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Least-Squares Estimator for Frequency-Shift Position Modulation in White Noise

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I. INTRODUCTION

Frequency-shift-position (FSP) modulation sends a sample $x \in (0, T)$ by changing the frequency of a wave from ω_0 to ω_1 at the time x. We consider the problem of optimal mean-square estimation of x from an FSP signal to which white noise has been added in transmission. The best estimator, given by a known formula of nonlinear filtering, contains stochastic integrals and is hard to implement. An approximation, obtained by omitting $0(\omega_i^{-1})$ terms, is readily implemented over the *next* interval $(0, T)$ by ordinary differential equations driven by the observed signal.

We pose the optimal least-squares demodulation problem for the following simple communications system: The *signal* to be sent consists of successive i.i.d. random variables x having a density b with support $(0, T)$; the *channel* provides an FM wave, of random initial phase θ, which shifts frequency from ω_0 to ω_1 at the point x during $(0, T)$; the resulting signal is

$$s_t = \begin{cases} R\cos(\theta + \omega_0 t, & t < x \\ R\cos(\theta + \omega_0 x + (t - x)\omega_1), & T \geq t \geq x; \end{cases}$$

this signal is observed through white noise, the *received signal* or *observation* being y_t given by

$$dy_t = s_t\, dt + db_t,$$

where b_t is a Brownian motion independent of x; the process is repeated over each interval of length T, and the problem is to guess or calculate a good estimate of x from the observation $\{y_t, 0 \leq t \leq T\}$.

This problem is resolved by using nonlinear filtering to give an exact but not recursive least-squares estimator $\hat{x}$ based on $\{y_t, 0 \leq t \leq T\}$. The estimator $\hat{x}$ is then analyzed for ways of calculating it in real time. First, it is re-expressed in terms of ordinary integrals by eliminating the stochastic differentials. Second, we find that by ignoring two terms that are $0(R^2/\min\{\omega_0, \omega_1\})$ we can use a Bessel integral and some simple differential equations to calculate an approximate version of $\hat{x}$ over the *next* interval of length T. This version will be very close to $\hat{x}$ when the carrier frequencies ω_0 and ω_1 are large compared to the amplitude R. To make the integrals appearing in the likelihoods we will use suitably dimensionless, we let R have unit $\sec^{-1/2}$.

II. FORMULA FOR $\hat{x}$

At time T, the best mean-square estimate of x given the observation $\{y_s, 0 \leq s \leq T\}$ is given by the Kallianpur-Striebel or Bayes formula[1]

$$\hat{x} = D^{-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \int_0^T tf(t)\, dt \exp\left\{R\int_0^t \cos(\theta + \omega_0 s)\, dy_s - \frac{R^2}{2}\int_0^t \cos^2(\theta + \omega_0 s)\, ds + R\int_t^T \cos(\theta + \omega_0 t + \omega_1(u - t))\, dy_u - \frac{R^2}{2}\int_t^T \cos^2(\theta + \omega_0 t + \omega_1(u - t))\, du\right\},$$

where D = normalizer = same integral with $f(t)$ for $tf(t)$. We first reduce this formula to ordinary integrals. Since the signal $s(t)$ is of bounded variation, the mixed quadratic variation $\langle ys \rangle$ is 0, and the differentiation formula justifies an integration by parts that expresses the exponent above as

$$Ry_t\cos(\theta + \omega_0 t) + R\omega_0\int_0^t y_s\sin(\theta + \omega_0 s)\, ds - \frac{R^2}{2}\int_0^t \cos^2(\theta + \omega_0 s)\, ds + Ry_T\cos(\theta + \omega_0 t + \omega_1(T - t) - Ry_t\cos(\theta + \omega_0 t) + R\omega_1\int_t^T y_u\sin(\theta + \omega_0 t + \omega_1(u - t))\, du - \frac{R^2}{2}\int_t^T \cos^2(\theta + \omega_0 t + \omega_1(u - t))\, du.$$

