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# On the Structure of Real-Time Source Coders

By H. S. WITSENHAUSEN

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*The outputs of a discrete time source with memory are to be encoded ("quantized" or "compressed") into a sequence of discrete variables. From this latter sequence, a receiver must attempt to approximate some features of the source sequence. Operation is in real time, and the distortion measure does not tolerate delays. Such a situation has been investigated over infinite time spans by B. McMillan. In the present work, only finite time spans are considered. The main result is the following. If the source is* $k$th-*order Markov, one may, without loss, assume that the encoder forms each output using only the last* $k$ *source symbols and the present state of the receiver's memory. An example is constructed, which shows that the Markov property is essential. The case of delay is also considered.*

## I. INTRODUCTION

The outputs of a discrete time source with memory are to be encoded ("quantized" or "compressed") into a sequence of discrete variables. From this latter sequence, a receiver must attempt to approximate some features of the source sequence. Operation is in real time, and the distortion measure does not tolerate delays. Such a situation has been investigated over infinite time spans in Ref. 1. In the present work, only finite time spans are considered.

The main result is the following. If the source is $k$th-order Markov, one may, without loss, assume that the encoder forms each output using only the last $k$ source symbols and the present state of the receiver's memory.

An example is constructed, which shows that the Markov property is essential.

## II. THE MODEL

### 2.1 The causal structure

A source produces a random sequence $X_1, X_2, \cdots, X_T$ where for each $t \in \{1, \cdots, T\}$, $X_t$ is a vector in $n_t$-dimensional real space. The source is characterized by the sequence distribution: A given probability measure on the Borel sets of the product space of dimension $\sum_{t=1}^{T} n_t$.

For each $t$, there is an opportunity for noiseless transmission of a signal $Y_t$ taking $q_t$ possible values. This signal is produced from the $X$ sequence by an encoder. As we consider the problem in real time, causality allows the encoder at $t$ to see only the values $X_1, \cdots, X_t$. The encoders are thus characterized by functions $f_t$: $R^{n_1+\cdots+n_t} \to \{1, \cdots, q_t\}$, Borel measurable, $t = 1, \cdots, T$.

At the receiving end, the most that could be accessible at stage $t$ is the subsequence $Y_1, \cdots, Y_t$. However, we also want to consider the case of limited memory, as the receiver might not be able to store this whole sequence for large $t$. The model will be the following.

At $t = 1$, only $Y_1$ is available, and a discrete variable $Z_1 = r_1(Y_1)$ taking $m_1$ values is stored in memory. For each $t > 1$, the memory is updated by

$$Z_t = r_t(Z_{t-1}, Y_t), \qquad t = 2, \cdots, T-1,$$

where $Z_t$ takes values in $\{1, \cdots, m_t\}$ and

$$r_t: \{1, \cdots, m_{t-1}\} \times \{1, \cdots, q_t\} \to \{1, \cdots, m_t\}$$

is the memory update function.

The purpose of the receiver is to generate a variable $V_t$ in $R^{s_t}$ by

$$V_1 = g_1(Y_1),$$

where $g_1$: $\{1, \cdots, q_1\} \to R^{s_1}$,
and for $t > 1$

$$V_t = g_t(Z_{t-1}, Y_t),$$

where

$$g_t: \{1, \cdots, m_{t-1}\} \times \{1, \cdots q_t\} \to R^{s_t}.$$

The interpretation of $V_t$ is that it represents an approximation to something we wish to know at the receiving end about $X_t$. In particular, one may have $s_t = n_t$ and consider $V_t$ as approximating $X_t$ itself.

The functional relationships described above are symbolized in Fig. 1.

The case of *full receiver memory* is included in this model. One need only identify $Z_t$ with $(Y_1, \cdots, Y_t)$ and $r_t$ with the concatenation function "append."

Furthermore, in this case,

$$m_t = \prod_{k=1}^{t} q_k.$$

### 2.2 The criteria

The performance of the system is defined by way of a sequence of distortion functions. For each $t$, a Borel measurable function

$$\psi_t\colon R^{n_t} \times R^{s_t} \to [0, \infty)$$

is given. Then

$$J_t = E\{\Psi_t(X_t, V_t)\}$$

measures the distortion at stage $t$. It is possible that $J_t$ be infinite, but it is always well defined, as the composition of Borel functions is Borel and $\Psi_t \geq 0$.

### 2.3 The optimization problem

The problem to be considered is the following:

*Given*: The integers: $T$; $n_1, \cdots, n_T$; $q_1, \cdots, q_T$; $m_1, \cdots, m_T$; $s_1, \cdots, s_T$.
The distribution of the $X$ sequence.
The distortion measures $\Psi_1, \cdots, \Psi_T$.

