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Steady-State Stability of a Synchronous Machine

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Mechanical oscillations of synchronous machines following the application of sudden shaft loads are important to operating engineers. This paper presents an analytical technique to determine the stability of a synchronous machine. We first decompose the nonlinear system representing the machine into an infinite number of subsystems connected in parallel. We then determine a bound on the input to the machine for which the number of subsystems will effectively be finite in number, then determine stability of the entire system by determining the stability of each subsystem.

I. INTRODUCTION

The problems associated with the maintenance of stability of synchronous machines and the factors affecting stability during transient disturbances have received considerable attention in recent years. If the sudden load application is such that synchronizing torque is less than the load torque, the machine will be thrown out of synchronism and instability will occur. A synchronous machine has steady-state stability if, after a small slow disturbance, it can regain and maintain synchronous speed. On the other hand, if the machine can regain and maintain synchronous speed after large sudden disturbances, it has transient stability. The maximum steady-state operating limit will depend upon the magnitude of instantaneous load changes and its ability to follow quickly any load changes. It is the purpose of this paper to determine an upper bound on the input load so that synchronism will never be lost.

II. MATHEMATICAL MODEL

The equation of motion of the synchronous machine is well known and has been extensively discussed in the literature.[1–6] However, the derived equation based on the electromechanical properties of the system is highly nonlinear, and the determination of the time-domain response of the system seems to be prohibitively difficult. Thus, to represent the system by some suitable mathematical model, the following simplifying assumptions are made.

(*i*) The twist in the shaft is negligibly small since the length of the shaft is small and the rotational stiffness is large.

(*ii*) All inductances are independent of current since nonsaturated conditions are considered.

(*iii*) The armature flux wave is sinusoidally distributed.

(*iv*) The currents in both the armature and field circuits are considered to be independent of a sudden change in operating conditions. Results obtained with an induction motor having a rotor winding excited with direct current appear to justify this assumption for those cases in which the time constants of the stator and rotor windings are relatively small or the moment of inertia is large enough to assume small acceleration.

(*v*) All resistances are negligible compared to inductances.

(*iv*) The direct-axis synchronous reactance is equal to the quadrature-axis synchronous reactance. This assumption is quite valid for a nonsalient pole machine but for a salient pole machine this assumption is only approximate.

In order to write the system equations, the following symbols are introduced.

J = polar moment of inertia of the rotor.
ω = instantaneous speed.
ω_s = synchronous speed.
δ = angle between the rotor pole position on load and its positions on no load (rotor angle).
T_i = externally applied torque.
T_e = torque of electrical origin.
D = angular viscous friction coefficient.
P_i = electrical power input.
P_d = system damping factor.
P_m = maximum static synchronous power.
P_o = output power.
M = inertia constant.

Based upon the simplifying assumptions, the equation of motion of the synchronous machine can now be derived. However, the detailed derivation will not be given here, since it can be found elsewhere.[1–7] Only an outline of the derivation which is necessary for the subsequent discussion is presented below.

$$\delta = (\omega - \omega_s)t, \tag{1}$$

$$\frac{d\delta}{dt} = \omega - \omega_s, \tag{2}$$

$$\frac{d^2\delta}{dt^2} = \frac{d\omega}{dt}. \tag{3}$$

The equation of motion is given in Ref. 7 as

$$T_i = J\frac{d^2\delta}{dt^2} + D\omega + T_e. \tag{4}$$

By substituting (2) and (3) in (4) and by converting the torque equation (4) into power equation

$$P_i = \omega J\frac{d^2\delta}{dt^2} + D\omega\left(\frac{d\delta}{dt} + \omega_s\right) + P_0$$

or

$$P_i = M\frac{d^2\delta}{dt^2} + P_d\frac{d\delta}{dt} + P_0, \tag{5}$$

where $P_d(d\delta/dt) = D\omega(d\delta/dt) + D\omega\omega_s$, P_d is the system damping factor and can only be determined empirically. P_o has been evaluated[6] as $P_m \sin\delta$. Equation (5) can now be put into dimensionless form by introducing the following change in variables:

$$t = \tau\sqrt{\frac{M}{P_m}}$$

$$p = \frac{P_i}{P_m}$$

$$2\xi = \frac{P_d}{\sqrt{(MP_m)}}.$$

By using the above transformation, (5) becomes

$$\frac{d^2\delta}{d\tau^2} + 2\xi\frac{d\delta}{d\tau} + \sin\delta = p$$

or

$$\ddot{\delta} + 2\xi\dot{\delta} + \delta + N(\delta) = p, \tag{6}$$

where $N(\delta)$ is a nonlinear function of δ and is given by

$$N(\delta) = \sum(-1)^n\frac{\delta^{2n+1}}{(2\,n+1)!}, \qquad n = 1, 2, 3, \cdots.$$

The behavior of the synchronous machine can now be determined from (6). The damping coefficient ξ cannot be evaluated using exact methods. Its value is usually approximated empirically from the knowledge of the system behavior. A value of $0 < \xi < 0.4$ seems to be a reasonable choice.