The stochastic integrals are thereby eliminated. Also,

$$R\cos(\theta + \omega_0 t + \omega_1(T - t)) = s(T),$$

and the $\cos^2$ integrals can be evaluated thus:

$$\int_0^t \cos^2(\theta + \omega_0 s)\, ds = \omega_0^{-1} \int_\theta^{\theta+\omega_0 t} \cos^2 u\, du$$

$$= \omega_0^{-1} \frac{1}{2} (u + \sin u \cos u)_\theta^{\theta+\omega_0 t} = \frac{t}{2} + \frac{\sin(2\theta + 2\omega_0 t) - \sin 2\theta}{4\omega_0},$$

and similarly

$$\int_t^T \cos^2(\theta + \omega_0 t + \omega_1(u - t))\, du = \omega_1^{-1} \frac{1}{2} (u + \sin u \cos u)_{\theta+\omega_0 t}^{\theta+\omega_0 t+\omega_1(T-t)}$$

$$= \frac{T-t}{2} + \frac{\sin(2\theta + 2\omega_0 t + 2\omega_1(T-t)) - \sin(2\theta + 2\omega_0 t)}{4\omega_1}.$$

We thus arrive at the formula for $\hat{x}$:

$$\hat{x} = D^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \int_0^T tf(t)\, dt \exp\Bigg\{ R y_T \cos(\theta + \omega_0 t + \omega_1(T-t))$$

$$+ R\omega_0 \int_0^t y_s \sin(\theta + \omega_0 s)\, ds + R\omega_1 \int_t^T y_s \sin(\theta + \omega_0 t$$

$$+ \omega_1(u - t))\, du - \frac{R^2 T}{4} - R^2 \frac{\sin(2\theta + 2\omega_0 t) - \sin 2\theta}{8\omega_0}$$

$$- R^2 \frac{\sin(2\theta + 2\omega_0 t + 2\omega_1(T-t)) - \sin(2\theta + 2\omega_0 t)}{8\omega_1}\Bigg),$$

again with D the same with $f(t)$ in place of $tf(t)$. Note that the R^2T term can be ignored, since it occurs also in D.

The $d\theta\, dt$ integration can be interchanged by absolute convergence, so we look at the dependence of the exponent on θ. This is of the form

$$\alpha_t \cos\theta + \beta_t \sin\theta + 0\left(\frac{R^2}{\omega_0 \wedge \omega_1}\right),$$

where with

$$A = \int_0^T y_s \sin \omega_1 s\, ds, \qquad B = \int_0^T y_s \cos \omega_1 s\, ds, \qquad \omega = \omega_0 - \omega_1$$

$$\alpha_t = R\omega_0 \int_0^t y_s \sin \omega_0 s\, ds + y_T R \cos(\omega_0 t + \omega_1(T-t))$$

$$+ R\omega_1(B \sin \omega t + A \cos \omega t)$$

$$- R\omega_1 \cos \omega t \int_0^t y_s \sin \omega_1 s\, ds - R\omega_1 \sin \omega t \int_0^t y_s \cos \omega_1 s\, ds$$

$$\beta_t = R\omega_0 \int_0^t y_s \cos \omega_0 s \, ds - y_T R \sin(\omega_0 t + \omega_1(T - t))$$
$$+ R\omega_1(B \cos \omega t - A \sin \omega t)$$
$$- R\omega_1 \cos \omega t \int_0^t y_s \cos \omega_1 s \, ds + R\omega_1 \sin \omega t \int_0^t y_s \sin \omega_1 s \, ds.$$

The vector (α, β) satisfies the differential equations:

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \omega R y_t \begin{pmatrix} \sin \omega_0 t \\ \cos \omega_0 t \end{pmatrix} - \omega R \omega_0 \int_0^t y_s \begin{pmatrix} \cos \\ -\sin \end{pmatrix} \omega_0 s \, ds.$$

It can be expressed in terms of the quantities

$$s(t) = \alpha_t - R\omega_0 \int_0^t y_s \sin \omega_0 s \, ds,$$
$$c(t) = \beta_t - R\omega_0 \int_0^t y_s \cos \omega_0 s \, ds,$$

which satisfy the nicer system

$$\begin{pmatrix} \dot{s} \\ \dot{c} \end{pmatrix} = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix} - R\omega_1 y_t \begin{pmatrix} \sin \omega_0 t \\ \cos \omega_0 t \end{pmatrix}.$$

The initial conditions for α, β, and also for s, c, are

$$s_0 = \alpha_0 = R\omega_1 A + R y_T \cos \omega_1 T$$
$$c_0 = \beta_0 = R\omega_1 B - R y_T \sin \omega_1 T.$$