*Choose*: The functions $f_1, \cdots, f_T$; $g_1, \cdots, g_T$; $r_1, \cdots, r_T$ (the latter are redundant in the full memory case, i.e., when $m_t \geq q_1 q_2 \cdots q_t$ for all $t$). A choice of a system of such functions will be called a "design."
*In order to*:

Minimize (exactly or within $\epsilon$) the sum

$$J = \sum_{t=1}^{T} J_t.$$

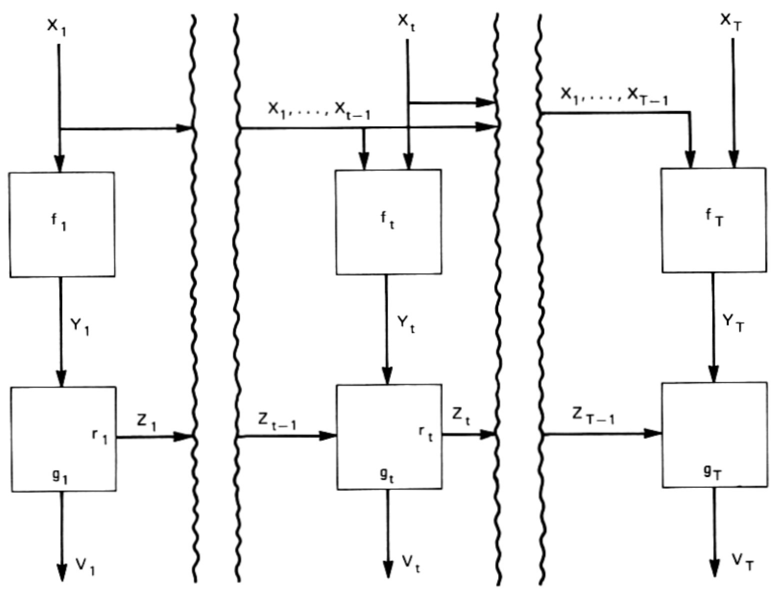


Fig. 1—General system.

Remark that nothing would be gained by having $J$ as a nonnegative linear combination $\sum c_t J_t$ (for instance, with $c_t = e^{-\lambda t}$, a discount factor) because such $c_t \geq 0$ can simply be absorbed into the definition of $\Psi_t$.

It should be said that the freedom of having $n_t$, $q_t$, $m_t$, $s_t$, $\psi_t$ depend upon $t$ is not a matter of extra generality, but is essential to the proof techniques used in the sequel.

A design producing the values $(J_1, \cdots, J_T)$ is *at least as good* as a design producing $(J_1', \cdots, J_T')$ when $J_t \leq J_t'$ for all $t \in \{1, \cdots, T\}$. This, of course, implies the much weaker statement that $J = \sum J_t \leq J' = \sum J_t'$.

A design may exist which is at least as good as any other; it is called a *dominant* design. In general, however, no dominant design exists because the set in $R^T$ of achievable vectors $(J_1, \cdots, J_T)$ does not have a corner $(J_1^*, \cdots, J_T^*)$ such that all other points of this set lie in the shifted orthant $J_t \geq J_t^*$, $t = 1, \cdots, T$. Instead, the set may have a *Pareto frontier* of "admissible" vectors, i.e., vectors $(J_1, \cdots, J_T)$ such that no vector $(J_1', \cdots, J_T')$ is achievable that has $J_t' \leq J_t$ for all $t$ with strict inequality for some $t$.

### 2.4 Special encoder structures

The encoder $f_t$ at a specific stage $t > k$ is said to have *memory structure of order* $k$, if there is a Borel function

$$\hat{f}_t \colon \{1, \cdots, m_{t-1}\} \times R^{n_{t-k+1}+\cdots+n_t} \rightarrow \{1, \cdots, q_t\}$$

such that

$$f_t(X_1, \cdots, X_t) = \hat{f}_t(Z_{t-1}, X_{t-k+1}, \cdots, X_t) \qquad \text{a.s.}$$

This is equivalent to the assertion that $Y_t$ is measurable on the $\sigma$-field generated by $Z_{t-1}, X_{t-k+1}, \cdots, X_t$. In other words, the encoder elaborates $Y_t$ using only the $k$ most recent source outputs $X_{t-k+1}, \cdots, X_t$ and the receiver's current memory $Z_{t-1}$.

## III. THE MAIN THEOREM

The sequence $X_1, X_2, \cdots, X_T$ is said to be $k$th-order Markov, when, given any block of $k$ consecutive $X_t$, the parts of the sequence preceding and following this block are conditionally independent. For $k = 1$, this is the ordinary Markov property. Note that the $k$th-order Markov property holds in a vacuous way if $T < k + 2$.