The input-output relationship of the synchronous machine can now be viewed in the form of a simplified block diagram as shown in Fig. 1. The input load ratio $p(\tau)$ is the input to the system, and its response is the rotor angle $\delta(\tau)$. Now by functional expansion,[8] $\delta(\tau)$ can be expressed in terms of $p(\tau)$ with all the limits of integrations ranging from 0 to ∞

$$\delta(\tau) = \int h_1(t_1)p(\tau - t_1)\,dt_1 + \iint h_2(t_1, t_2)p(\tau - t_1)p(\tau - t_2)\,dt_1\,dt_2$$
$$+ \iiint h_3(t_1, t_2, t_3)p(\tau - t_1)p(\tau - t_2)p(\tau - t_3)\,dt_1\,dt_2\,dt_3 + \cdots. \quad (7)$$

Let

$$\delta_n(\tau) = \int \cdots \int h_n(t_1, t_2, \cdots, t_n)p(\tau - t_1) \cdots p(\tau - t_n)\,dt_1 \cdots dt_n. \quad (8)$$

Thus

$$\delta(\tau) = \sum_{n=1}^{\infty} \delta_n(\tau).$$

Hence the nonlinear system in Fig. 1 can be broken down into an infinite number of subsystems connected in parallel as shown in Fig. 2. The response of the first-order or linear subsystem is $\delta_1(\tau)$, and its impulse response is $h_1(t_1)$. The response of the nth order subystem is $\delta_n(\tau)$, and its impulse response is $h_n(t_1, t_2, \cdots, t_n)$.

Thus the system shown in Fig. 1 will be stable if

(*i*) Equation (7) is convergent, which also implies $|\delta_i| > |\delta_{i+1}|$.

(*ii*) The linear subsystem is stable, which also implies $s^2 + 2\xi s + 1 = 0$ does not have any root with positive real part, $s = d/dt$.

Condition (*ii*) can easily be checked from the system equation. To prove the condition of convergence, we proceed as follows.

Let $h(t)$ be the impulse response of the first order or the linear subsystem, i.e.,

$$h(t) \leftrightarrow \frac{1}{s^2 + 2\,\xi s + 1}$$

and

$$H = \int |h(t)|\,dt.$$

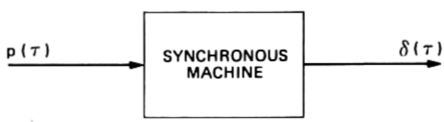

Fig. 1—Input-output relationship of a synchronous machine.

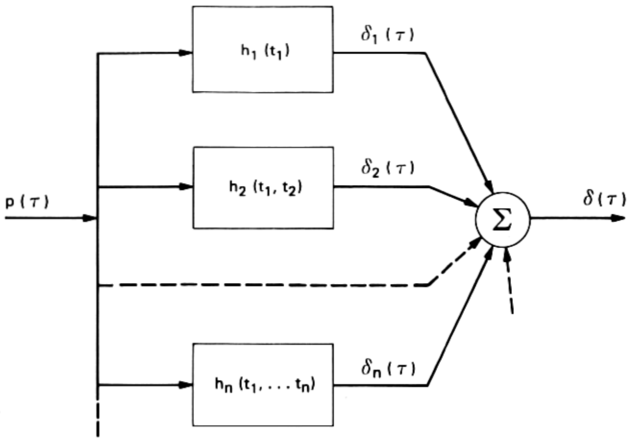

Fig. 2—Decomposition of a synchronous machine into infinite subsystems.

Let

$$\max_{0 \le \tau < \infty} |p(\tau)| < P.$$

With the above assumptions, by the technique of series reversion method,[9] Barrett has shown that the functional expansion expressed in (7) will be convergent if the input has an upper bound given by

$$|p(\tau)| < P = \frac{1+H}{H} \cosh^{-1} \frac{1+H}{H} - \sinh \cosh^{-1} \frac{1+H}{H}.$$

The integral $H = \int |h(t)|\, dt$ can be shown[12] to have the value

$$H = \coth\left(\frac{\pi\xi}{2\sqrt{1-\xi^2}}\right).$$

By knowing H, P can be calculated. The values of ξ, H, and P are shown in Table I.

