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Peak Signal-to-Noise Ratio Formulas for Multistage Delta Modulation With RC-Shaped Gaussian Input Signals

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A multistage delta modulation (MSDM) encoder contains a number of delta modulation (DM) stages, where each delta modulator encodes the band-limited error of the previous delta modulator. The DM binary outputs are then multiplexed for transmission. By this technique, substantial gains in s/n *compared to a single-stage DM can be achieved at high transmitted bit rate to message bandwidth ratios (f_p/f_c). For Gaussian input signals having band-limited resistance-capacitance (RC) spectra, the peak* s/n *performance of MSDM as a function of (f_p/f_c) and the number of DM stages is presented. It is shown that like PCM, MSDM exchanges* s/n *with (f_p/f_c) on an exponential basis.*

I. INTRODUCTION

Delta modulation (DM) has been extensively studied,[1–8] and following its integration onto a chip,[9] it is being increasingly used in industrial applications. The salient advantages of DM are robustness to transmission errors; tolerance to clock jitter; simple filtering requirements; suitability for encryption; and low complexity resulting in inexpensive implementation. In typical applications, the ratio f_p/f_c is <10, where f_p is the transmitted bit rate and f_c is the bandwidth of the message signal. However, DM does not efficiently improve its s/n with increasing f_p/f_c, particularly when compared to pulse code modulation (PCM). As a consequence, DM is rarely used to encode high-quality audio signals because of the excessive f_p/f_c ratios required.

In DM, the quantization noise is dependent on the error $e(t)$ between the input signal $x(t)$ and a locally reconstructed version $y(t)$ (formed by locally decoding the transmitted bit stream). The $y(t)$ signal essentially tracks $x(t)$, and the polarity of the transmitted bit is identical to the polarity of the tracking error $e(t)$ at any sampling instant. To increase s/n by increasing f_p, requires $e(t)$ to decrease. It might be supposed that the reduction in $e(t)$ would be enhanced if adaptive delta modulation[1] (ADM) is used rather than linear DM.[1] In ADM, the changes in $y(t)$ per clock period, i.e., the step sizes, are not constant as in linear DM. Various step-size algorithms have been used in ADM, but all occasionally produce inappropriate step-sizes resulting in "overshoot-noise."[10] In fact, ADM is generally not used to increase peak s/n but, rather, to greatly extend the dynamic range of nonadaptive DM. However, with care the peak s/n can be enhanced at high clock rates, but not by significant amounts.[11]

The basic problem with any form of DM is that the encoder generates information which is only dependent on the polarity of the error. No description of the magnitude of the error is available at the receiver. Das and Chatterjee[12] made a proposal to overcome this defect by conceiving an encoder composed of many DM stages, each encoding the band-limited error signal of the preceding stage. In this way, a more accurate description of the tracking error is available at the receiver, and the exchange of s/n with transmitted bit-rate is greatly enhanced. This method of modulation is multistage delta modulation (MSDM).

Initially it was claimed[12] that MSDM had a better coding efficiency than conventional PCM for the same information rate, but subsequent work[13] using computer simulation up to 3-DM stages showed that although MSDM is better than DM, it does not perform as well as PCM operating with "4σ loading." A theory of MSDM was presented by Franks, Schachter, and Shilling,[14] together with computer simulation of a two-stage MSDM. Chakravarthy and Faruqui[15] constructed a two-stage MSDM using adaptive DM, and refined the expressions for s/n previously propounded.[14]

In spite of these endeavors, there appeared to be a need for more precise formulations of the peak s/n of MSDM. These expressions were found to involve the summation of the peak s/n of numerous DM stages; therefore, it became necessary to develop accurate, yet simple, expressions of s/n for a DM encoder. This was done, and the findings published separately[8] as part of the theory on linear DM. We now use these results for their intended purpose, namely, to provide simple equations for the peak s/n of MSDM when encoding Gaussian input signals with band-limited RC spectra. In the pursuit of this goal, we hope to provide new insight into the behavior of MSDM. Later in Section IV, we compare the s/n performance of MSDM with DM, PCM, and differential pulse code modulation (DPCM). It should be noted that we do not play advocate for MSDM, but merely endeavor to place its s/n performance in perspective. No attempt is made to judge its relative complexity. We commence with a description of the principle of MSDM.

II. PRINCIPLE OF MSDM

The simplest form of MSDM uses two delta modulators in the encoder as shown in Fig. 1. The analog input signal is band-limited by filter F_1 to give the MSDM input signal $x(t)$. The first delta modulator DM_1 encodes $x(t)$ producing binary information representing the polarity of the tracking error $e_1(t)$. In delta modulation, as distinct from MSDM, the error in the recovered signal is the filtered $e_1(t)$ signal, namely $\tilde{e}_1(t)$. Two-stage MSDM reduces this error by encoding $\tilde{e}_1(t)$ by a second delta modulator DM_2 and, thereby, making it available at the receiver. By this means, the overall error is reduced to the in-band error $\tilde{e}_2(t)$ of DM_2. Observe that in our nomenclature a tilde ($\tilde{}$) above a symbol means that it has been low-pass filtered by a filter having a linear phase-frequency characteristic.