Most sequences can be approximated by $k$th-order Markov sequences for sufficiently large $k$. If this $k$ is small compared to $T$, then the following main theorem provides a substantial simplification of the encoder optimization problem.

*Theorem* 1: *Suppose the source is* $k$th-*order Markov. Then, given any design, there is another design with the following properties:*

(*i*) *The new design differs from the given one only in the choice of encoders.*

(*ii*) *All encoders of the new design have memory structure of order $k$. (The last encoder $f_T$ can even be made to have memory structure of order* 1.)

(*iii*) *The performance index $J$ of the new design does not exceed the index of the old design.*

Postponing the proof of Theorem 1 to Section V, we comment here on its significance. It says, in particular, that for a Markov source and a receiver with perfect memory, one need only consider encoders which generate each code symbol $Y_t$ using only the current source symbol $X_t$ and the past code sequence $Y_1, Y_2, \cdots, Y_{t-1}$. This result is essentially dependent on the Markov property of the source as can be seen from the following example.

Take $T = 3$ and, for $t = 1, 2, 3$, let $n_t = s_t = 1$, $q_t = 2$, $\psi_t(X_t, V_t) = (X_t - V_t)^2$. Suppose the receiver has perfect memory. Suppose that the source sequence $(X_1, X_2, X_3)$ takes just eight equally probable values, namely (13, 1, 3), (12, 1, 2), (11, 1, 1), (10, 1, 0), (−10, −1, 0), (−11, −1, 1), (−12, −1, 2), (−13, −1, 3).

At the first stage, if one considers only the minimization of $J_1$, one has a classical quantization problem for $X_1$. As $X_1$ takes its values in two separate equiprobable clusters, the minimum of $J_1$ is attained by letting $Y_1$ signal the sign of $X_1$ to identify the cluster. Then $V_1 = \pm 11.5$ and $J_1 = 1.25$. Any other choice of the first encoder yields a strictly larger value for $J_1$. Furthermore, $Y_1$ is already sufficient for the attainment of $J_2 = 0$, the second-stage receiver need not even look at $Y_2$. However, $Y_2$ can be used to help the third-stage receiver. If one lets $Y_2$ signal the parity of $X_1$, then $J_3 = 0$ is attainable by letting $Y_3$ signal whether $|X_3| \leq 1$ or not.

The design so obtained minimizes $J_t$ for each $t$ (it is "dominant"); *a fortiori*, it minimizes $J = \sum_1^3 J_t$, giving $J = 1.25$. However, the second-stage encoder does not have memory structure of order one.

Is it possible to achieve $J = 1.25$ with memory structure of order one although the source is not Markov? The answer is no for, if one changes the first-stage encoder, this alone will drive $J_1$ and, *a fortiori*, $J$ above 1.25. But if the first encoder signals the sign of $X_1$ and the second encoder must have first-order structure, then the second encoder is useless. Indeed, $X_2$ contains no information not already contained in $Y_1$, and the receiver remembers $Y_1$. Now $Y_1$ is useless to the third-stage receiver,† and a single binary signal $Y_3$ is insufficient to distinguish among the four possible values of $X_3$. The best that can be

† $Y_1 = \operatorname{sgn} X_1$ and $X_3$ are independent.

done is to form $Y_3$ as in the previous design, giving $J_3 = 0.25$; hence, $J = 1.5$.

The optimum design requires encoder $f_2$ to "signal ahead" features of $X_1$ for the later benefit of receiver $g_3$. This phenomenon is ruled out for sources with the Markov property.

## IV. TWO BASIC LEMMATA

All the results in this paper will be derived from two basic lemmata: a "two-stage lemma" and a more complex "three-stage lemma." Once these are obtained, the use of induction and of the technique of "repackaging" random variables will suffice.

### 4.1 The two-stage lemma

This lemma uses what is, in fact, the basic line of reasoning in Ref. 1. Consider a system with $T = 2$ and any joint distribution of the pair of random vectors $(X_1, X_2)$. Observe that the content $Z_1$ of the receiver's memory at the beginning of stage 2 is a certain function of $X_1$; that is,

$$Z_1 = \phi(X_1),$$

where $\phi$ is a Borel function (in fact, it is the composition of $f_1$ and $r_1$). The second (and last) stage is characterized by the functions $f_2$ and $g_2$ with (Fig. 2)

$$Y_2 = f_2(X_1, X_2),$$

$$V_2 = g_2(Z_1, Y_2).$$

*Lemma* 1: *Given a two-stage system with a design in which $f_2$ does not have memory structure of order* 1, *one can change $f_2$ (and only $f_2$) so that it has this structure and the new design is at least as good as the given design.*