It can also be shown[12] that, for the system described by (6), $h_2(t_1, t_2)$, $h_4(t_1, \cdots, t_4)$ are zeros. Hence

$$\delta(\tau) = \delta_1(\tau) + \delta_3(\tau) + \delta_5(\tau), \cdots.$$

III. DETERMINATION OF THE STEADY-STATE RESPONSE

A sudden change in the load on the synchronous machine is equivalent to a step change in the prime mover torque applied to the shaft. Hence, the system stability can thus be investigated on the basis of a step input. It is also clear from Fig. 2 that the steady-state response of the entire system is the summation of the steady-state responses of each of the subsystems. Let

$$\Delta_n = \lim_{\tau \to \infty} \delta_n(\tau)$$

Table I—Input as a function of damping coefficient

Damping Coefficient ξ	$\int_0^\infty \lvert h(t) \rvert dt$ H	Upper Bound of Input P
0.01	63.664	0.002
0.02	31.835	0.005
0.03	21.227	0.010
0.04	15.924	0.015
0.05	12.743	0.021
0.06	10.623	0.027
0.07	9.109	0.034
0.08	7.974	0.042
0.09	7.092	0.050
0.10	6.387	0.058
0.11	5.810	0.067
0.12	5.330	0.076
0.13	4.924	0.085
0.14	4.576	0.095
0.15	4.275	0.105
0.16	4.012	0.116
0.17	3.780	0.127
0.18	3.574	0.138
0.19	3.390	0.149
0.20	3.225	0.160
0.25	2.599	0.221
0.30	2.186	0.285
0.35	1.895	0.353
0.40	1.680	0.421
0.45	1.517	0.490
0.50	1.390	0.557

and the input to the system be a step function $p(\tau) = p_i$. It has already been determined from Table I that, for stability the values p_i will be in the range $p_i \le P \le 0.421$ for $0 < \xi \le 0.4$. The values of $\Delta_1, \Delta_3, \cdots$ can now be determined by the application of the final-value theorem. Δ_1 is given as

$$\Delta_1 = \lim_{s \to 0} \frac{sp_i}{s(s^2 + 2\xi s + 1)} = p_i.$$

To determine Δ_3, we proceed as follows: First we take a three-dimensional transform[10] of the third subsystem corresponding to the output $\delta_3(\tau)$. Next, we apply the technique of association of variables[11] and make a transformation from three dimensions to a single dimension. Now we are ready to apply the final-value theorem to find Δ_3.

$$h_3(t_1, t_2, t_3) \leftrightarrow H_3(s_1, s_2, s_3).$$

Let A stand for associate variables $s_1, s_2, s_3 \to s$. Then

$$\delta_3(s) = A\left[\frac{H_3(s_1, s_2, s_3)p_i^3}{s_1 s_2 s_3}\right]$$

$$\Delta_3 = \lim_{s \to 0} sA\left[\frac{H_3(s_1, s_2, s_3)p_i^3}{s_1 s_2 s_3}\right].$$

This limit has been evaluated[12] to be

$$\Delta_3 = \frac{p_i^3}{6}, \qquad \Delta_1 \gg \Delta_3.$$

Higher limits Δ_5, Δ_7, etc., will further be smaller, and the evaluation of these terms is not necessary since their contributions will be insignificant. Thus

$$\Delta = \lim_{\tau \to \infty} \delta(\tau) = \Delta_1 + \Delta_3$$

is a very good approximation, and

$$\max \Delta = p_i \left(1 + \frac{p_i^3}{6}\right) < \pi/2,$$

since $0 < p_i < 0.421$. Physically, it means that if the magnitude of the abrupt load change is such that $0 < P_i/P_m < 0.421$, the system will never lose synchronism and will remain stable. Thus, for $p(\tau) < 0.421$ the system can be described by the linear transfer function within a reasonable degree of accuracy.

With the above assumption, the output is given by Ref. 13:

$$\delta(\tau) = p_i \left[1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\tau} \sin(\sqrt{1-\xi^2}\,\tau + \cos^{-1}\xi) \right].$$

$\delta(\tau)$ can be expressed in terms of t only by using the relationship $t = \tau\sqrt{M/P_m}$.

Now by separating p into two parts, the initial load ratio p_{oo} and the abrupt load ratio p_L, the maximum value of p_L can be determined before the system goes out of synchronism.

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