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Frequency-Hopped Multilevel FSK for Mobile Radio

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A multiple-access modulation technique that uses multilevel frequency shift keying (FSK) to modulate frequency-hopped, spread-spectrum carriers is examined for possible application to digital mobile radiotelephony. This technique, in which all users employ the full system bandwidth simultaneously, would be resistant to the frequency-selective fading so troublesome in mobile radio. We have studied base-to-mobile communication of 32 kb/s per user in the 20-MHz (one-way) bandwidth of the 850-MHz mobile radio band. The number of users that can be served within a given bit error rate criterion depends on the quality of the radio channel. For perfect transmission, where the only degradation is mutual interference, an error rate less than 10^{-3} can be maintained with up to 209 simultaneous users. Transmission impairments, consisting of white Gaussian noise and frequency-selective Rayleigh fading with an average rf signal-to-noise ratio of 25 dB, reduce the number of simultaneous users to about 170. This capacity is roughly three times that of a phase-shift-keying spread-spectrum system recently proposed for mobile radio. For mobile-to-base transmission of FH-FSK, we have yet to study impairments resulting from delay spread in a synchronous system or, alternatively, the penalty for operating asynchronously. These effects would reduce the number of possible users from the estimates we have given for base-to-mobile transmission.

I. INTRODUCTION

In exploring the possibility of providing digital mobile communications service to a large number of users, we have investigated a spread-spectrum modulation technique using frequency-hopped frequency shift keying (FH-FSK) and a majority logic receiver. This modulation is an extension of a technique proposed by Viterbi[1] for multiple access by low-rate mobile users employing a satellite transponder.

By assigning a distinct tone sequence called an *address* to each user, FH-FSK allows many users to share the same frequency band. This approach differs markedly from time-division or frequency-division channel assignment in that each address is a carrier spread over the entire frequency band of the system and the entire time interval of each code word. Code words are transmitted by means of uniform frequency shifts of the address. Because the address is spread in frequency, FH-FSK is resistant to the frequency-selective fading that degrades channelized systems in which each user operates within a narrow frequency band.

On the other hand, performance of FH-FSK is limited by mutual interference among system users. Even if all transmitted tones could be perfectly detected, transmission errors would still occur when several users communicate simultaneously. Assuming that adequate speech quality is provided by 32 kb/s transmission with a binary error rate less than 10^{-3}, we find that, with perfect transmission, up to 209 users can share the 20-MHz (per direction) bandwidth proposed for mobile telephony.

Multipath propagation and noise lead to imperfect tone detection and therefore additional errors. Taking a simplified view of transmission impairments, we estimate the maximum number of users in an isolated mobile telephone system to be about 170. This exceeds the capacity of a frequency-hopped, phase-shift-keying, spread-spectrum system proposed by Cooper and Nettleton.[2–4]

The main contribution of this paper is an analysis of the error rate of FH-FSK as a function of the system bandwidth, the transmission rate per channel, the number of simultaneous users, and two statistics of the noise and propagation conditions. From this analysis, we derive an optimum design (256 tones in all, 19 tones per address) for the 20-MHz bandwidth and 32-kb/s transmission rate.

We then introduce a simplified transmission model (white Gaussian noise, Rayleigh-distributed fading) to show how information about fading and noise can be combined with our modulation analysis to determine system performance.

Our calculated performance measures are preliminary indications of system capabilities for base-to-mobile communication. In a two-way system, FH-FSK is vulnerable to further degradations due to synchronization problems not considered in this paper. Our results have stimulated further study of: (*i*) advantages of sophisticated address assignment and improved detection,[6, 11, 12] (*ii*) synchronization problems in mobile-to-base transmission, (*iii*) the effects of additional interference that would arise with FH-FSK used in a cellular mobile radio environment,[5] and (*iv*) implementation problems that appear formidable with current technology.

II. SYSTEM DESCRIPTION

2.1. Transmitter and receiver

The operation of the system may be understood by referring to Figs. 1 and 2. Every T seconds, K message bits are loaded into a shift register and transferred as a K-bit word X_m to the buffer. (The subscript m denotes one link in a multiuser system.) Assume for the moment that the modulo-2^K adder does nothing, so that X_m appears at the adder output, where it is used to select one of the 2^K different frequencies available from the tone generator. At the receiver, the spectrum of each T-second transmission is analyzed to determine which frequency, and hence which K-bit word X_m, is sent.

The 2^K-ary FSK system just described is clearly not suitable for multiple-user operation. If a second transmitter were to generate X_n, neither receiver m nor receiver n would know whether to detect X_n or X_m. However, by using address generators and modulo-2^K adders and subtractors, many links can share the same spectral band.

Consider first an isolated link. During the basic signaling interval $T = L\tau$ seconds, an address generator in the transmitter generates a sequence of L numbers, each K bits long:

