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On Switching Networks and Block Designs, II

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An important objective in designing switching networks is to minimize the probability of calls being blocked. A number of methods have been developed in the past for designing efficient switching networks that satisfy various constraints on their parameters. In this paper we investigate a special class of subnetworks of a switching network, called channel graphs. It is known that, under the usual assumptions made for calculating blocking probabilities, the blocking probability of a switching network is small if blocking probabilities of its channel graphs are small. With the use of certain combinatorial structures, known as block designs, we construct a large class of nearly optimal channel graphs.

I. INTRODUCTION

Two switching networks having the same number of crosspoints but with different linking patterns can, in general, have different blocking performance. One widely used method to estimate the blocking performance of a switching network is to calculate the (Lee) blocking probabilities[1] of the channel graphs (also called the linear graphs), each of which is the union of all paths that can be used to connect an input terminal and an output terminal.*

Two channel graphs G_1, G_2 of two balanced[2] k-stage networks N_1, N_2, respectively, having the same number of crosspoints but with different linking patterns, often satisfy the following properties:

(*i*) The number of distinct paths in G_1 is equal to the number of distinct paths in G_2.

(*ii*) Suppose we use Lee's probability model together with the assumption that the probabilities of being busy for links in successive stages are independent. If we assume N_1 and N_2 carry the same traffic

* The reader is referred to Refs. 2 and 3 for graph representations of switching networks or undefined graph-theoretic terminology.

loads then for a fixed i, $1 \le i \le k - 1$, a link between stage i and stage $i + 1$ in G_1 has the same probability of being blocked as that of a link between stage i and stage $i + 1$ in G_2. In other words, G_1 and G_2 have the same link occupancies.

For example, in Fig. 1, we have 4-stage channel graphs. The first and last stage each consist of one node, called the source and the sink, respectively. In Fig. 1a, m nodes are in stages 2 and 3. Each node in stage 2 is connected by n links to exactly one node in stage 3. (We note that this multilink 4-stage channel graph can sometimes be viewed as a simple link k-stage channel graph for $k > 4$ after replacing each link between stage 2 and 3 by a path of length > 1.) In Fig. 1b, m nodes are in stage 2, n nodes in stage 3, and every node in stage 2 is connected (by one link) to all nodes in stage 3. In Fig. 1c, m nodes are in stage 2 and m' nodes in stage 3. Every node in stage 2 is connected (by one link) to n distinct nodes in stage 3, and every node in stage 3 is connected to n' distinct nodes in stage 2. Furthermore, $mn = m'n'$.

We note that the channel graph G_1 in Fig. 1a is series-parallel[2], but the channel graphs G_2 and G_3 in Figs. 1b and 1c are not series-parallel. They are often called *spider-web* channel graphs. Recent studies have shown, either by analysis or by simulation,[4-6] that spider-web channel graphs can sometimes significantly reduce blocking probabilities over series-parallel channel graphs for given switching network hardware. The history of this can be traced back to Le Gall[7] who showed that,