Thus, in the two-stage MSDM, the signals $x(t)$ and $\tilde{e}_1(t)$ are encoded by DM_1 and DM_2 to yield binary signals $L_1(t)$ and $L_2(t)$. Typically, both delta modulators will be clocked at the same rate f_s, resulting in a transmitted bit-rate of $f_p = 2f_s$. At the receiver, the $L_1(t)$ and $L_2(t)$ signals are demultiplexed and decoded to give $y_1(t)$ and $y_2(t)$, respectively. A delay D is introduced in the first channel to compensate for the delay resulting from the band-limiting of $e_1(t)$ by filter F_I at the input to DM_2. The delayed decoded signal $z_1(t)$ is added to $y_2(t)$, and the noise residing outside the highest frequency f_c in $x(t)$ is removed by the final filter F_o to yield a recovered signal $m(t)$ that is a close approximation to $x(t)$. We assume that the binary signals $L_1(t)$ and $L_2(t)$ are generated without error.

The scheme is extendable to N delta modulators, each encoding the band-limited error signal of the preceding modulator and operating at a clock rate f_{si}, $i = 1, 2, \cdots, N$. Figure 2 shows an N-stage MSDM system. Signals $x(t)$ and $\tilde{e}_k(t)$, $k = 1, 2, \cdots, N-1$, are encoded by delta modulators DM_k, $k = 1, 2, \cdots, N$, into binary signals $L_k(t)$, $k = 1, 2, \cdots, N$. The binary signals are multiplexed, and transmitted at a bit-rate f_p whose value is the sum of the DM sampling frequencies. After demultiplexing, each of the $L_k(t)$ signals are decoded into $y_k(t)$, $k = 1, 2, \cdots, N$ and delayed by the networks designated D_k, $k = 1, 2, \cdots, N-1$, in Fig. 2, with the exception of $y_N(t)$. We will assume that filters F_1, F_I, and F_o are identical linear filters that impose a signal delay of t_o seconds, whereas the delays associated with the networks D_k are integer multiples of t_o, being $(N-1)t_o$ for the first channel ($k = 1$), and reducing by t_o for subsequent channels. The locally decoded signals in each channel, suitably compensated by the

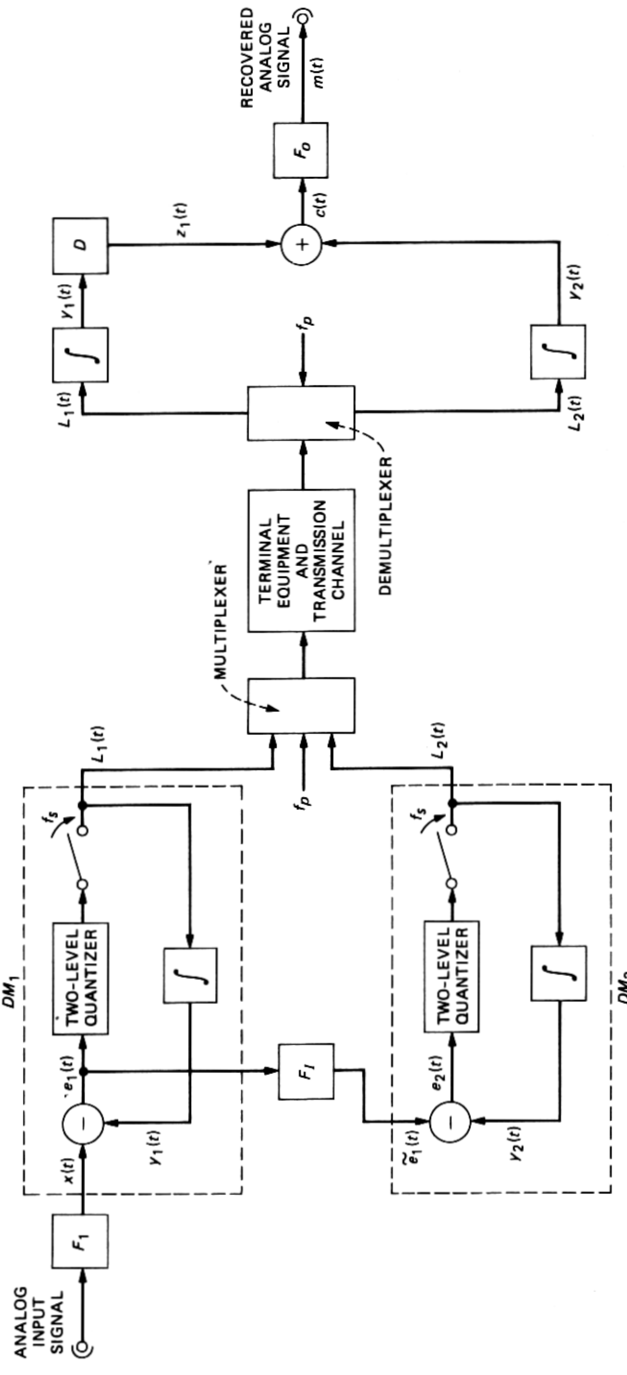

Fig. 1—Two-stage MSDM system.