*Proof*: If only $f_2$ is changed, then $J_1$, $\phi$, and $g_2$ remain as given. We have to show that, for a suitable change in $f_2$, $J_2$ can only decrease. Consider the function

$$F((Z_1, X_2), Y_2) \equiv \psi_2(X_2, g_2(Z_1, Y_2)).$$

As $Y_2$ is discrete and $F$ is measurable (by its construction), a measurable function $\hat{f}_2$ exists (see the appendix) such that

$$F((Z_1, X_2), \hat{f}_2(Z_1, X_2)) \le F((Z_1, X_2), Y_2)$$

for all values of $Z_1$, $X_2$, $Y_2$. Hence, by the substitution

$$Z_1 = \phi(X_1)$$

$$Y_2 = f_2(X_1, X_2)$$

$$\psi_2(X_2, g_2(\phi(X_1), \hat{f}_2(\phi(X_1), X_2))) \le \psi_2(X_2, g_2(\phi(X_1), f_2(X_1, X_2))$$

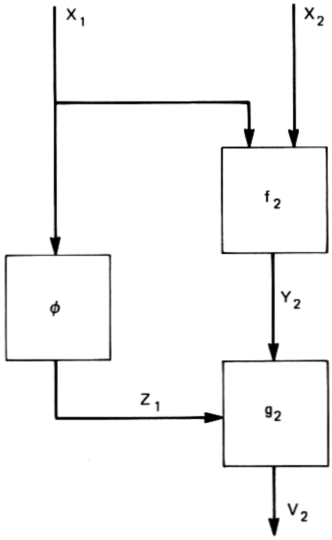


Fig. 2—Two-stage lemma.

holds for all $X_1$, $X_2$. As the functions $\phi_1$, $f_2$ and $\hat{f}_2$ are measurable, both sides of the inequality are measurable. Since they are also nonnegative, the inequality persists when taking the expectation of both sides, whether finite or not. This establishes that $J_2$ can only decrease as claimed.

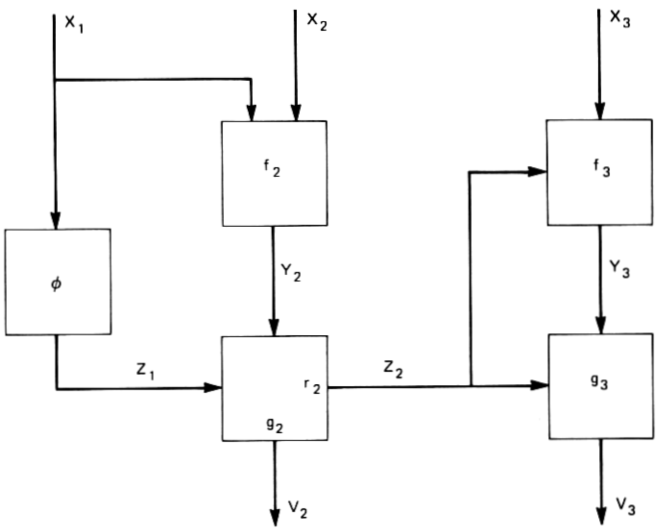


Fig. 3—Three-stage lemma.

### *4.2 The three-stage lemma*

Consider a three-stage system ($T = 3$) with a Markov source. Assume that the last encoder $f_3$ already has first-order memory structure, while $f_2$ does not (Fig. 3).

*Lemma 2: Under the above assumptions, one can replace $f_2$ by an encoder $\hat{f}_2$ having memory structure of order one, without increasing the total cost $J = J_1 + J_2 + J_3$.*

*Proof*: The first-stage cost $J_1$ is unaffected by changes in $f_2$ and the effect of the first-stage design is to generate the receiver memory $Z_1$ as a certain measurable function $Z_1 = \phi(X_1)$, where $\phi$ is the composition $r_1 \circ f_1$. By assumption, $f_3$ can be written in the form

$$Y_3 = f_3(Z_2, X_3)$$

where
$$Z_2 = r_2(Z_1, Y_2).$$

The cost incurred in the last two stages can thus be written

$$\psi_2(X_2, g_2(Z_1, Y_2)) + \psi_3(X_3, g_3(r_2(Z_1, Y_2), f_3(r_2(Z_1, Y_2), X_3))) \equiv F(Z_1, X_2, X_3, Y_2),$$

defining the measurable function $F$.