$$R_{m,1},\ R_{m,2},\ \cdots,\ R_{m,L},$$

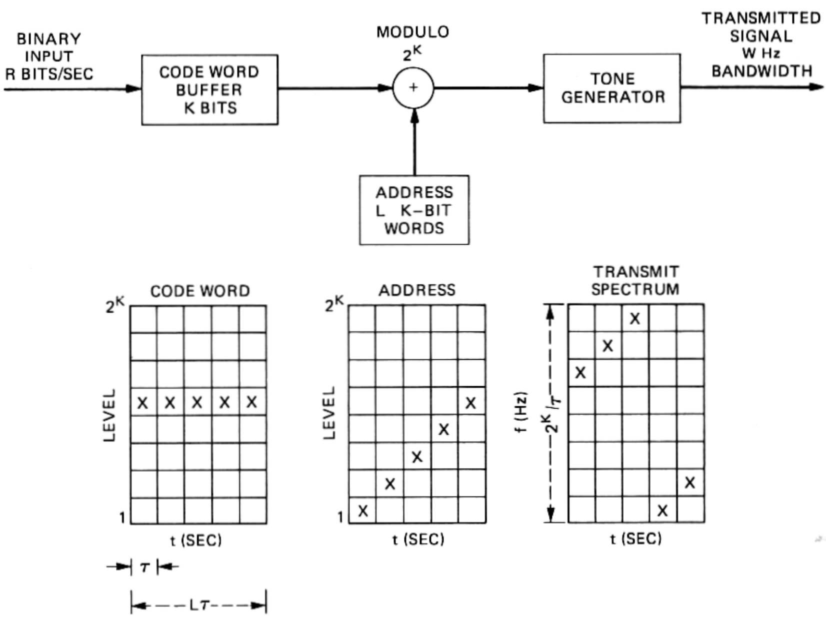

Fig. 1—Transmitter. The block diagram indicates signal processing operations. The matrices show sequences of logic levels (code word, address) or frequencies (transmitted spectrum) within the transmitter. Each user has a different address sequence consisting of L K-bit words.

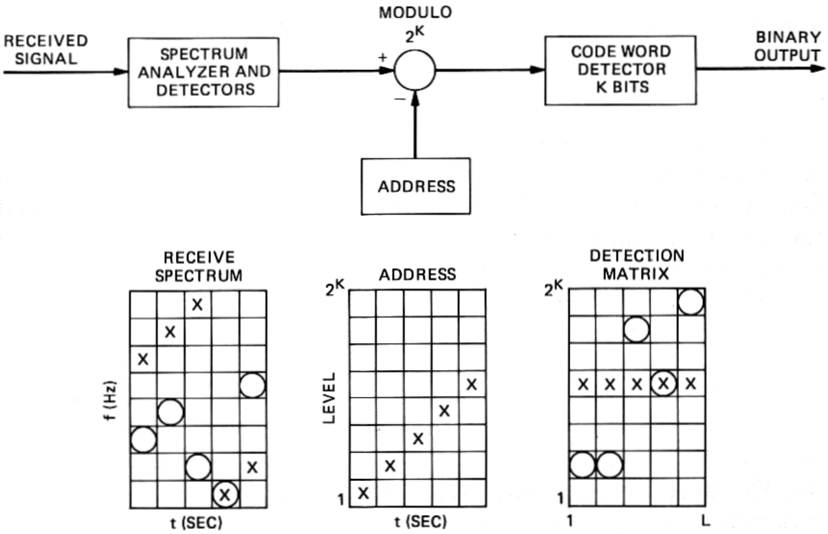

Fig. 2—Receiver block diagram and signal matrices. The matrices show reception of the frequencies in Fig. 1 (X) and reception of a set of tones (0) transmitted by one other user.

at a rate of one number every $\tau = T/L$ seconds. Each $R_{m,l}$ is added modulo-2^K to X_m to produce a new K-bit number

$$Y_{m,l} = X_m \oplus R_{m,l} \tag{1}$$

which is used to select a transmitter frequency.* At the receiver, demodulation and modulo-2^K subtraction by the same number $R_{m,l}$ are performed every τ seconds, yielding

$$Z_{m,l} = Y_{m,l} \ominus R_{m,l} = X_m. \tag{2}$$

The sequence of operations is illustrated by the matrices of Figs. 1 and 2. Each matrix is either a sequence of K-bit numbers (code word, address, detection matrix) or a frequency-time spectrogram (transmit spectrum, receive spectrum). The matrices pertain to one link in a multiuser system. Crosses show numbers and frequencies generated in that link; circles show the contributions of another link.

If M links share the same frequency band, the detection and modulo-2^K subtraction produce extraneous entries in the 2^K-by-L matrix of detected energy. Thus, as shown in Fig. 2, a word X_n transmitted over the nth link will be decoded by the receiver m as

$$Z'_{m,l} = X_n \oplus R_{n,l} \ominus R_{m,l}. \tag{3}$$

When the address sequences $R_{n,l}$ and $R_{m,l}$ are distinct, the elements of

* A circle around + or − indicates the appropriate operation modulo 2^K.

$Z'_{m,l}$ are scattered over different rows. The desired transmission, on the other hand, is readily identified, because it produces a complete row of entries in the detection matrix. With many users, however, detection errors can occur in link m, even when all transmitted tones are received perfectly. This is because the tones of other users can combine to produce a complete row other than X_m in the detection matrix.

2.2 Majority logic decision rule

Noise and multipath propagation can influence the detection matrix in Fig. 2 by causing a tone to be detected when none has been transmitted (false alarm). In addition, the receiver can omit a transmitted tone from the detection matrix (miss). The effects of a false alarm and a miss are shown in Fig. 3. The squares indicate a missing tone in the transmitted code word; the solid circles indicate a false alarm. As shown in Fig. 3, a miss can cause a detection matrix to have no complete row. To allow for this possibility, we use the majority logic decision rule: *Choose the code word associated with the row containing the greatest number of entries.*

Under this decision rule, an error *will* occur when insertions (tones due to other users and false alarms) combine to form a row with more entries than the row corresponding to the transmitted code word. An error *can* occur when insertions combine to form a row containing the same number of entries as the row corresponding to the transmitted code word.

2.3 Error rate formulas

Table I contains a set of formulas, derived in the appendix, which give a tight upper bound on average bit error rate as a function of the following quantities:

(*i*) K bits per code word.

(*ii*) L transmitted tones per code word.

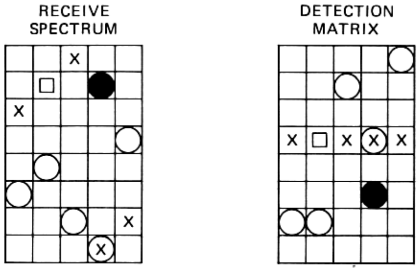

Fig. 3—The effects of a deletion and false alarm on signals in the receiver of Fig. 2. The deletion (□) has removed one entry from the row of X's in the detection matrix of Fig. 2. The false alarm has introduced a spurious entry (●).