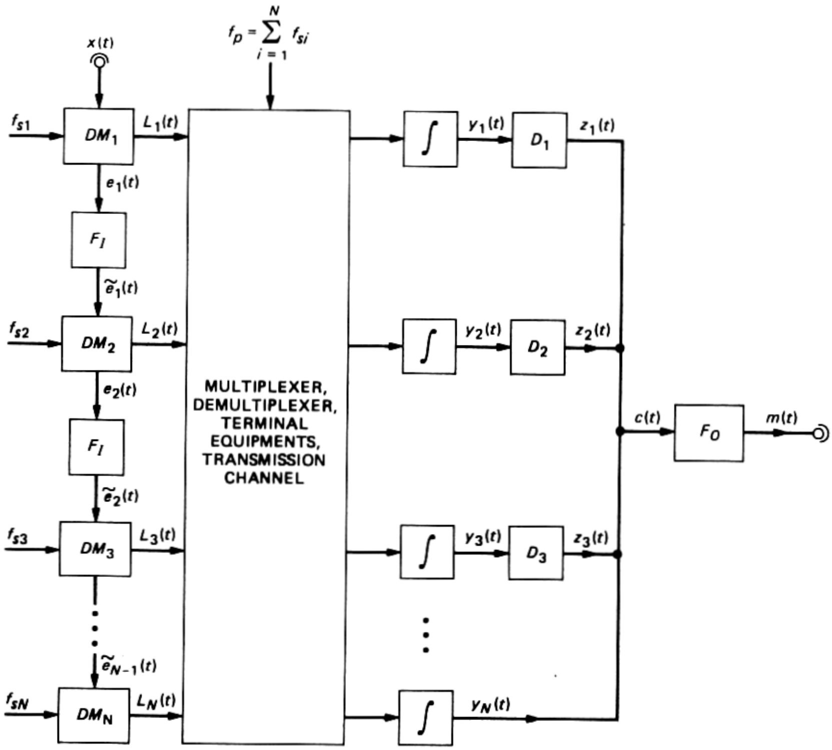

Fig. 2—N-stage MSDM system.

delays D_k for the delays caused by filters F_I in the encoder, are combined to give

$$c(t) = y_N(t) + \sum_{k=1}^{N-1} z_k(t), \tag{1}$$

where $z_k(t)$ is the signal at the output of D_k. Now the locally decoded signals at the outputs of the integrators are

$$y_1(t) = x(t) - e_1(t)$$

and

$$y_k(t) = \tilde{e}_{k-1}(t) - e_k(t), \qquad k = 2, 3, \cdots, N-1$$

and after delay compensation they become

$$z_1(t) = x(t - (N-1)t_o) - e_1(t - (N-1)t_o) \tag{2}$$

and

$$z_k(t) = \tilde{e}_{k-1}(t - (N-k)t_o) - e_k(t - (N-k)t_o),$$
$$k = 2, 3, \cdots, N-1, \tag{3}$$

respectively. Substituting $z_1(t)$ and $z_k(t)$ from eqs. (2) and (3) into eq. (1) yields

$$c(t) = x[t - (N-1)t_o] - e_N(t) - \sum_{k=1}^{N-1} \{e_k[t - (N-k)t_o] - \tilde{e}_k[t - (N-k-1]t_o)\}, \quad (4)$$

and upon filtering $c(t)$ by filter F_o to remove out-of-band noise, the recovered signal is obtained,

$$m(t) = x(t - Nt_o) - \tilde{e}_N(t). \quad (5)$$

This signal $m(t)$ is composed of the input signal $x(t)$ delayed by Nt_o, and a noise component that is the filtered error signal of the Nth stage DM. Observe that the error signals in each DM stage, with the exception of the last stage, cancel out because of the choice of delays D_k in the MSDM decoder channels.

Therefore, the s/n of MSDM is

$$\text{s/n} = \frac{\langle x^2(t) \rangle}{\langle \tilde{e}_N^2(t) \rangle}, \quad (6)$$

where $\langle (\cdot) \rangle$ means time averaging of $(\cdot)$. Alternatively, eq. (6) can be expressed as

$$\text{s/n} = \frac{\langle x^2(t) \rangle}{\langle \tilde{e}_1(t) \rangle} \frac{\langle \tilde{e}_1(t) \rangle}{\langle \tilde{e}_2(t) \rangle} \cdots \frac{\langle \tilde{e}_{N-1}(t) \rangle}{\langle \tilde{e}_N(t) \rangle} = \prod_{i=1}^{N} \text{s/n}_i, \quad (7)$$

the product of the s/n's of each DM stage. The s/n in dBs is

$$\text{S/N} = \sum_{i=1}^{N} \text{S/N}_i, \quad (8)$$

where

$$\text{S/N}_i = 10 \log_{10} \text{s/n}_i, \quad (9)$$

The upper case S/N in eq. (9) is in dBs, and the lower case s/n is a ratio. Thus, the S/N of an MSDM system is the sum of the S/N's of each DM stage. The next problem is to determine these S/N's in terms of DM parameters.