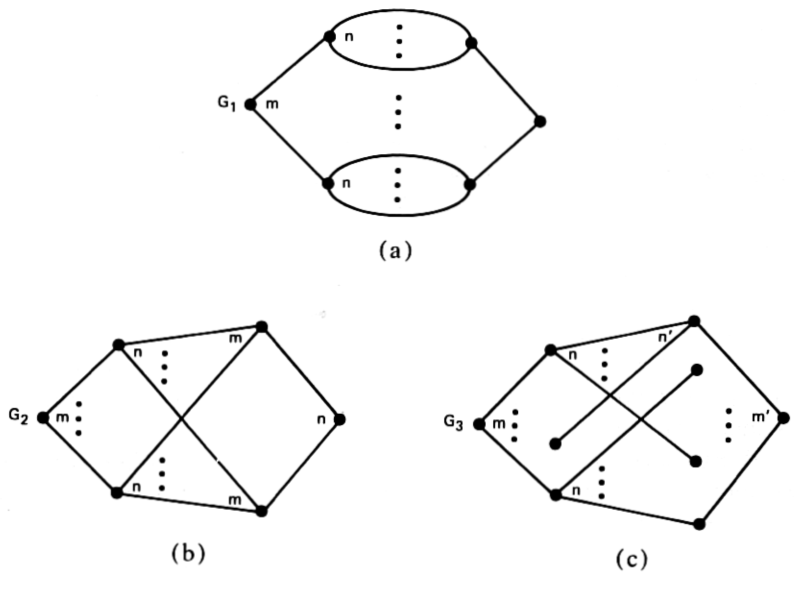

Fig. 1—Three channel graphs.

under certain independence assumptions, the channel graph in Fig. 1b is superior to that of Fig. 1a if $n \geq m$, i.e., the blocking probability of G_2 does not exceed that of G_1 for any given link occupancies. Takagi[6] and Van Bosse[8] strengthened this result by relaxing the independence assumption in different directions. Chung and Hwang[9] generalized this result by showing that the channel graph G_3 in Fig. 1c is superior to the channel graph G_1 in Fig. 1a for $n \geq m$. Hwang[10] also showed that G_3 is, in fact, superior to G_2.

We note that all the graphs in Fig. 1 have the property that each of the m nodes in stage 2 is connected by the same number of links, say n links, to nodes in stage 3. Also, each of the m' nodes in stage 3 is connected by the same number of links, say n' links to nodes in stage 2. This set of channel graphs will be denoted by $C(m, n, m', n')$ where $mn = m'n'$, $m' \geq m$. Graphs in $C(m, n, m', n')$ often come from switching networks with the same number of crosspoints but having different linking patterns. The main problem is to determine the best linking pattern in graphs of $C(m, n, m', n')$. In this paper, we intend to study this problem by the use of block designs.

II. BLOCK DESIGNS

The following is a generalized version of block designs.

A $(\nu, b, r, k, \lambda; t)$-design is a family of subsets $X_1, \cdots, X_b$ of a ν-element set X satisfying the following conditions.

(i) Each X_i has k elements.

(ii) Each t-subset of X is a subset of exactly λ of the sets $X_1, \cdots, X_b$.

Properties (iii), (iv) and (v) follow immediately from (i) and (ii).

(iii) Each element of X is in exactly r of the sets $X_1, \cdots, X_b$ and $bk = \nu r$.

(iv) $b\binom{k}{t} = \lambda\binom{\nu}{t}$.

(v) $X_1, \cdots, X_b$ are also a $(\nu, b, r, k, \lambda_i; i)$-design where

$$\lambda_i = \lambda\binom{\nu - i}{t - i}\Big/\binom{k - i}{t - i} \quad \text{for} \quad i = 1, \cdots, t - 1.$$

As an example, let us set $X_i = \{i, i + 1, i + 3\}$ (mod 7) for $i = 1, \cdots, 7$. It is easy to check that X_i, $1 \leq i \leq 7$, is a (7, 7, 3, 3, 1 ; 2)-design.

The reader is referred to Refs. 11 to 13 for the existence and construction of various classes of block designs, and referred to Refs. 2 and 3 for some applications of block designs in switching networks.

III. AN EXAMPLE

To show the use of block designs in designing good linking patterns for switching networks, let us first have an example. Figure 2 has two

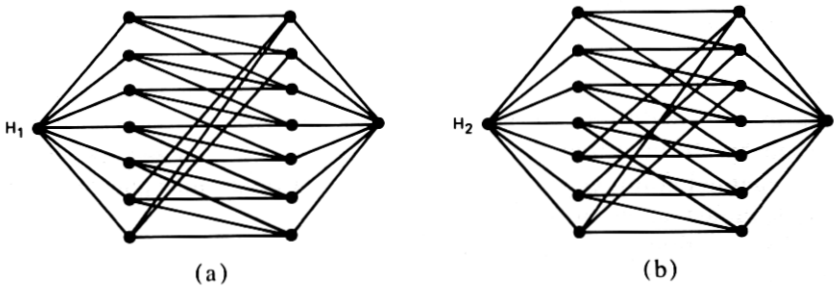

Fig. 2—Two channel graphs in C (7, 3, 7, 3).