Table I—Error rate formulas

Probability of insertion due to interference:

$$p = [1 - (1 - 2^{-K})^{M-1}](1 - p_D)$$

Probability of insertion due to interference or false alarm:

$$p_I = p + p_F - pp_F.$$

Probability of m entries in a spurious row:

$$P_S(m) = \binom{L}{m} p_I^m (1 - p_I)^{L-m}.$$

Probability that no unwanted row has as many as n entries:

$$P(n, 0) = \left(\sum_{m=0}^{n-1} P_S(m)\right)^{2^K-1}; \qquad n > 0.$$

Probability that n is the maximum number of entries in an unwanted row and only one unwanted row has n entries:

$$P(n, 1) = (2^K - 1)P_S(n) \left(\sum_{m=0}^{n-1} P_S(m)\right)^{2^K-2}; \qquad n = 1, 2, \cdots, L.$$

Probability of i entries in the correct row:

$$P_C(i) = \binom{L}{i} (1 - p_D)^i p_D^{L-i}.$$

Upper bound on bit-error rate:

$$P_B \approx \frac{2^{K-1}}{2^K - 1}\left(1 - \sum_{i=1}^{L} P_C(i)\left[P(i, 0) + \frac{1}{2} P(i, 1)\right]\right).$$

(*iii*) M simultaneous users.

(*iv*) p_D miss (deletion) probability.

(*v*) p_F false alarm probability.

A key assumption in the derivation of these formulas is that the address sequence of each user is chosen at random. That is, each $R_{m,l}$ is uniformly distributed between 0 and $2^K - 1$ and is independent of all other address words. The formulas give the average error rate over an ensemble of systems with random addresses. Clearly, a set of addresses exists for which the average error rate is at least as good as this ensemble average and, in fact, Einarsson[6] has derived an address set for which the error rate of *each link* is slightly better than the error rate given by our formulas.

Applications of the formulas to system design and performance evaluation are the subjects of Sections III and IV.

III. DESIGN PROCEDURE

The modulation scheme requires a bandwidth that supports 2^K tones each of duration τ seconds. Tone orthogonality is desirable and requires a bandwidth of at least

$$W = \frac{2^K}{\tau} \text{ Hz.} \tag{4}$$

(In mobile-to-base transmission, the multipath delay spread will lead to some nonorthogonality and a higher bandwidth requirement, as discussed in Section VI.) The transmission rate per user is K bits every $L\tau$ seconds so that

$$R = \frac{K}{L\tau} \text{ b/s.} \tag{5}$$

Ordinarily, W and R will be specified for a particular system application, along with $\bar{P}_B$, the maximum tolerable bit error rate. The goal of the design procedure is to find the values of K and L which maximize the number of users served by the system.

A useful quantity for characterizing various systems is the dimensionless parameter r defined by

$$r = \frac{W}{R} = \frac{2^K L}{K}. \tag{6}$$

Computing (by a search method) $(\hat{K}, \hat{L})$ the optimum (K, L) combination for various values of $(r, \bar{P}_B, p_D, p_F)$, we find that $\hat{K}$ depends only weakly on p_D and p_F. For a given $(r, \bar{P}_B)$, the $\hat{K}$ corresponding to perfect transmission $(p_D = p_F = 0)$ also applies to moderately high (p_D, p_F). Only when (p_D, p_F) is high enough to reduce the number of users by about 50 percent (relative to $p_D = p_F = 0$), does $\hat{K}$ go down and $\hat{L}$ go up. Then the additional redundancy of longer code sequences is justified by the difficult environment.

This is illustrated in Fig. 4 which pertains to possible mobile radio parameters $W = 20$ MHz, $R = 32$ kb/s, $\bar{P}_B = 10^{-3}$, and various values of p_D and p_F. For each K, $\hat{M}$ is the maximum number of users for which the error rate is less than 10^{-3} and L is the greatest integer not exceeding $rK2^{-K}$ [see eq. (6)]. In Fig. 4, we see that $\hat{K} = 8$, corresponding to $\hat{L} = 19$.

In general, for any r and $\bar{P}_B$ we can compute $\hat{M}$ as a function of K with $p_D = p_F = 0$ and find $(\hat{K}, \hat{L})$. These parameters remain optimum for levels of p_D and p_F that are not high enough to erode efficiency severely. Usually, $\hat{K} \approx \log_2 r - 1$ when $p_D = p_F = 0$. In the mobile radio context, we assume that, for $r = 625$ and $\bar{P}_B = 10^{-3}$, $\hat{K}$ will be 8 over the range of p_D and p_F of practical interest.

Figure 5 contains *modulation capacity curves*. To compute them, we let $K = 8$, $L = 19$, $\bar{P}_B = 10^{-3}$, and plot in the p_F, p_D plane contours of constant $\hat{M}$, the maximum number of users for which $P_B \le \bar{P}_B$. The area below and to the left of each contour contains values of p_D and p_F