III. APPROXIMATION FOR SMALL $R^2/(\omega_0 \wedge \omega_1)$

The "double-phase" $0(R^2/\omega_0 \wedge \omega_1)$ term in the exponent is independent of y, and considerable simplification results if we ignore it. The numerator of $\hat{x}$ is then nearly

$$\int_0^T t f(t) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\alpha_t \cos \theta + \beta_t \sin \theta} \, d\theta \, dt,$$

and we can use the Bessel function integral representation

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\alpha \cos \theta + \beta \sin \theta} \, d\theta = I_0(\sqrt{\alpha^2 + \beta^2})$$

to "do" the $d\theta$ integral approximately, to get

$$\hat{x} \cong D^{-1} \int_0^T tf(t) I_0(\sqrt{\alpha_t^2 + \beta_t^2})\, dt, \qquad (1)$$

again with D the same integral with the t factor left out.

IV. IMPLEMENTATION

We can now use the differential equations for α_t and β_t to suggest how to realize the approximate estimator (1) in real time. Since the initial values α_0 and β_0 depend on both

$$A = \int_0^T y_s \sin \omega_1 s\, ds$$

$$B = \int_0^T y_s \cos \omega_1 s\, ds$$

as well as on y_T, we shall first wait till the end of the current interval of length T, all the while "beating" the received signal y_s with the "second" FM frequency ω_1 and integrating, to end up with A and B at time T. Noting the value of y_T, we obtain α_0 and β_0 and start to solve the differential equations for α_t and β_t during the next interval of length T. This is achieved by having, in the preceding interval, beaten y_s with the "first" FM frequency ω_0, integrated, and passed the results through a delay of size T to obtain, at time $T + t$, each of

$$\int_0^t y_s \begin{pmatrix} \cos \\ -\sin \end{pmatrix} \omega_0 s\, ds,$$

this vector being precisely what we need to add to the solution of the differential equations

$$\begin{pmatrix} \dot{s} \\ \dot{c} \end{pmatrix} = \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s \\ c \end{pmatrix} - R\omega_1 y_t \begin{pmatrix} \cos \\ -\sin \end{pmatrix} \omega_0 t$$

to get α_t and β_t. We also delay y_t by T so as to have it available in real time as soon as s_0 and c_0 are known, so as to use it to drive the system for s, c in the last equation.

Once α_t, β_t are available, we can calculate $I_0(\sqrt{\alpha_t^2 + \beta_t^2})$, and thence the integrals making up the ratio (1) that approximates $\hat{x}$. Figure 1 is a block diagram for this implementation.

Since the approximate estimator depends on y_s only through the "magnitude" $r = \sqrt{\alpha^2 + \beta^2}$, some simplifications occur upon a change to polar coordinates. Putting also $\phi = \arctan \alpha/\beta$, we obtain these