channel graphs in the class of graphs $C(m, n, m', n')$, where $m = m' = 7$, $n = n' = 3$, one of which is connected by the use of block designs and will be subsequently shown to be superior to the other. The graph H_1 in Fig. 2a is connected according to a cyclic rule: the ith, node in stage 2 is connected to the ith, $(i + 1)$th, $(i + 2)$th nodes (modulo 7) in stage 3. The graph H_2 in Fig. 2b is connected according to the (7, 7, 3, 3, 1 ; 2)-design: the ith node in stage 2 is connected to the ith, $(i + 1)$th, $(i + 3)$th nodes in stage 3, modulo 7.

We will show that, for any given link occupancy, $\{p_1, p_2, p_3\}$ (i.e., any link between stage i and $i + 1$ has probability p_i of being blocked), the blocking probability of H_1 exceeds that of H_2. First we will have an auxiliary lemma:

Lemma 1: In a channel graph H in C(m, n, m′, n′), let S be a state (i.e., a set of unblocked links) on the set of all links between stages 1 and 2. Suppose S contains | S | unblocked links. We define $x_j(S)$ to be the number of nodes in the second stage which are joined to the jth node in stage 3 and are connected by unblocked links in S to the source. Then the (Lee) blocking probability of H is

$$P(H) = \sum_S p_1^{m-|S|}(1 - p_1)^{|S|} \prod_j (1 - (1 - p_2^{x_j(S)})(1 - p_3)).$$

Proof: We note that the blocking probability of H under S is

$$P(H, S) = \prod_j (1 - (1 - p_2^{x_j(S)})(1 - p_3)).$$

$P(H)$ is just the sum of $P(H, S)$ over all states S. Lemma 1 is proved.

To compare the blocking probability of H_1 with that of H_2, we only have to consider, for each fixed integer q, $1 \le q \le m$,

$$P_q(H_i) = \sum_{|S|=q} \prod_j (1 - (1 - p_2^{x_j(S)})(1 - p_3))$$

for $i = 1, 2$.

Using the inequality for arithmetic and geometric means, one can show that

$$P_1(H_1) = P_1(H_2) = 7(1 - (1 - p_2)(1 - p_3))^3$$

$$\begin{aligned} P_2(H_1) &= 7(1 - (1 - p_2^2)(1 - p_3))^2(1 - (1 - p_2)(1 - p_3))^2 \\ &\quad + 7(1 - (1 - p_2^2)(1 - p_3))(1 - (1 - p_2)(1 - p_3))^4 \\ &\quad + 7(1 - (1 - p_2)(1 - p_3))^6 \\ &\geq \binom{7}{2} (1 - (1 - p_2^2)(1 - p_3))(1 - (1 - p_2)(1 - p_3))^4 \\ &= P_2(H_2). \end{aligned}$$

It can be verified (also see Ref. 14) that $P_3(H_1) \geq P_3(H_2)$. However, the detail is a little messy and will be omitted here.

IV. BLOCK DESIGN GRAPHS IN $C(m, n, m', n')$

To simplify the expressions, let us first define some notation.

For a graph H in $C(m, n, m', n')$, let S_H denote the set of states on the set of links between stages 1 and 2. Let $S_{H,q}$ be a subset of S_H consisting of all states containing q unblocked links. Also, as defined in the above section, $x_j(S)$, $S \epsilon S_H$, is the number of nodes of the second stage which are joined to the jth node in the third stage and connected by unblocked links in S to the source. From Lemma 1 we know that the blocking probability of H is

$$P(H) = \sum_q p_1^{m-q}(1 - p_1)^q P_q(H),$$

where

$$P_q(H) = \sum_{S \epsilon S_{H,q}} \prod_j (1 - (1 - p_2^{x_j(S)})(1 - p_3)).$$

We will prove the following

Main Theorem: Suppose we have a (v, b, r, k, λ ; t)-design. Then in C(m, n, m′, n′), with m = v, n = r, m′ = b, n′ = k, let H be connected according to this t-design, i.e., the ith node in stage 2 is connected to the jth node in stage 3 if and only if the ith element of X belongs to the jth block X_j. Then for any $q \leq t$ or $q \geq m - t$ we have $P_q(H) \leq P_q(H')$ for any graph H′ in C(m, n, m′, n′).