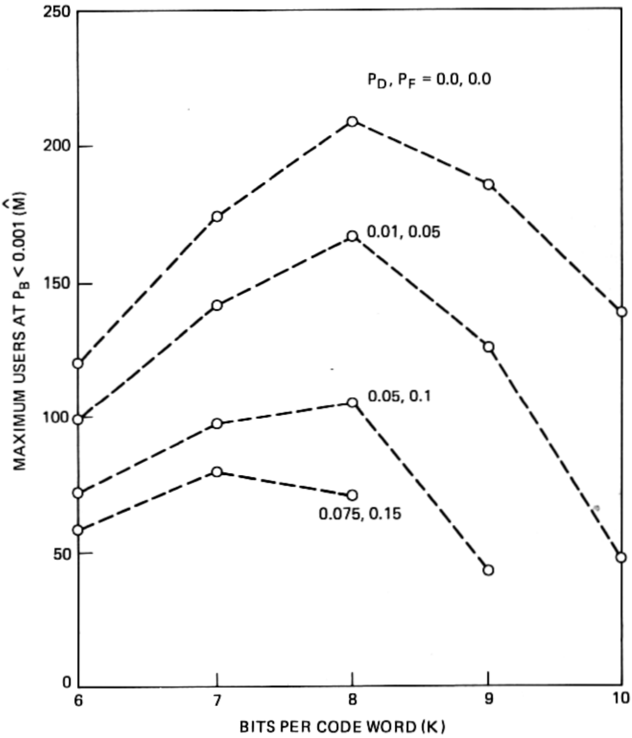

Fig. 4—For a bit error rate of 10^{-3}, a 20-MHz total bandwidth and 32 kb/s bit rate per user, each point shows the maximum number of users. For each combination of miss and false alarm probabilities (p_D, p_F), there is an optimum number of bits/code word. This optimum (8 for $p_D = p_F = 0$) is insensitive to p_D and p_F. Only when p_D, p_F are high enough to drive M below 100, does the optimum K go down to 7.

for which the system will operate properly with $\hat{M}$ users. The contours shown are for $\hat{M} = 209$, the maximum number of users under perfect transmission conditions, and $\hat{M} = 188$, 167, and 146 which allow 10-, 20-, and 30-percent reductions in capacity to accommodate transmission impairments.

IV. PERFORMANCE

4.1 Transmission model

We view transmission to each square in the tone detection matrix as an example of noncoherent on-off keying. To estimate the performance of the $K = 8$, $L = 19$ design, we consider the *receiver operating curves* of Fig. 6, which pertain to a white Gaussian noise channel with Rayleigh fading. Each curve is a contour of false alarm probability p_F and average miss probability p_D for a certain average signal-to-noise ratio $\bar{\rho}$. The two probabilities depend on the detection threshold, represented here by the normalized variable, β, which is the actual threshold divided by the rms receiver noise. We obtained Fig. 6 from the textbook formulas:[10]

$$p_F = \exp\left(-\frac{\beta^2}{2}\right) \tag{7}$$

$$p_D = 1 - \exp\left(-\frac{\beta^2}{2(1+\bar{\rho})}\right). \tag{8}$$

To apply the receiver operating curves to our analysis of FH-FSK, we have assumed that the amplitudes of received tones are all mutually independent (see Section A.5) and plotted Fig. 7, in which the capacity curves of Fig. 5 (solid) and receiver operating curves of Fig. 6 (broken) are superimposed. Figure 7 reveals how many users can be accommodated at a given average signal-to-noise ratio, $\bar{\rho}$. It also indicates the best threshold. (In principle, for each $\bar{\rho}$, an $\hat{M}$ contour is precisely tangent to the $\bar{\rho}$ contour. The value of β at the tangent point is the optimum threshold.)

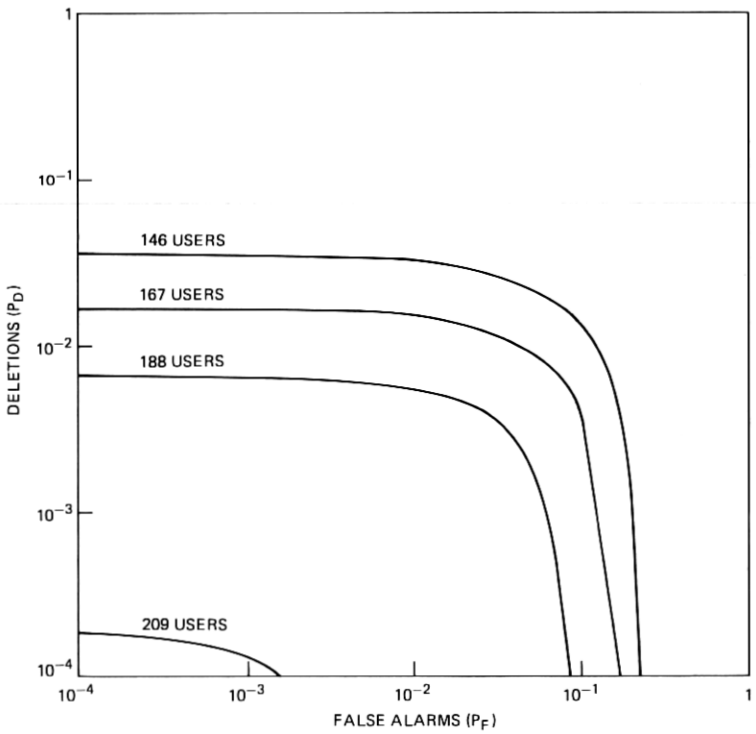

Fig. 5—Modulation capacity curves. Each contour shows p_D, p_F combinations that lead to bit error rate, $P_B = 10^{-3}$ with the number of users indicated by the contour label. For p_D and p_F below or to the left of a contour, $P_B < 10^{-3}$ with the indicated number of users. There are $K = 8$ bits/code word and $L = 19$ words/address sequence.