Consider now the conditional expectation

$$E\{F(Z_1, X_2, X_3, Y_2) | X_1, X_2\}.$$

Because $X_3$ is a finite dimensional random vector, a regular conditional distribution of $X_3$ exists for any condition. In view of the Markov property, conditioning on the pair $(X_1, X_2)$ is equivalent to conditioning on $X_2$ only. Let $P(dX_3 | X_2)$ be a regular version of this conditional distribution.

Then the conditional expectation under consideration can be written

$$\int P(dX_3 | X_2) F(Z_1, X_2, X_3, Y_2),$$

where $Z_1$ and $Y_2$, which depend only on the conditioning variables $X_1$, $X_2$, can be treated as fixed. This integral defines a measurable function (nonnegative and possibly extended real-valued)

$$G(Z_1, X_2, Y_2).$$

For any choice of $f_2$, the sum $J_2 + J_3$ will be given by the expectation of $G$. Note that $X_1$ enters $G$ *only* by way of $Z_1$ and $Y_2$.

As in Lemma 1, a measurable function $\hat{f}_2$ exists such that, for all $Z_1$, $X_2$ and $Y_2$,

$$G(Z_1, X_2, \hat{f}_2(Z_1, X_2)) \leq G(Z_1, X_2, Y_2).$$

Substituting $Z_1 = \phi(X_1)$, $Y_2 = f_2(X_1, X_2)$ and taking the expectations of both sides of this inequality, implies, by the chain rule, that $J_2 + J_3$ cannot increase when $f_2$ is replaced by $\hat{f}_2$.

## V. PROOF OF THE MAIN THEOREM

To begin with, the situation of the last stage is always a special one, as the following lemma shows.

*Lemma* 3: *For any source statistics and any design, one can replace the last encoder by one having memory structure of order one, without performance loss.*

*Proof*: The given $T$-stage system can be considered as a two-stage system, by setting

$$\tilde{X}_1 = (X_1, X_2, \cdots, X_{T-1})$$

$$\tilde{X}_2 = X_T$$

$$\tilde{Z}_1 = Z_{T-1} = \phi(\tilde{X}_1)$$

$$\tilde{Y}_2 = Y_T$$

$$\tilde{f}_2(\tilde{X}_1, \tilde{X}_2) = f_T(X_1, X_2, \cdots, X_{T-1}, X_T)$$

$$\tilde{g}_2(\tilde{Z}_1, \tilde{Y}_2) = g_T(Z_{T-1}, Y_T)$$

$$\tilde{V}_1 = (V_1, \cdots, V_{T-1})$$

$$\tilde{V}_2 = V_T$$

$$\tilde{\psi}_1(\tilde{X}_1, \tilde{V}_1) = \sum_{t=1}^{T-1} \psi_t(X_t, V_t)$$

$$\tilde{\psi}_2(X_2, V_2) = \psi_T(X_T, V_T),$$

which amounts to a change in notation. Of course,

$$\tilde{n}_1 = \sum_{t=1}^{T-1} n_t,$$

a substantial increase in dimension.

By Lemma 1, there exists an encoder $\hat{\tilde{f}}_2$ which has the structure

$$\tilde{Y}_2 = \hat{\tilde{f}}_2 (\tilde{Z}_1, \tilde{X}_2)$$

and whose use does not increase $\tilde{J}_2$. Reverting to the original notation, this corresponds to an encoder $\hat{f}_T$ with the structure

$$Y_T = \hat{f}_T(Z_{T-1}, X_T)$$

whose use does not increase $J_T$. As the other $J_t$ are unchanged, the lemma is proved.

The above fact is the starting point for the proof of the main theorem with $k = 1$.

*Lemma 4: The main theorem holds for $k = 1$. That is, for a Markov source and any design, one can replace the encoders by appropriate encoders having first-order memory structure without increase in the expected cost $J$.*

*Proof*: Using backward induction, one can first replace $f_T$ by $\hat{f}_T$, as in Lemma 3. Now suppose the encoders for stages $t + 1, t + 2, \cdots, T$ already have memory structure of order one. It must be shown that $f_t$ can be replaced by $\hat{f}_t$ with such structure, without increase in expected total cost. To this effect, the $T$-stage system can be considered as a three-stage system, in which the third stage has first-order memory structure and the source is Markov, as follows. Set

$$\tilde{X}_1 = (X_1, \cdots, X_{t-1}) \left(\text{thus, } \tilde{n}_1 = \sum_{i=1}^{t-1} n_i\right)$$

$$\tilde{Z}_1 = Z_{t-1} = \phi(\tilde{X}_1)$$

$$\tilde{X}_2 = X_t$$

$$\tilde{Y}_2 = Y_t$$

$$\tilde{Z}_2 = Z_t = \tilde{r}_2(\tilde{Z}_1, \tilde{Y}_2) = r_t(Z_{t-1}, Y_t)$$