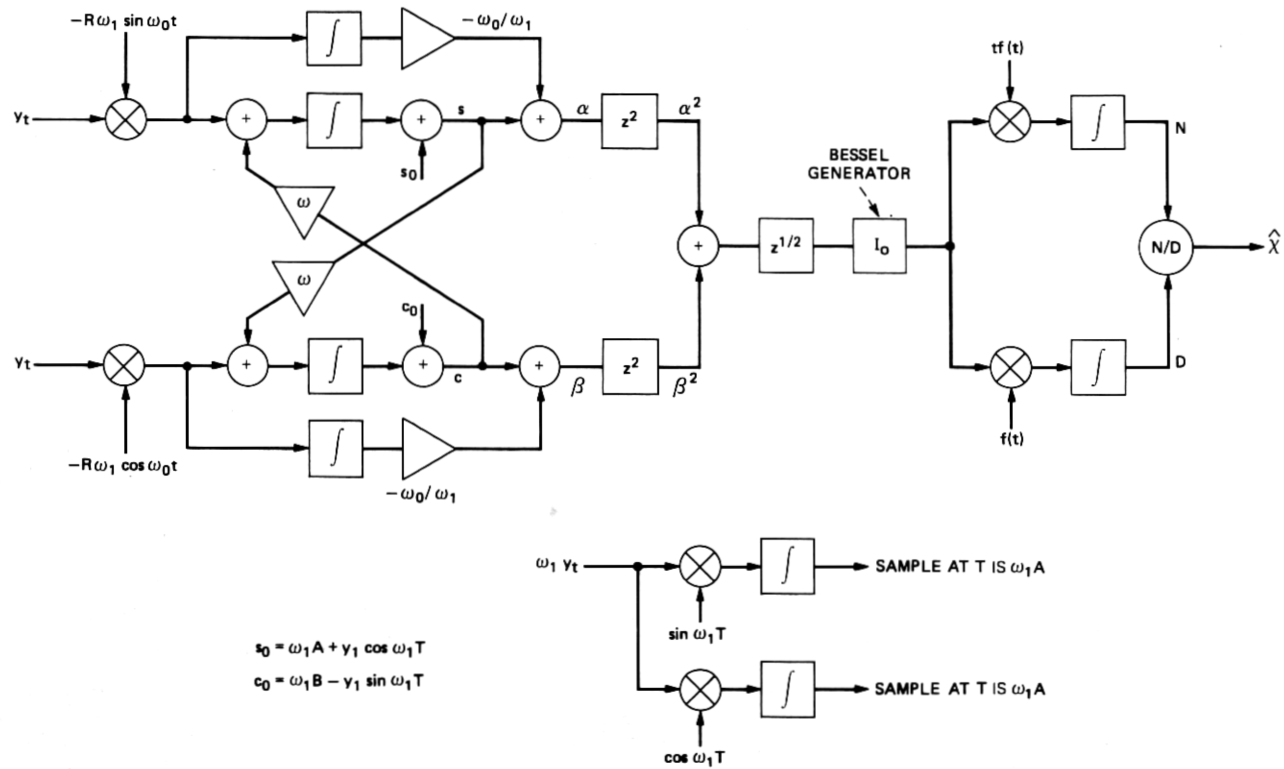

Fig. 1—Block diagram for implementing approximate estimator.

equivalent equations in terms of the new variables $r = r_t$ and $\phi = \phi_t$:

$$\dot{r} = -\omega R y_t \cos(\phi - \omega_0 t) - \omega R \omega_0 \int_0^t y_s \sin(\omega - \omega_0 s)\, ds$$

$$\dot{\phi} = -\omega R y_t \sin(\omega - \omega_0 t) - \omega R \omega_0 \int_0^t y_s \cos(\phi - \omega_0 s)\, ds.$$

It is perhaps not surprising that these equations should bear some resemblance to those arising in the study of the pendulum, the phase-locked loop, and the FM-with-feedback circuit.

V. BASEBAND VERSION

Instead of shifting frequency from ω_0 to ω_1 at the sample value $x \in [0, T]$, we can imagine changing a signal from the constant value r_0 to r_1 at the sample value (time) x. This leads to a "baseband" version of the problem just studied. The signal is now

$$s_t = \begin{cases} r_0 & 0 \le t \le x \\ r_1 & x \le t \le T, \end{cases}$$

and the observation is $dy_t = s_t\, dt + db_t$ again. The optimal mean-square estimate $\hat{x}$ is

$$\hat{x}\, D^{-1} \int_0^T t f(t) e^{r_0 y_t - 1/2 r_0^2 t + r_1 (y_T - y_t) - 1/2 r_1^2 (T-t)}\, dt,$$

where the normalizer D is the same integral with $f(t)$ for $tf(t)$. Figure 2 is an exact implementation of this estimator in real time. The terms in y_T and $r_1^2 T$ cancel with the same ones in D, so can be ignored.

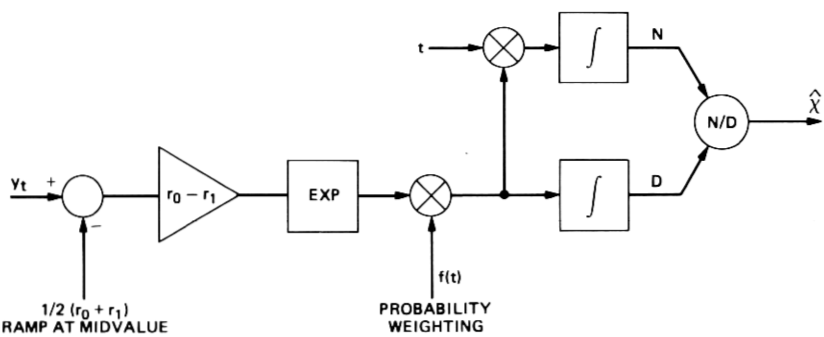

Fig. 2—Baseband version of estimator.

VI. ACKNOWLEDGMENT

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