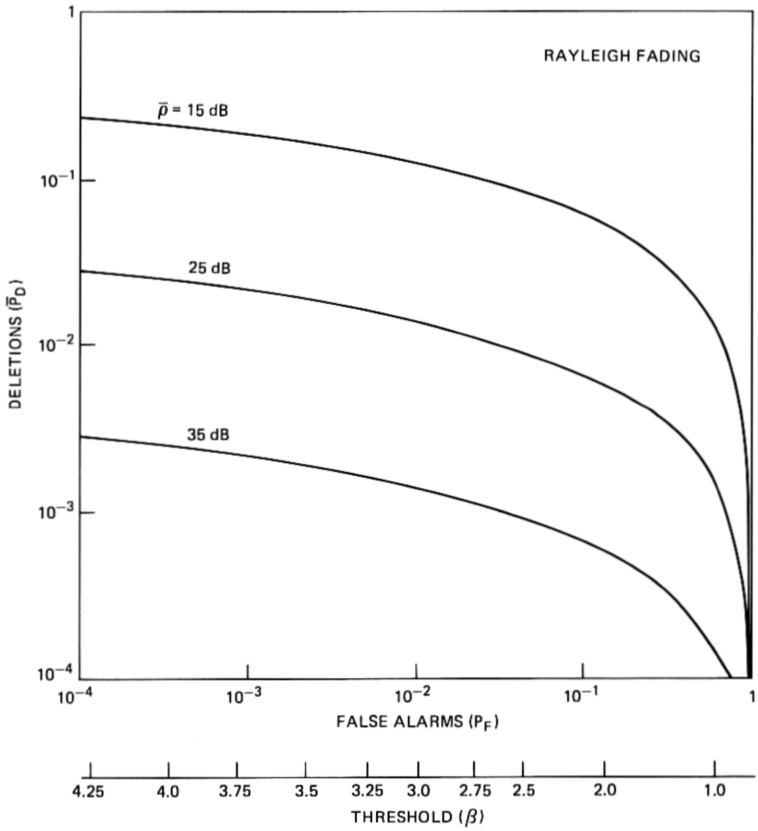

Fig. 6—Receiver operating curves for noncoherent, on-off keying in a Rayleigh fading environment.

To evaluate a specific system, we start with $\bar{\rho} = 25$ dB, which we assume to be a practical average signal-to-noise ratio. Then we observe in Fig. 7 that $\beta = 2.75$ is nearly optimum. With this threshold, up to 170 links can operate with $P_B < 10^{-3}$.

4.2 Degradation due to overload

In a working system, the performance at each time and mobile location is strongly influenced by the number of users and the average signal-to-noise ratio; together, they determine the error probability which in turn affects speech quality. For $K = 8$, $L = 19$, and $\beta = 2.75$, Fig. 8 shows how P_B varies with M and with $\bar{\rho}$ for the case of selective fading.

These curves suggest that performance degrades gradually as M increases. For example, Fig. 8 reveals that, with $\bar{\rho} = 25$ dB, the system can accommodate up to 170 users before $P_B > 10^{-3}$. Then, in an "overload" situation, as additional users seek service, it is possible to invoke an operating strategy that permits some compromise between blocking probability and voice quality by allowing the error probability to rise above $\bar{P}_B$ in busy intervals. For example, 214 users (a 26-percent overload) can use the system with $\bar{P}_B = 10^{-2}$. Provided the digital code is carefully chosen, this error rate would lead to received speech that is somewhat noisy yet quite intelligible.[7,8]

4.3 Refined performance estimates

The technique of using modulation capacity curves and receiver operating curves to determine the maximum number of users is not restricted to the simple Rayleigh fading case illustrated in Section 4.1. Receiver curves can be generated from more refined fading models or from actual measurements of p_D and p_F, and these curves can be

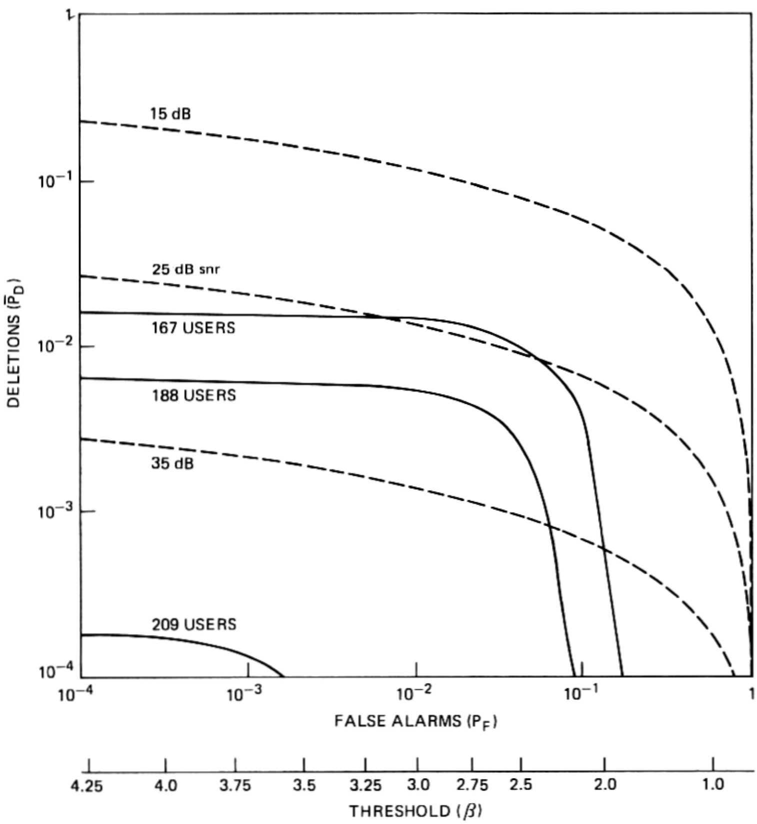

Fig. 7—Capacity curves from Fig. 5 (solid) and operating curves from Fig. 6 (broken). When an s/n ratio contour falls below or to the left of a capacity contour corresponding to M users, the system can operate with $P_B < 10^{-3}$ at that s/n ratio with M or fewer users. Again, $K = 8$, $L = 19$.

