $\hat{\mathbf{s}}$.

We shall develop a first-order perturbation theory of the general eigenproblem. The result is directly applicable to the calculation of group velocity. When the permeability and permittivity are assumed to be frequency independent, the group velocity must be parallel to the ray velocity.

Suppose we wish to perturb the eigenproblem:

$$\overleftrightarrow{A}\mathbf{v}_i = \lambda_i \overleftrightarrow{B}\mathbf{v}_i, \tag{73}$$

where $\overleftrightarrow{A}$ and $\overleftrightarrow{B}$ are Hermitian. In addition, $\overleftrightarrow{B}$ is positive definite so that a complete set of eigensolutions with real λ_i is expected.

Our perturbation assumption is that the replacement of

$$\overleftrightarrow{A} \rightarrow \overleftrightarrow{A} + \epsilon\overleftrightarrow{A}' \tag{74}$$

requires the corresponding substitutions:

$$\lambda_i \rightarrow \lambda_i + \epsilon\lambda_i' + \cdots \tag{75}$$

$$\mathbf{v}_i \rightarrow \mathbf{v}_i + \epsilon\mathbf{v}_i' + \cdots \tag{76}$$

Therefore,

$$(\overleftrightarrow{A} + \epsilon\overleftrightarrow{A}')(\mathbf{v}_i + \epsilon\mathbf{v}_i' + \cdots) = (\lambda_i + \epsilon\lambda_i' + \cdots)\overleftrightarrow{B}(\mathbf{v}_i + \epsilon\mathbf{v}_i' + \cdots). \tag{77}$$

Expanding and collecting terms with common powers of ϵ, we obtain for ϵ^0 the original (73); we obtain for ϵ^1:

$$\overleftrightarrow{A}\mathbf{v}_i' + \overleftrightarrow{A}'\mathbf{v}_i = \lambda_i\overleftrightarrow{B}\mathbf{v}_i' + \lambda_i'\overleftrightarrow{B}\mathbf{v}_i. \tag{78}$$

Taking the dot product with $\mathbf{v}_i^*$ and cancelling terms:

$$\mathbf{v}_i^* \cdot \overleftrightarrow{A}' \mathbf{v}_i = \lambda' \mathbf{v}_i^* \cdot \overleftrightarrow{B} \mathbf{v}_i. \tag{79}$$

Therefore, we can calculate λ_i' from

$$\lambda_i' = \frac{\mathbf{v}_i^* \cdot \overleftrightarrow{A} \mathbf{v}_i}{\mathbf{v}_i^* \cdot \overleftrightarrow{B} \mathbf{v}_i}. \tag{80}$$

The group velocity is calculated by differentiating $\omega(\mathbf{k})$ with respect to the components of $\mathbf{k}$.[10] The ω dependence is determined from an eigenproblem of the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & k_z & -k_y \\ 0 & 0 & 0 & -k_z & 0 & k_x \\ 0 & 0 & 0 & k_y & -k_x & 0 \\ 0 & -k_z & k_y & 0 & 0 & 0 \\ k_z & 0 & -k_x & 0 & 0 & 0 \\ -k_y & k_x & 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{\nu} = \omega \overleftrightarrow{\beta} \boldsymbol{\nu}. \tag{81}$$

Thus, for example, the x-component of the group velocity, v_{gx} is determined from (80) and (81):

$$\mathrm{v}_{gx} = \partial\omega/\partial k_x = \frac{1}{2U} \boldsymbol{\nu}^* \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{\nu}$$

$$= (E_y^* H_z - E_z^* H_y - H_y^* E_z + H_z^* E_y)/2U = S_x/U. \tag{82}$$

Thus, we have demonstrated:

$$\mathbf{v}_g = \frac{\mathbf{S}}{U} = \mathbf{v}_r. \tag{83}$$

REFERENCES

1. M. Born and E. Wolf, *Principles of Optics*, 5th ed., New York: Pergamon Press, 1975, Chapter 14.
2. B. Lax and K. J. Button, *Microwave Ferrites and Ferrimagnetics*, New York: McGraw-Hill, Chapter 7, 1962.
3. J. F Dillon, Jr., J. Appl. Phys., *39,* No. 2 (February 1968), p. 922.
4. D. Polder, "On the Theory of Ferromagnetic Resonance," Phil. Mag., *40* (1949), pp. 99–115.
5. R. W. Ditchburn, *Light*, 3rd ed., New York: Academic Press, 1976, p. 463.
6. G. R. Fowles, "*Introduction to Modern Optics*," 2nd ed., New York: Holt, Rinehart, and Winston, 1975, pp. 33–7.
7. J. A. Kong, Proc. IEEE, *60* (September 1972), p. 1036.
8. R. W. Ditchburn, "*Light*," 3rd ed., New York: Academic Press, 1976, p. 575.
9. Ibid., p. 578.
10. E. E. Bergmann, "Comments on Precise Definition of Group Velocity," Amer. J. Phys., *44,* No. 9 (September 1976), p. 890.