III. PEAK S/N OF MSDM

We will assume that each DM stage has a step-size which produces peak s/n for that stage. For a Gaussian input signal $x(t)$ band-limited to frequency f_c, the peak s/n for the first DM stage in the MSDM encoder may be expressed as[8]

$$\hat{s/n}_{RC} = \frac{0.059}{C^2_{opt,RC}}\left(\frac{\mu}{\beta}\right)\left(\frac{f_{si}}{f_c}\right)^3, \qquad 0.01 \le \beta \le 0.5 \tag{10}$$

and

$$\hat{s/n}_F = \frac{0.177}{C^2_{opt,F}}\left(\frac{f_{si}}{f_c}\right)^3, \tag{11}$$

where the subscripts RC and F refer to RC and flat Gaussian input signals, respectively. The flat Gaussian signal is band-limited white noise occupying the frequency band $\pm f_c$, and by RC filtering this signal, the RC Gaussian signal is obtained. The sampling frequency in eqs. (10) and (11) is f_{si}; the break frequency of the RC Gaussian input signal is f_1; and β and μ are given by

$$\beta = \frac{f_1}{f_c} \tag{12}$$

and

$$\mu = \frac{\tan^{-1}(1/\beta)}{1 - \beta\tan^{-1}(1/\beta)}. \tag{13}$$

The optimum slope loading factors are

$$C_{opt,RC} = 1.3\left[\log_e\left(\frac{f_{si}}{f_c}\right)\right]^{0.72} \tag{14}$$

and

$$C_{opt,F} = 0.5 + 0.722\log_e\left(\frac{f_{si}}{f_c}\right). \tag{15}$$

Consequently, if the first delta modulator stage DM_1 is encoding an RC Gaussian input signal the peak s/n can be expressed from eq. (10) as

$$\hat{s/n}_{RC} = \theta_{RC} f^3_{s1}, \tag{16}$$

where θ_{RC} is a DM parameter whose value is evident from eq. (10), and f_{s1} is the clock frequency for DM_1. Observe that θ_{RC} has a relatively weak dependence on f_{s1} [see eq. (14)] and this dependency will be considered later in the calculation of the peak s/n of MSDM.

Irrespective of the spectrum of $x_1(t)$, the signals applied to subsequent DM stages will be assumed to be flat Gaussian signals. This is a reasonable assumption as each DM, excluding the first, encodes the band-limited error signal from the preceding stage. To substantiate this assumption, we make the following points. Granular noise is known[8] to dominate slope overload noise when a delta modulator operates at its maximum s/n. Computer simulation results for granular noise when the s/n is close to its peak value have shown[7] that the noise spectrum is flat over the message bandwidth. Models have also been proposed[16] for DM that have flat noise spectra when the encoder is optimally loaded. Let us now consider the probability density function (PDF) of the filtered error signal $\tilde{e}_1(t)$. Although the unfiltered DM error signal $e(t)$ is uniformly distributed over the range of twice the DM step-size,[8] the act of filtering $e(t)$ by a filter F_I of bandwidth f_c is to produce a signal $\tilde{e}(t)$ whose PDF becomes increasingly Gaussian as the ratio of f_{si}/f_c is increased. For the values of f_{si}/f_c considered here, the Gaussian PDF assumption of the $\tilde{e}(t)$ signal is reasonable, and this, coupled with its flat spectral properties, supports our assertion that $\tilde{e}(t)$ is a flat Gaussian signal. Finally, we note that even if the PDF of $\tilde{e}(t)$ were not Gaussian, we would still be justified in treating $\tilde{e}(t)$ as a flat Gaussian signal. This is because we only use the power properties of the signal in our calculations. The PDF of $\tilde{e}(t)$ need not be considered because of the DM noise being predominately granular.

From eq. (11), we will express the peak s/n of a delta modulator encoding a flat Gaussian input signal as

$$\widehat{s/n}_{F,j} = \theta_F f_{sj}^3, \qquad j = 2, 3, \cdots, N, \tag{17}$$

where θ_F is a DM parameter having a weak dependence on f_{sj} [see eq. (15)]. The value of j in eq. (17) denotes the DM stage number and f_{sj} is the sampling frequency for the jth stage.

The s/n in dB will be written in upper case letters. Thus, from eq. (8) and the preceding discussion, we have for the N stage MSDM,

$$S/N = S/N_{RC} + \sum_{j=2}^{N} S/N_{F,j} \tag{18}$$

and from eqs. (16), (17), and (18)

$$\begin{aligned} S/N &= 10 \log_{10}\theta_{RC} + (N-1)10 \log_{10}\theta_F + \sum_{i=1}^{N} 30 \log_{10} f_{si} \\ &= 10 \log_{10}\theta_{RC} + (N-1)10 \log_{10}\theta_F + 30 \log_{10}\lambda, \end{aligned} \tag{19}$$

where

$$\lambda = \prod_{i=1}^{N} f_{si}. \tag{20}$$

The transmitted bit rate is

$$f_p = \sum_{i=1}^{N} f_{si}. \tag{21}$$

If θ_{RC} and θ_F were independent of the DM sampling rates f_{si}, $i = 1, 2, \cdots, N$, then the S/N of the MSDM would be maximized by maximizing λ, subject to the constraint that f_p in eq. (21) is constant. This would occur when