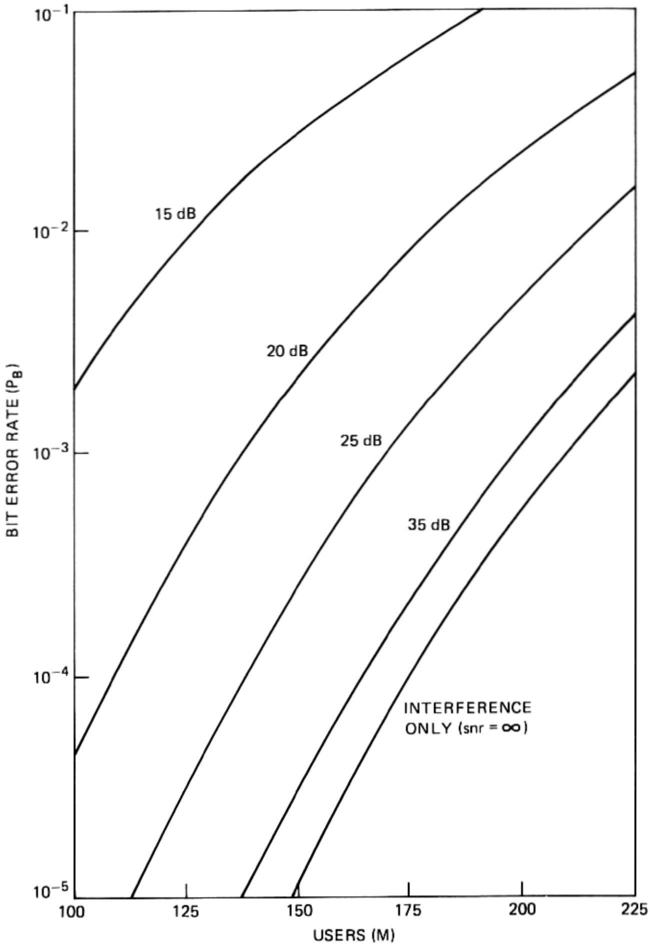

Fig. 8—Performance curves for transmission in an environment of completely selective Rayleigh fading. Note the gradual degradation as the system is loaded beyond the maximum number of users for $P_B < 10^{-3}$.

combined with the capacity curves (Fig. 5) to yield an estimate of system performance. The effects of degradations such as frequency drift and synchronization errors can also be incorporated into the receiver curves. In general, what is needed is a description of the p_D, p_F performance that can be achieved by given equipment over a particular channel. This performance can then be translated, via Fig. 5, into user capacity.

V. ALTERNATIVE MODULATION TECHNIQUE

A spread-spectrum technique which uses PSK instead of FSK has also been proposed for mobile radio communication.[2] With this technique, each code word conveys 5 bits and is transmitted as a sequence of 32 tones. The user address is specified by the tone frequencies, and the source information is carried by the phases. Figure 9 shows the performance of this method[3] as well as the performance of FH-FSK. The PSK curve pertains to errors caused solely by interference among system users; it does not include the effects of receiver noise. However, the PSK system is very tolerant of such noise; only with $\bar{\rho} < 9$ dB would PSK efficiency be 20 percent lower than that indicated by Fig. 9. In fact, it would appear that the PSK system has been overdesigned in the sense that its redundancy allows it to perform adequately in poorer channels than might be expected in mobile radio. The price is an unnecessary limitation on the user capacity. For reasonable s/n ratio

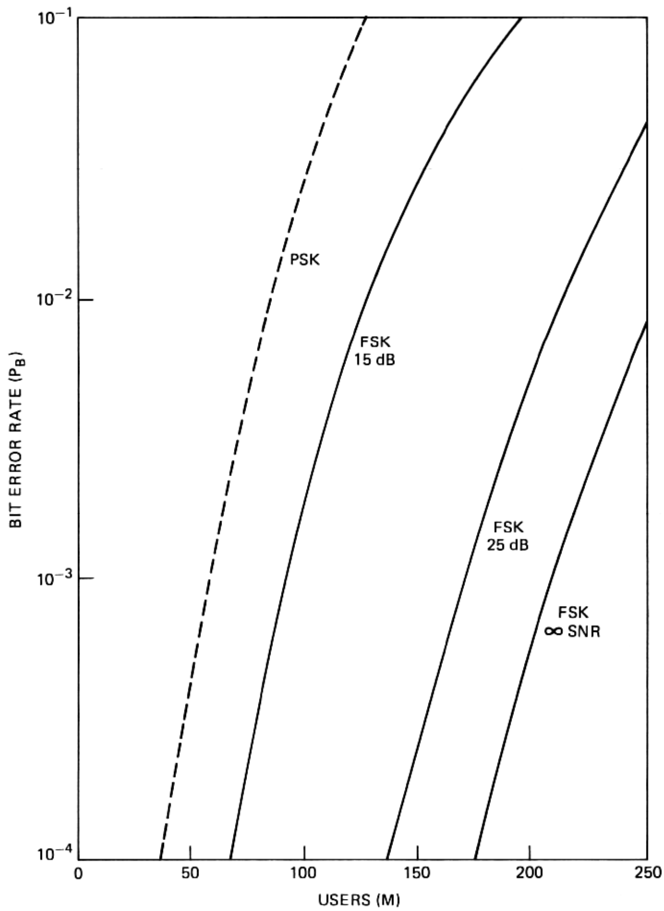

Fig. 9—Comparison of FSK and PSK spread-spectrum performance in selective Rayleigh fading. Although the PSK curve pertains to infinite s/n ratio, that system is quite immune to noise and the curve would be changed little by inclusion of receiver noise unless the average s/n ratio were below about 9 dB.

($\bar{\rho} \approx 25$ dB), the FSK system has roughly three times the capacity of the PSK system.

VI. SUMMARY AND EVALUATION

Our analysis of FH-FSK goes beyond previous work[1] in two respects. It incorporates majority logic detection to deal with transmission effects and it provides precise error rate formulas for all values of K, L, M, p_D, and p_F. Previously, only bounds were available which are quite loose in some regions of interest. We have placed FH-FSK in the context of mobile telephony by substituting representative numerical values for the variables in the error rate formulas. Although our transmission model (white Gaussian noise, Rayleigh fading) is oversimplified, the technique for combining receiver operating curves and modulation capacity curves is applicable to models and measurements of more realistic hardware and transmission environments.