Before proving the main theorem, we will first prove several auxiliary lemmas.

Lemma 2: For a graph G in C(m, n, m′, n′), we have

$$\sum_{S \epsilon S_{G,q}} \sum_j \binom{x_j(S)}{t} = m' \binom{n'}{t}\binom{m - t}{q - t}$$

for any q.

Proof: Let T be a set of t nodes of the second stage. We define $g(j, S, T)$ to be 1 if all nodes in T are joined to the jth node of the third stage and connected by unblocked links in S to the source, and to be 0 otherwise. It is easy to see that

$$\sum_S \sum_j \sum_T g(j, S, T) = \sum_S \sum_j \binom{x_j(S)}{t},$$

which is equal to the following

$$\sum_j \sum_T \sum_S g(j, S, T) = \sum_j \sum_T \binom{m-t}{q-t} = \sum_j \binom{n'}{t} \binom{m-t}{q-t}$$
$$= m' \binom{n'}{t} \binom{m-t}{q-t}.$$

Thus, Lemma 2 is proved.

Lemma 3: Let G and G' be two graphs in $C(m, n, m', n')$. Let $X(S)$ denote the set $\{x_j(S)$: $j = 1, \cdots, m\}$. Then the union (counting multiplicity) of all $X(S)$, $S \epsilon S_{G,q}$, is the same as the union (counting multiplicity) of all $X(S')$, $S' \epsilon S_{G';q}$.

Proof: From Lemma 2 we have

$$\sum_{S \epsilon S_{G,q}} \sum_{x \epsilon X(S)} \binom{x}{t} = \sum_{S' \epsilon S_{G',q}} \sum_{x' \epsilon X(S')} \binom{x'}{t} \quad \text{for} \quad t \leq q.$$

Let $\bar{x} = \max\{x : x \epsilon X(S) \text{ or } x \epsilon X(S')\}$.

If we choose $t = \bar{x}$, in the above equality we have the number of x in the union of $X(S)$ over all $S \epsilon S_{G,q}$ having value x' is equal to the number of x' in the union of $X(S')$ over all $S' \epsilon S_{G',q}$ having value $\bar{x}$. It follows easily from the above equality (by induction) that the union of $X(S)$ over all $S \epsilon S_{G,q}$ is the same as the union of $X(S')$ over all $S' \epsilon S_{G',q}$.

Lemma 4: Let H be a graph in $C(m, n, m', n')$ which is connected according to a t-design. Let S and S' be two states in $S_{H,q}$ where $q \leq t$. Then it follows that the set $\{x_j(S) : j = 1, \cdots, m'\} = \{x_j(S') : j = 1, \cdots, m'\}$. In other words, $x_j(S)$ is a permutation of $x_j(S')$.

Proof: Let T be a set of i nodes of the second stage and assume $i \leq q$. We define $g(j, T)$ to be 1 if all nodes in T are joined to the jth node of the third stage and are connected by unblocked lines in S to the source; and 0 otherwise. It is easy to see that

$$\sum_j \sum_T g(j, T) = \sum_j \binom{x_j(S)}{i},$$

which is equal to

$$\sum_T \sum_j g(j, T) = \lambda_i \binom{q}{i}.$$

Thus we have

$$\sum_j \binom{x_j(S)}{i} = \sum_j \binom{x_j(S')}{i}$$

for any $i \leq q$.