$$f_{s1} = f_{s2} = \cdots = f_{sN}. \tag{22}$$

However, θ_{RC} and θ_F depend on $C_{\mathrm{opt,RC}}$ and $C_{\mathrm{opt},F}$, and although these slope loading factors are functions of the sampling frequency f_{si}, the variation of θ_{RC} and θ_F with f_{si} has only a minor effect on S/N. Thus, with each DM operating at the same sampling frequency, the S/N for the MSDM is close to its peak value. To each DM stage we will, therefore, assign the clock frequency

$$f_s = \frac{f_p}{N}, \tag{23}$$

enabling the S/N of eq. (19) to have a peak value of

$$\widehat{\mathrm{S/N}} = \widehat{\mathrm{S/N}}_{\mathrm{RC}} + (N-1)\widehat{\mathrm{S/N}}_F. \tag{24}$$

The $\widehat{\mathrm{S/N}}_{\mathrm{RC}}$ and $\widehat{\mathrm{S/N}}_F$ terms are the values of $\widehat{\mathrm{s/n}}_{\mathrm{RC}}$ and $\widehat{\mathrm{s/n}}_F$ in dBs, where f_{si} in eqs. (10) and (11) is replaced by f_p/N. Observe that $\widehat{\mathrm{S/N}}_F$ is independent of the RC Gaussian input signal $x(t)$ applied to the MSDM encoder, provided that DM_1 tracks it to produce a flat error signal spectrum. Thus, the $\widehat{\mathrm{S/N}}$ of MSDM is calculable from eqs. (10) and (11).

Substituting the result of eq. (23) into eq. (19) enables the $\widehat{\mathrm{S/N}}$ to be expressed in terms of parameters θ_{RC} and θ_F, the transmitted bit rate f_p, and the number of stages N, namely,

$$\begin{aligned}\widehat{\mathrm{S/N}} &= 10\log_{10}\theta_{\mathrm{RC}} + (N-1)10\log_{10}\theta_F \\ &\quad + 30\log_{10}\left(\frac{f_p}{N}\right)^N \\ &= 10\log_{10}\theta_{\mathrm{RC}} + (N-1)10\log_{10}\theta_F \\ &\quad + 30N\log_{10} f_p - 30N\log_{10} N.\end{aligned} \tag{25}$$

Chakravarthy and Faruqui[15] assumed the $\widehat{\mathrm{S/N}}$ to be N times the $\widehat{\mathrm{S/N}}$ for each DM stage. In their presentation, θ was not explicitly derived; instead it was a system parameter. From their measurements, they concluded that the assumed $\widehat{\mathrm{S/N}}$ was too high and, accordingly, it was reduced by a factor $H(N-1)$, where H is an empirical constant, namely,

$$\widehat{\mathrm{S/N}} = 10N\log_{10}\left\{\theta_{\mathrm{RC}}\left(\frac{f_p}{N}\right)^3\right\} - H(N-1). \tag{26}$$

The exponent 3 is a system parameter α in their formula as they considered delta modulators having local decoders composed of either single or double integration stages. From eqs. (25) and (26) we can specify their value of H for the MSDM using linear DM stages as

$$H = 10\log_{10}\left(\frac{\theta_{\mathrm{RC}}}{\theta_F}\right) = 10\log_{10}\left(\frac{\widehat{\mathrm{S/N}}_{\mathrm{RC}}}{\widehat{\mathrm{S/N}}_F}\right), \tag{27}$$

or from eqs. (10) and (11),

$$H = 10 \log_{10}\left[\frac{1}{3}\left(\frac{\mu}{\beta}\right)\left(\frac{C_{\mathrm{opt},F}}{C_{\mathrm{opt,RC}}}\right)^2\right]. \tag{28}$$

As μ/β is independent of f_{si}/f_c, and $C_{\mathrm{opt},F}/C_{\mathrm{opt,RC}}$ is nearly independent[8] of f_{si}/f_c, we can appreciate why H was introduced as a constant.

Returning to eq. (25) and replacing θ_{RC} and θ_F by the DM parameters, the peak S/N of MSDM can be expressed in terms of the normalized transmitted bit rate f_p/f_c, namely,

$$\widehat{\mathrm{S/N}} = -4.77 - 7.52N - 20 \log_{10}C_{\mathrm{opt,RC}} - 20(N-1)\log_{10}C_{\mathrm{opt},F} + 10 \log_{10}\left(\frac{\mu}{\beta}\right) + 30N \log_{10}\left(\frac{f_p}{f_c}\right) - 30N \log_{10}N, \tag{29}$$

where $C_{\mathrm{opt,RC}}$ and $C_{\mathrm{opt},F}$, given by eqs. (14) and (15) have sampling frequencies f_{si} equal to f_p/N. The variation of the peak S/N of MSDM as expressed by eq. (29), is presented in Fig. 3 as a function of f_p/f_c for different values of N. The value of β used is 0.235, corresponding to frequency parameters f_1 and f_c of 800 and 3,400 Hz, respectively. This value of β is often used[3,8] when RC Gaussian signals are employed as