Our theoretical results have stimulated further study of opportunities for enhancing performance and of obstacles to practical realization. Recent results include derivations of attractive address sets and decoding schemes that exploit the structure of the addresses and the redundancy of the modulation to increase efficiency by about 50 percent. In common with other digital modulations, FH-FSK would benefit from sophisticated speech signal processing. Good speech quality can be obtained at bit rates below 32 kb/s, and it may be possible[9] to tolerate higher error rates than 10^{-3}. Any decrease in R below 32 kb/s or increase in $\bar{P}_B$ above 10^{-3} would admit more simultaneous users of FH-FSK.

Balanced against opportunities for performance improvement are the deleterious effects of impairments that are not analyzed in this paper. Shadow fading is one such impairment. It can significantly degrade mobile telephone performance and must be taken into account before any assessment or comparison can be regarded as definitive. Another impairment is the delay spread of multipath propagation. This delay spread makes it impossible to precisely synchronize the arrival times at the base stations of the tones from all mobiles. Consequently, in mobile-to-base transmission the orthogonality assumed in our transmission model cannot be achieved within the bandwidth indicated by eq. (4). The effect, a higher p_F and/or a bandwidth penalty, will offset some performance gains discussed above. Alternatively, asynchronous transmission from mobile to base would also produce impairments to offset these gains.

If FH-FSK is embedded in a matrix of cells covering a large geographical area,[5] intercell interference substantially reduces the number of users per cell. As yet, there is no precise evaluation of this reduction.

Finally, it should be noted that the problem of hardware implementation is formidable, particularly in base-to-mobile transmission where the signal reception and decoding techniques face stringent economic, power, and environmental constraints.

APPENDIX

Error Rate Formulas

A.1. The approach

To assess bit error rate, we refer to the detection matrix of Fig. 3. For a specific link, we derive $P_C(i)$, the probability of i entries in the correct row of the matrix (i.e., the row corresponding to the transmitted code word). Similarly, for a spurious row, we derive $P_S(m)$, the probability of m entries. There are $2^K - 1$ spurious rows, and if all of them have the distribution $P_S(m)$, we can obtain $P(n, k)$, the probability that n is the maximum number of entries in any spurious row and that k spurious rows have n entries. If $n > i$, an error will certainly occur; if $n = i$, the receiver guesses (with probability $1/(k + 1)$ of being correct) which row among those with n entries was transmitted. These observations lead to the formula for P_W, the word error probability, and finally to P_B, the bit error probability.

A.2 Addressing scheme, probability space

The system has M simultaneous users, each with an address that is a sequence of L K-bit words. We assume that each word of an address is selected at random (with equal probability over the 2^K possibilities) and that all addresses in the system are selected independently of each other. We also assume that each user transmits code words that are equally likely over the 2^K possibilities and independent of each other. We then compute probabilities over an ensemble of systems with addresses and code words randomized this way.

A.3 Derivations

The implication of these assumptions is found in the first step of the error rate derivation. We consider time slot l in spurious row j of the detection matrix and consider one interfering user. The probability of that user not sending a tone to the j, l position in the detection matrix is $1 - 2^{-K}$ and the probability that none of the $M - 1$ interfering users will send a tone to j, l is

$$(1 - 2^{-K})^{M-1}.$$

Therefore, the probability of a tone transmitted to position j, l is

$$1 - (1 - 2^{-K})^{M-1}.$$

The miss (deletion) probability, p_D, is defined as the probability of no tone detected in some matrix position, conditioned on a tone transmitted to that position. It follows that the probability of a tone in one position of a spurious row in the detection matrix is

$$p = [1 - (1 - 2^{-K})^{M-1}](1 - p_D). \tag{9}$$

A false alarm (spurious detection) can also cause an entry in position j, l. The false alarm probability p_F is defined as the probability of detecting a tone when none was sent by any user of the system. If p is the probability of an entry due to other users and p_F the probability of an entry due to a false alarm, the overall insertion probability for position j, l is

$$p_I = p + p_F - pp_F. \tag{10}$$

The probability of m entries in row j is

$$P_S(m) = \binom{L}{m} p_I^m (1 - p_I)^{L-m}. \tag{11}$$

Over the $2^K - 1$ incorrect rows, the probability that n is the maximum number of entries and that exactly k unwanted rows contain n entries is

$$P(n, k) = \binom{2^K - 1}{k} [P_S(n)]^k \left(\sum_{m=0}^{n-1} P_S(m) \right)^{2^K - 1 - k}; \qquad n > 0$$

$$P(0, 2^K - 1) = [P_S(0)]^{2^K - 1}$$

$$P(0, k) = 0; \; k \neq 2^K - 1. \tag{12}$$

Note that if we set $k = 0$ in (12), we obtain

$$P(n, 0) = \left(\sum_{m=0}^{n-1} P_S(m) \right)^{2^K - 1}, \tag{12a}$$

which is just the probability that the maximum number of entries in an incorrect row is less than n.

Now consider the correct row. The probability of an entry at some position of this row is $1 - p_D$ and the probability of i entries is

$$P_C(i) = \binom{L}{i} (1 - p_D)^i p_D^{L-i}. \tag{13}$$

The correct word is detected with certainty if $n < i$ [which occurs with probability $P(i, 0)$]; it is detected with probability 1/2 if $n = i$ and only one incorrect row contains i entries [which occurs with probability $P(i, 1)$]; it is detected with probability 1/3 if $n = i$ and two incorrect rows contain i entries [which occurs with probability $P(i, 2)$]; etc. Thus, with i given, the probability of correct detection is

$$P(i, 0) + \frac{1}{2} P(i, 1) + \frac{1}{3} P(i, 2) + \cdots + 2^{-K} P(i, 2^K - 1)$$

$$= \sum_{k=0}^{2^K - 1} \frac{1}{k + 1} P(i, k).$$

The overall probability of correct detection is

$$1 - P_W = \sum_{i=0}^{L} P_C(i) \sum_{k=0}^{2^K-1} \frac{1}{k+1} P(i, k). \tag{14}$$

Thus the word error probability is P_W and the bit error rate is

$$P_B = \frac{2^{K-1}}{2^K - 1} P_W, \tag{15}$$

where the factor multiplying P_W is the expected fraction of bit errors in a wrong K-bit word.