$$\tilde{V}_2 = V_t = \tilde{g}_2(\tilde{Z}_1, \tilde{Y}_2) = g_t(Z_{t-1}, Y_t)$$

$$\tilde{\psi}_2(\tilde{X}_2, \tilde{V}_2) = \psi_t(X_t, V_t)$$

$$\tilde{X}_3 = (X_{t+1}, \cdots, X_T), \qquad \left(\tilde{n}_3 = \sum_{i=t+1}^{T} n_i\right)$$

$$\tilde{Y}_3 = (Y_{t+1}, \cdots, Y_T), \qquad \left(\tilde{q}_3 = \prod_{i=t+1}^{T} q_i\right)$$

$$\tilde{V}_3 = (V_{t+1}, \cdots, V_t) = \tilde{g}_3(\tilde{Z}_2, \tilde{Y}_3), \qquad \left(\tilde{s}_3 = \sum_{i=t+1}^{T} s_t\right).$$

The latter relation follows from the fact that each $V_\theta$, $\theta > t$, is given by $g_\theta(Z_{\theta-1}, Y_\theta)$ and the variables $Z_{\theta-1}$, $Y_\theta$ are known functions of $Z_t$, $Y_{t+1}$, $Y_{t+2}, \cdots, Y_T$ using the memory update functions. Then

$$\tilde{\psi}_3(\tilde{X}_3, \tilde{V}_3) = \sum_{\theta=t+1}^{T} \psi_\theta(X_\theta, V_\theta).$$

As the encoders for stages $t + 1, \cdots, T$ already have first-order memory structure, their effect is to define an encoder

$$\tilde{Y}_3 = \tilde{f}_3(\tilde{Z}_2, \tilde{X}_3)$$

because each of the $Y_\theta$, $\theta > t$, included in $\tilde{Y}_3$ is given by a function $f_\theta(Z_{\theta-1}, X_\theta)$ and the variables $Z_{\theta-1}$, $X_\theta$ are known functions of $\tilde{Z}_2$, $\tilde{X}_3$; i.e., of $Z_t, X_{t+1}, \cdots, X_T$ using the memory update functions and recursion. The given encoder at stage $t$ has the general form $Y_t = f_t(X_1, X_2, \cdots, X_{t-1}, X_t)$ which translates to

$$\tilde{Y}_2 = \tilde{f}_2(\tilde{X}_1, \tilde{X}_2).$$

The new source $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3)$ is Markov since $\tilde{X}_1 = (X_1, \cdots, X_{t-1})$ and $\tilde{X}_3 = (X_{t+1}, \cdots X_T)$ are conditionally independent given $\tilde{X}_2 = X_t$, by the assumed Markov property of the original source.

Thus, the three-stage system satisfies the assumptions of Lemma 2, and $\tilde{f}_2$ can be replaced without loss of total expected cost by $\hat{\tilde{f}}_2$, which has the structure

$$\tilde{Y}_2 = \hat{\tilde{f}}_2(\tilde{Z}_1, \tilde{X}_2).$$

This translates to an encoder

$$Y_t = \hat{f}_t(Z_{t-1}, X_t)$$

for the original problem. Since the notational changes do not influence total cost, the inductive step, and therefore the lemma, is proved.

Note that the above induction is carried out down to $t = 2$ because $f_1$ cannot help but have the desired structure, albeit trivially so.

Turning to the case of general $k$, observe that the encoders for the first $k$ stages have memory structure of order $k$ in a trivial way, whatever their design, and for the last stage, Lemma 3 applies. Thus the conclusion of the main theorem holds for $T \le k + 1$ trivially, as does the assumption on the source. Hence, assume $T \ge k + 2$.

The essence of the proof is a "sliding block" repackaging of the source variables.

Let $$\tilde{X}_t = (X_t, X_{t+1}, \cdots, X_{t+k-1})$$

for $\quad t = 1, \cdots, \tilde{T} \quad$ where $\quad \tilde{T} = T - k + 1 \ge 3.$