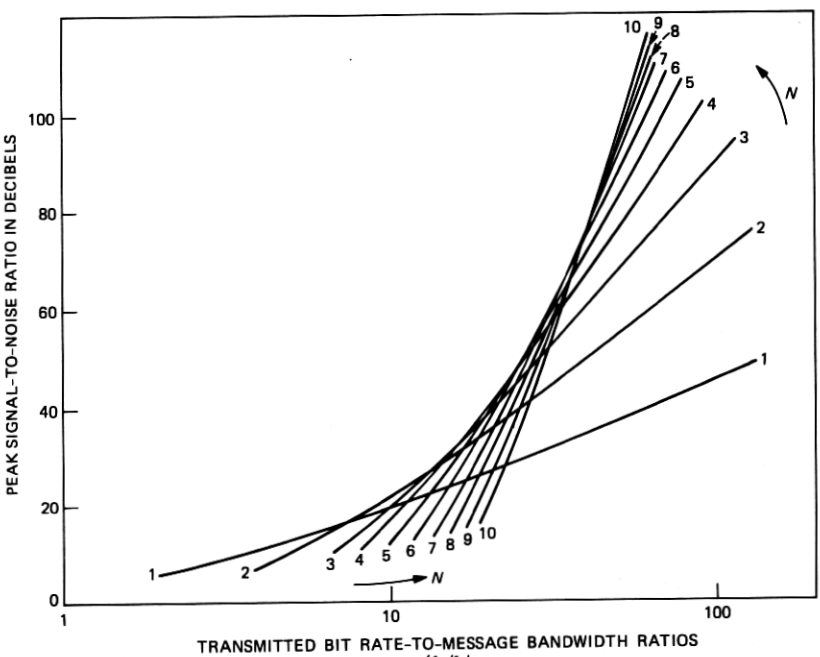

Fig. 3—The $\widehat{\mathrm{S/N}}$ of MSDM as a function of f_p/f_c for N having values 1 to 10, and $\beta = 0.235$.

an approximation to band-limited speech signals in telephony. For monochrome luminance signals, $\beta = 0.01$ may be used,[3,8] and although this smaller value of β increases the S/N of DM_1, the shape of the curves in Fig. 3 are essentially the same. Thus, it is f_p/f_c and N that are the cardinal parameters governing the S/N of MSDM.

3.1 Selecting N for maximum S/N

From Fig. 3, we observe that for any f_p/f_c there is a particular number of DM stages to maximize the S/N. When $f_p/f_c < 7.5$, only one stage should be used (linear DM), and as f_p/f_c is increased the number of stages N should be increased appropriately. The maximum value of the normalized frequency associated with a given N for the maximum $\widehat{S/N}$ is, for $\beta = 0.235$,

$$\left(\frac{f_p}{f_c}\right)_{\max} = \frac{N + 0.4}{0.187}; \qquad 1 \le N \le 4. \tag{30}$$

For example, $N = 3$ gives the maximum $\widehat{S/N}$ over the frequency range $12.8 < f_p/f_c < 18.2$. The range of f_p/f_c associated with a value of N to give maximum $\widehat{S/N}$ becomes progressively smaller as N is increased. Practical MSDM systems are unlikely to be produced with $N > 4$. Thus, given that N can be varied to maximize the $\widehat{S/N}$ for any f_p/f_c, we have from the curves of Fig. 3,

$$\widehat{S/N}_{\max} = 5 + 1.7\left(\frac{f_p}{f_c}\right). \tag{31}$$

IV. COMPARING S/N OF MSDM WITH DM, PCM, AND DPCM

4.1 Multistage delta modulation (MSDM)

The $\widehat{S/N}$ of MSDM, given by Eq. (29), increases at a rate of approximately $7N$ dB/octave increase in f_p/f_c, for $32 < f_p/f_c < 128$. At lower values of f_p/f_c, the variation of $\widehat{S/N}$ with f_p/f_c is approximately $5.3N$ dB/octave. Selecting N to peak the S/N for any f_p/f_c, yields the $\widehat{S/N}_{\max}$ of eq. (31).

4.2 Delta modulation (DM)

The $\widehat{S/N}$ of linear DM is found by putting $N = 1$ in eq. (29),

$$\widehat{S/N}_{DM} = -12.3 + 10\log_{10}\left(\frac{\mu}{\beta}\right) - 20\log_{10}C_{RC} + 30\log_{10}\left(\frac{f_p}{f_c}\right), \tag{32}$$

i.e., we may view DM as a special case of MSDM. In DM, the rate of improvement of $\widehat{S/N}$ with f_p/f_c varies from 5 dB/octave for $2 < f_p/f_c < 5$, to 9 dB/octave for $32 < f_p/f_c < 128$. This rate of improvement is significantly less than in MSDM [see eq. (31)].