A.4. Computation, bounds

Of the seven numbered formulas in Section A.3, only (12) causes computational problems over the parameter range of interest in mobile radio. The number of bits per symbol, K, can be as high as 10, requiring in (12) the computation of $\binom{1023}{k}$. To avoid this task, we can refer to (14) and derive the following bounds on the second summation

$$P(i, 0) + \frac{1}{2} P(i, 1) < \sum_{k=0}^{2^K-1} \frac{1}{k+1} P(i, k) < P(i, 0) + \frac{1}{2} \sum_{k=1}^{2^K-1} P(i, k).$$

The lower bound assigns zero to the probability of correct detection when at least two or more unwanted rows have i entries, and i is the maximum number of entries in an unwanted row. The upper bound assigns ½ to the probability of correct detection in this event. To compute the upper bound, note that

$$\sum_{k=1}^{2^K-1} P(i, k) = \left(\sum_{m=0}^{i} P_S(m) \right)^{2^K-1} - \left(\sum_{m=0}^{i-1} P_S(m) \right)^{2^K-1}; \qquad i \neq 0$$

$$= [P_S(0)]^{2^K-1}; \qquad i = 0, \tag{16}$$

which is the probability that i is the maximum number of entries in a spurious row. The sums in (16) appear in (12a), allowing us to write

$$\sum_{k=1}^{2^K-1} P(i, k) = P(i+1, 0) - P(i, 0); \qquad i = 0, 1, \cdots, L,$$

where $P(L + 1, 0) = 1$. Thus we have the upper bound

$$P(i, 0) + \frac{1}{2} \sum_{k=1}^{2^K-1} P(i, k) = \frac{1}{2} P(i+1, 0) + \frac{1}{2} P(i, 0)$$

and

$$\sum_{i=0}^{L} P_C(i) \left(P(i, 0) + \frac{1}{2} P(i, 1) \right) < 1 - P_W$$

$$< \frac{1}{2} \sum_{i=0}^{L} P_C(i) \left(P(i+1, 0) + P(i, 0) \right).$$

In the numerical results presented in this paper, P_B is the upper bound on the bit error rate which is computed with the lower bound on the probability of a correct word. Thus

$$P_B < \frac{2^{K-1}}{2^K - 1}\left(1 - \sum_{i=0}^{L} P_C(i)\left[P(i, 0) + \frac{1}{2} P(i, 1)\right]\right). \tag{17}$$

The formulas used in the computation of this bound are listed in Table I.

Note that (17) is a tight bound.* The only approximation is that there is no chance of correct detection when two or more spurious rows have the same number of entries as the correct row. This is an unlikely event. [The bound can be further tightened by the addition of easily computable terms ⅓ $P(i, 2)$, ¼ $P(i, 3)$ etc. to the sum in (17)].

A.5 Fading

While in Sections A.3 and A.4 p_D and p_F are treated as constants, in mobile radio they are time and frequency dependent. Due to fading, they can vary within the detection matrix and change as a new matrix is formed for each code word.

The error rate formulas can be applied in a straightforward manner to two idealized kinds of fading. In *flat fading*, the signal amplitude is constant over an entire code word and varies from word to word. In *selective fading*, the basis of our numerical examples, all signal elements have mutually independent amplitudes.

For flat fading, we can view P_B in Table I as a function of a single signal-to-noise ratio, ρ. Since ρ is constant for each code word, the average P_B is simply the expected value of this function with respect to the probability density function of ρ.

On the other hand, in selective fading, P_B depends on many different s/n ratios, one for each square in the detection matrix. Furthermore, in mobile-to-base transmission, each link has a different s/n ratio even in a single square. The different s/n ratios are mutually independent and, to find the average of P_B over their joint distribution, we have considered how each of them affects (17). Our conclusion is that, for the numerical examples presented here, it is accurate to compute P_B as indicated in Table I, using for p_D the *average* miss probability (8). The reasoning begins with the observation that, in (17), P_C depends on s/n ratios of elements in the correct row of the detection matrix, while P depends on s/n ratios in other rows. It follows that P and P_C are independent, and that their average product is the product of their averages.

Consider $P_C(i)$. This probability of i entries in the correct row is

* Over the range of error rates appearing in the figures, the upper and lower bounds are within 10 percent of each other and usually much closer.

(13) for constant s/n ratio. In a fading situation, it is the sum of $\binom{L}{i}$ products, each product containing L factors. Each factor is p_D or $1 - p_D$ of a different signal element. Since the signal elements have independent amplitudes, the average product is the product of the averages of the individual factors. Therefore, each average product is $(1 - p_D)^{L-i}\, p_D^i$, where the p_D is now an average miss probability. It follows that, for selective fading, the average P_C can be computed by using in (13) the average value of p_D which in our simplified transmission model is (8).

It remains to find the average of P. To do this rigorously, one must find the average p_I in (10), a procedure that is different for base-to-mobile and mobile-to-base transmission, because, in base-to-mobile transmission, the $M - 1$ p_D terms in (9) have the same s/n ratio, while in mobile-to-base transmission they all have independent s/n ratios. In our numerical work, we assumed independence in both directions (a conservative assumption for base-to-mobile transmission) and computed P using the average p_D in (9). In our model, p_F is independent of s/n ratio so that it has no variability to be considered in assessing performance in a fading environment.

It follows that we can use (7) and (8) to find p_F and the average p_D and then use those values in Table I to compute the average bit error rate.

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