Then the sequence $(\tilde{X}, \tilde{X}_2, \cdots, \tilde{X}_{\tilde{T}})$ is Markov. For the variables, let

$$\tilde{Y}_1 = (Y_1, \cdots, Y_k),$$

$$\tilde{Y}_t = Y_{t+k-1}, \quad \text{for } t = 2, \cdots, \tilde{T}$$

$$\tilde{Z}_t = Z_{t+k-1}, \quad \text{for } t = 1, \cdots, \tilde{T}$$

$$\tilde{V}_1 = (V_1, \cdots, V_k),$$

and $$\tilde{V}_t = V_{t+k-1}, \quad \text{for } t = 2, \cdots, \tilde{T}.$$

The functions relating these variables are written as follows:

$$\tilde{Y}_1 = \tilde{f}_1(\tilde{X}_1)$$

summarizes the action of the first $k$ encoders, which will remain unchanged as they already (trivially) have memory structure of order $k$. For $t > 1$,

$$\tilde{Y}_t = \tilde{f}_t(\tilde{X}_1, \tilde{X}_2 \cdots, \tilde{X}_t) = f_{t+k-1}(X_1, \cdots, X_{t+k-1})$$

where $\tilde{f}_t$ is not uniquely defined by this relation. It can be made unique and measurable by requiring (for example) that for $\theta = 2, \cdots, t$, the function $\tilde{f}_t$ depends on the argument $\tilde{X}_\theta = (X_\theta, \cdots, X_{\theta+k-1})$ only through its last component $X_{\theta+k-1}$. However, any measurable $\tilde{f}_t$ satisfying the identity is acceptable.

The receivers are characterized by their memory updating functions:

$$\tilde{Z}_1 = \tilde{r}_1(\tilde{Y}_1)$$

summarizes the recursive buildup of $Z_k$ from $(Y_1, \cdots, Y_k)$ using $r_1, \cdots, r_k$. For $t > 2$, $\tilde{r}_t$ is defined by

$$\tilde{Z}_t = \tilde{r}_t(\tilde{Z}_{t-1}, \tilde{Y}_t) = r_{t+k-1}(Z_{t+k-2}, Y_{t+k-1}).$$

Likewise,

$$\tilde{V}_1 = \tilde{g}_1(\tilde{Y}_1)$$

summarizes the action of the first $k$ decoders (including their memory updating). For $t > 2$, $\tilde{g}_t$ is defined by

$$\tilde{V}_t = \tilde{g}_t(\tilde{Z}_{t-1}, \tilde{Y}_t) = g_{t+k-1}(Z_{t+k-2}, Y_{t+k-1}).$$

Finally, $\tilde{\psi}_1(\tilde{X}_1, \tilde{V}_1) = \sum_{t=1}^{k} \psi_t(X_t, V_t)$ and for $t > 2$

$$\tilde{\psi}_t(\tilde{X}_t, \tilde{V}_t) = \psi_{t+k-1}(X_{t+k-1}, V_{t+k-1}),$$

where $\tilde{\psi}_t$ depends on argument $\tilde{X}_t$ only through the component $X_{t+k-1}$. Now Lemma 4 can be applied to this $\tilde{T}$ stage system with Markov source. Without increase in total cost, the encoders $\tilde{f}_2, \cdots, \tilde{f}_T$ can be replaced by encoders $\hat{\tilde{f}}_t$ with first-order memory structure, i.e.,

$$\tilde{Y}_t = \hat{\tilde{f}}_t(\tilde{Z}_{t-1}, \tilde{X}_t), \qquad \text{for } t = 2, \cdots, \tilde{T}.$$

Expressing this in terms of the original variables, the functions $f_t$ for $t = k + 1, \cdots, T$ are replaced by functions $\hat{f}_t$ satisfying

$$Y_{t+k-1} = \hat{f}_{t+k-1}(Z_{t+k-2}, X_t, X_{t+1}, \cdots X_{t+k-1}) \qquad \text{for } t = 2, \cdots, T - k + 1$$

or equivalently

$$Y_t = \hat{f}_t(Z_{t-1}, X_{t-k+1}, \cdots, X_t) \qquad \text{for } t = k + 1, \cdots, T.$$

These $\hat{f}_t$ exhibit memory structure of order $k$, so that the main theorem is proved.

## VI. THE CASE OF DELAYED DISTORTION MEASURES

The basic model of Section II can be modified as follows for the case in which a delay of $\delta > 0$ steps is included in the definition of distortion. The first change is that the variables $V_1, \cdots, V_\delta$ are simply not generated, the receiver spends its first $\delta$ periods just accumulating observations of $Y_1, \cdots, Y_\delta$ and updating its memory accordingly.

The second change is that, for $t > \delta$, distortion is measured by a function $\psi_t(X_{t-\delta}, V_t)$ whose expectation defines $J_t$. The design objective is to minimize

$$J = \sum_{t=\delta+1}^{T} J_t.$$

For this situation, the following structure simplifying result holds.