4.3 *Pulse code modulation (PCM)*

The input signal is sampled at the Nyquist rate of $2f_c$, and the transmitted bit rate f_p is $2f_c n$, where n is the number of bits in the code words. For "4σ loading," the $\widehat{\text{S/N}}$ in dB of linear PCM is[2]

$$\widehat{\text{S/N}}_{\text{PCM}} = -7.3 + 6n = -7.3 + 3\left(\frac{f_p}{f_c}\right). \tag{33}$$

We observe that PCM is more efficient at exchanging $\widehat{\text{S/N}}$ with f_p/f_c, or n, compared to both DM and MSDM. However, at low values of f_p/f_c, PCM has a lower $\widehat{\text{S/N}}$ than DM.

4.4 *Differential pulse code modulation (DPCM)*

The $\widehat{\text{S/N}}$ in dB of linear DPCM is

$$\widehat{\text{S/N}}_{\text{DPCM}} = -A + 6n + G_p = -A + 3\left(\frac{f_p}{f_c}\right) + G_p, \tag{34}$$

where A is a constant, and G_p is the prediction gain factor. The constant A depends on the probability density function of the error signal and can often be taken as $\simeq 7$ dB, i.e.,[17]

$$\widehat{\text{S/N}}_{\text{DPCM}} = \widehat{\text{S/N}}_{\text{PCM}} + G_p. \tag{35}$$

The prediction gain factor is

$$G_p = 10 \log_{10}(\text{SFM}), \tag{36}$$

and SFM is the ratio of the arithmetic mean to the geometric mean of the discrete spectrum of the input signal.[17] For a first-order optimum predictor

$$G_p = -10 \log_{10}(1 - \rho^2), \tag{37}$$

where ρ is the correlation between adjacent samples of the input signal. The autocovariance function of an RC Gaussian input signal band-limited to f_c has been found by O'Neal,[2] enabling us to formulate an approximate expression for ρ as

$$\rho = \left[\exp(-2\pi f_1/f_s) - 4\frac{f_1}{f_s}\left\{\frac{\cos(2\pi f_c/f_s)}{(2\pi f_c/f_s)} + \text{Si}(2\pi f_c/f_s) - \frac{\pi}{2}\right\}\right] \Big/ 1 - (2/\pi)\beta, \tag{38}$$

where β is defined by eq. (12), f_s is the sampling rate, and Si is the sine integral function. When the input signal is sampled at the Nyquist rate, i.e., $f_s = 2f_c$, eq. (38) reduces to

$$\rho = \frac{e^{-\pi\beta} + 0.0751\beta}{1 - (2/\pi)\beta}. \tag{39}$$

For a value of β of 0.235, $\rho = 0.54$, resulting in $G_p = 1.12$ dB.

Thus, linear DPCM has an advantage G_p over linear PCM, but it has the same type of dependence on f_p/f_c. Therefore, we conclude that MSDM, where the number of stages are selected according to eq. (30), has an $\widehat{\text{S/N}}$ of the same form as PCM and DPCM, namely,

$$\widehat{\text{S/N}} = P_1 + P_2\left(\frac{f_p}{f_c}\right), \tag{40}$$

where P_1 and P_2 are parameters that depend on the type of modulation, and for MSDM, DM, and DPCM, on the input signal. The MSDM operating with the optimum number of stages to peak the S/N is, therefore, more efficient at exchanging $\widehat{\text{S/N}}$ with f_p/f_c than DM [see eq. (32)].

The variation of peak $\widehat{\text{S/N}}$ as a function of f_p/f_c for an RC Gaussian input signal having $\beta = 0.235$ is shown in Fig. 4 for DM; MSDM with $N = 4$; MSDM employing the optimum number of stages; and PCM with "4σ loading" and sampling at the Nyquist rate. The curve of DPCM for $\beta = 0.235$ and Nyquist sampling is not displayed as it has the same shape as that of PCM, but with $\widehat{\text{S/N}}$ increased by G_p.

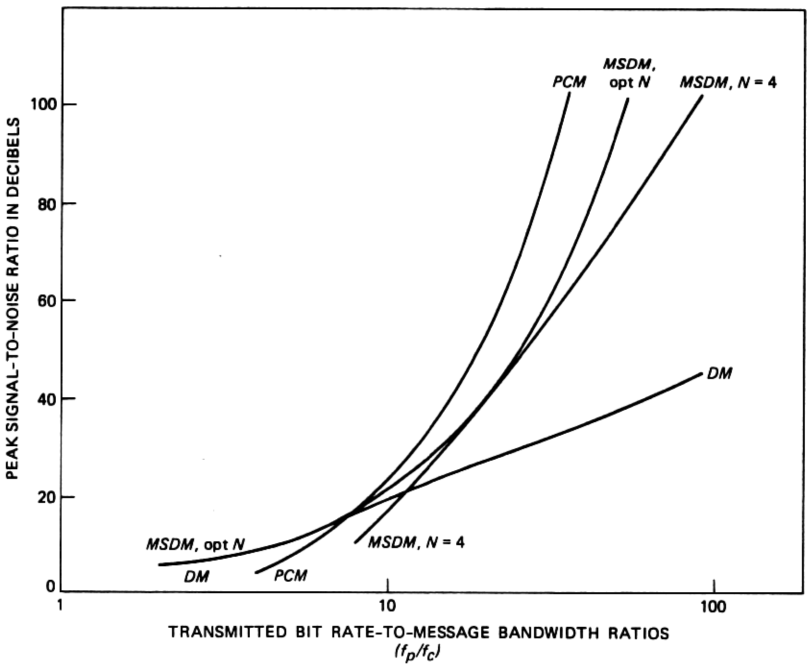

Fig. 4—The $\widehat{\text{S/N}}$ as a function of f_p/f_c for linear DM, $\beta = 0.235$; MSDM, $N = 4$, $\beta = 0.235$; MSDM with optimum N, $\beta = 0.235$; and PCM, $f_p = 2f_c$, "4σ loading."