It is easy to see that the number of x_j with value q in $\{x_j(S') : j = 1, \cdots, m\}$ is equal to λ_q. It can be easily derived from the above equations (by induction) that

$$\{x_j(S) : j = 1, \cdots, m'\} = \{x_j(S') : j = 1, \cdots, m'\}.$$

Lemma 5: Let H be a graph in C(m, n, m′, n′) which is connected according to a t-design. Let S and S′ be two states in $S_{H,q}$ *where* $q \geq m - t$. *Then we have* $\{x_j(S) : j = 1, \cdots, m'\} = \{x_j(S') : j = 1, \cdots, m'\}$.

Proof: Let T be a set of i nodes of the second stage and assume $i \leq m - q$. We define $g'(j, T)$ to be 1 if all nodes in T are joined to the jth node of the third stage and are connected by blocked lines in S to the source, and to be 0 otherwise. It is easy to see that

$$\sum_j \sum_T g'(j, T) = \sum_j \binom{n' - x_j(S)}{i},$$

which is equal to

$$\sum_T \sum_j g'(j, T) = \lambda_i \binom{q}{i}.$$

Thus we have

$$\sum_j \binom{n' - x_j(S)}{i} = \sum_j \binom{n' - x_j(S')}{i}$$

for any $i \leq m - q$.

Similar to the proof in Lemma 4, we have

$$\{x_j(S) : j = 1, \cdots, m'\} = \{x_j(S') : j = 1, \cdots, m'\}.$$

Lemma 6: Suppose sets Z and Z_i, $i = 1, \cdots, w$, *consist of positive integers (repetition allowed). If the union of* Z_i *(counting multiplicity) over* $i = 1, \cdots, w$ *is the same as the union of w copies (counting multiplicity) of Z, then we have*

$$\sum_i \prod_{z_{ij} \in Z_i} z_{ij} \geq w \prod_{z_j \in Z} z_j.$$

Proof: Since z_{ij} and z_j are all positive, it follows from the arithmetic-geometric mean inequality.

Proof of the Main Theorem: Let us define

$$f(x) = 1 - (1 - p_2^x)(1 - p_3).$$

Then for a graph G in $C(m, n, m', n')$, we have

$$P_q(G) = \sum_{S \epsilon S_{G,q}} \prod_{x \epsilon X(S)} f(x).$$

Suppose H and H' are two graphs in $C(m, n, m', n')$ and let H be constructed according to a t-design and let $q \leq t$ or $q \geq m - t$. From Lemma 3 we have the union (counting multiplicity here) of all $X(S')$, $S \epsilon S_{H',q}$, is the same as the union of all $X(S)$, $S \epsilon S_{H,q}$. From Lemma 4, we have $X(S_1) = X(S_2)$ for any S_1, $S_2 \epsilon S_{H,q}$. By Lemmas 5 and 6, we have

$$\begin{aligned} P_q(H') &= \sum_{S' \epsilon S_{H,q}} \prod_{x \epsilon X(S')} f(x) \\ &\geq \binom{m}{q} \cdot \prod_{x \epsilon X(S)} f(x) = P_q(H). \end{aligned}$$

This completes the proof of the main theorem.

We remark that, in case of $2t + 1 \geq m$, the graph connected according to the t-design has the least blocking probability in $C(m, n, m', n')$.

V. CONCLUDING REMARKS

We note that the channel graphs connected according to t-designs are, somehow, more "evenly distributed." Roughly speaking, by "evenly distributed" we mean the following: Over all possible q-node combinations of the second-stage nodes, the number of distinct third-stage nodes connected to any given q-node combination of the second-stage nodes varies less for block design graphs than for other graphs. In other words, the vector sets $\{x_j(S): j = 1, \cdots, m\}$ for $|S| = q$ have likely similar patterns. We note the probability $P_q(G)$ can be written as a sum over all states S, $|S| = q$. Since the arithmetic mean is always greater than or equal to the geometric mean, $P_q(G)$ tends to be minimized when the graph is "evenly distributed." We also note that the interconnection pattern of the channel graph will then help to determine the linking pattern of the multi-stage switching network (see Refs. 2 to 4).

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