*Delay Theorem: Suppose that the source is k*th *order Markov and that the distortion is defined with delay δ. Then any given design can be replaced, without loss, by one in which the encoders have memory structure of order max(k, δ + 1).*

*Proof*: In case $k \geq \delta + 1$, one can perform the same transformation of the point of view as in the proof of the main theorem. Indeed, this transformation gives cost functions of the form

$$\tilde{\psi}_t(\tilde{X}_t, \tilde{V}_t), \qquad t = 1, \cdots, T - k + 1,$$

where

$$\tilde{X}_t = (X_t, \cdots, X_{t+k-1}),\ \tilde{V}_1 = (V_1, \cdots, V_k),\ \tilde{V}_t = V_{t+k-1}.$$

This is compatible with the delay criterion, as follows:

$$\tilde{\psi}_1(\tilde{X}_1, \tilde{V}_1) = \sum_{t=1}^{k-\delta} \psi_t(X_t, V_{t+\delta})$$

and for $t = 2, \cdots, T - k + 1$

$$\tilde{\psi}_t(\tilde{X}_t, \tilde{V}_t) = \psi_{t+k-1}(X_{t+k-\delta-1}, V_{t+k-1}),$$

where it happens that $\tilde{\psi}_t$ depends upon $\tilde{X}_t$ only through the component $X_{t+k-\delta-1}$.

Therefore the argument of the main theorem applies: One can use encoders with memory structure of order $k$. In fact, the above shows that this conclusion is valid for any criteria of the form

$$\sum_t \psi_t(X_t, X_{t+1}, \cdots, X_{t+k-1}, V_{t+k-1}).$$

As for the case $k < \delta + 1$, observe that the source is then, *a fortiori*, Markov of order $\delta + 1$. Hence, the first case applies to yield memory structure of order $\delta + 1$, as claimed.

## VII. CONCLUDING REMARKS

A few extensions of the results are of interest.

(*i*) The proof of the three-stage lemma goes through under the weaker assumption that $f_3$ depends upon $Z_2$, $X_2$, and $X_3$.

(*ii*) All the results in this paper remain true for $V_t$ restricted to given subsets of $R^{s_t}$. This would correspond to quantization levels fixed in advance, as opposed to their selection as part of the design.

(*iii*) Suppose $\delta = 0$, $k = 1$, and the encoder is restricted *a priori* to be a finite state machine of the type

$$W_t = h_t(W_{t-1}, X_t),$$

$$Y_t = f_t(W_{t-1}, X_t),$$

where $W_t$ is a discrete variable representing the contents of the encoder's memory. Then the main theorem implies that it is optimal to take $Z_t = W_t$ and $h_t = r_t$ since this simulation of the receiver's memory produces the argument required for the generation of $Y_t$. This result was obtained independently by N. T. Gaarder.

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## APPENDIX

Let $X$ be a set and $\mathscr{B}$ a $\sigma$-algebra of subsets of $X$. Let $Y$ be a finite set $\{1, \cdots, q\}$. A function $F: X \times Y \rightarrow \mathbb{R}$ is called measurable if for each $y$ in $Y$, the function $F(\cdot, y): X \rightarrow \mathbb{R}$ is $\mathscr{B}$-measurable.

Then it follows that a function $f: X \rightarrow Y$ exists such that

$$F(x, f(x)) \leq F(x, y)$$

holds for all $x \in X$ and $y \in Y$ and the function $f$ is $\mathscr{B}$-measurable (which means that $\{x \mid f(x) = y\}$ is in $\mathscr{B}$ for each $y$).

Since $y$ takes only finitely many values, it is evident that, for each $x$, one can select an $f(x)$ to satisfy the inequality. However, there may be many $x$ for which the minimizing $y$ is not unique. This creates the need for a choice of values in defining $f$, and if such a choice were made in a totally arbitrary manner, it is possible that the resulting $f$ not be $\mathscr{B}$-measurable. What is needed is the (elementary) proof that, for a reasonable way to resolve ambiguous choices, the resulting $f$ is automatically $\mathscr{B}$-measurable.

Given $y \in Y$, consider the set $A_y$ of all $x$ for which $y$ is among the minimizing values, this set is measurable because it is defined by a finite number of inequalities among measurable functions, namely, for each $y' \in Y$,

$$F(x, y) \leq F(x, y').$$

The sets $A_y$ cover $X$ but with overlaps. To remove the overlaps, use the numerical indexing of $Y$ to define

$$B_1 = A_1$$

and, for $y > 1$,

$$B_y = A_y - \bigcup_{i=1}^{y-1} A_i.$$

This construction preserves measurability and removes overlap. Thus, if $f$ is defined to take value $y$ on $B_y$, the desired result is attained. This amounts to stipulating that, when the minimum is attained for more than one element of $Y$, $f(x)$ is defined as the element with the smallest label.

## REFERENCE


1. B. McMillan, "Communicating Systems Which Minimize Coding Noise," B.S.T.J., *48*, No. 9 (November 1969), pp. 3091–3113.