V. THE MSDM USING ADM STAGES

An MSDM codec is required to operate over a wide range of input levels. In Section III, we considered the delta modulators to be linear with their step sizes adjusted to maximize their S/N values. When the input power of such an MSDM encoder is reduced causing DM_1 to operate in the granular region with a slope loading factor $C_{RC} < 8$, the noise generated in DM_1 is substantially independent of the input signal power.[1] As the filtered error signal of DM_1 is the input to DM_2, and the power of this signal is approximately unaltered by the reduction of the input power to DM_1, the S/N of DM_2 remains the same. Because the power level to DM_2 is unchanged, the S/N's of subsequent DM stages are therefore similar, particularly for $N \leq 4$. As the input power to DM_1 is further reduced, the noise generated in DM_1 will eventually increase,[3] causing some overloading in DM_2 which, in turn, will reduce the S/N's in subsequent DM stages. Thus, if excessive granular noise is generated in DM_1, slope-overloading in the remaining delta modulators ensues. However, this slope-overloading of the second and subsequent DM stages for a reduction in input level to DM_1 operating in the granular region, is far less severe than when the input power is increased, causing DM_1 to become slope-overloaded. When DM_1 is slope-overloaded, all the stages in the MSDM experience slope-overload, and the S/N of the MSDM rapidly deteriorates with increasing input power.

The range of input power over which the MSDM can operate, while providing an acceptable S/N, i.e., the dynamic range, can be greatly enhanced by using adaptive delta modulators (ADMs) instead of linear delta modulators. The diversity of ADMs is considerable,[1] but all have the property of extending the dynamic range, while retaining an approximately constant S/N. The quantization noise power in ADM is, therefore, proportional to the signal power. As the input signal power varies, the filtered error signal applied to DM_2 varies, and if this encoder is not to be overloaded, it must also be adaptive. The same argument applies to subsequent stages, and hence the complete MSDM codec is composed of ADM stages.

If the N ADM encoders in the MSDM codec are syllabically companded delta modulators[1] [currently available on a chip[9] in the form of continuously variable slope delta (CVSD) codecs], and they use single integrators in the local decoding process, then the peak S/N of each ADM stage is a close approximation to that given by eqs. (10) or (11). Further, the dynamic range of each codec where the S/N is maintained near its peak S/N is wide,[15] typically 40 dB. Thus, the maximum S/N of MSDM given by eq. (29) is a good approximation for MSDM having ADM stages over a wide range of input power.

VI. DISCUSSION

Easily evaluated equations of peak S/N for MSDM have been derived in terms of normalized bit rate, number of DM stages, and the shape of the spectrum of the RC Gaussian input signal. A suitable choice for the sampling rate for each delta modulator was found to be f_p/N. For a given input signal bandwidth f_c, and transmission bit rate f_p, there is an optimum number of stages which maximizes S/N. The variation of this $\widehat{S/N}$ with f_p/f_c is given by eq. (29), and we showed in Section IV that this has the same form as for PCM and DPCM. Thus, MSDM is more efficient than DM at exchanging $\widehat{S/N}$ for f_p/f_c, but the $\widehat{S/N}$ of MSDM is generally lower than that of PCM and DPCM. At very low values of f_p/f_c, DM performs better than the other modulation methods considered (see Fig. 4). In Section V, we discussed MSDM with CVSD stages, concluding that the $\widehat{S/N}$ of eq. (29) remains valid, provided that single-stage integrators are used in the CVSD codecs. Further gains in $\widehat{S/N}$ are attainable if double stage integrators[1,15] replace the single-stage integrators. The CVSD codecs enable the MSDM to have a wide dynamic range and, therefore, we envisage MSDM codecs to be constructed[15] with adaptive, rather than linear, delta modulators.

Although this work has been concerned with the derivation of $\widehat{S/N}$ of MSDM, we conclude with the observation that MSDM having four CVSD codecs might have a role to play in a variable bit-rate transmission system. For example, each CVSD stage could operate at 16 kb/s giving a transmitted bit rate of 64 kb/s when the four stages are in use. In a time division multiplex (TDM) system, the MSDM codecs would attempt to operate at 64 kb/s, but as traffic increased they could discard the higher order CVSD stages, decreasing their bit rates from 64 to 48 to 32 kb/s, until when the system is at maximum capacity, each MSDM would behave as a 16-kb/s CVSD codec. By using MSDM instead of a single-stage DM operating at the same bit rate, we are able to enhance the quality of the recovered signal as the bit rate increases from 16 kb/s.

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