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A Counterexample to a Conjecture on the Blocking Probabilities of Linear Graphs

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It was conjectured by Chung and Hwang that a series-parallel regular linear graph is superior to another if its degree sequence majorizes the degree sequence of the other. A counterexample to this conjecture is given.

The following definitions are taken from Refs. 1 and 2. Consider a t-stage linear graph with a source (the vertex of the first stage) and a sink (the vertex of the last stage). All the vertices are arranged in a sequence of stages such that, for each edge, one vertex is in stage i and the other vertex is in stage $i + 1$, for some i. Each edge is in one of two states, *busy* or *idle*. A linear graph is *blocked* if every path joining the source and the sink contains a busy edge. Assume that any edge connecting a vertex in stage i with a vertex in stage $i + 1$ has probability p_i of being busy for $1 \leq i \leq t - 1$. For a t-stage linear graph, the sequence $(p_1, p_2, \cdots, p_{t-1})$ is called the *link occupancies* for that graph. One t-stage linear graph is *superior* to another if, for any given link occupancies, the blocking probability of the first graph does not exceed that of the second.

Let e be an edge from a vertex a in stage i to a vertex b in stage $i + 1$. Define $\lambda(e)$ to be the ratio of the outdegree of a to the indegree of b. A t-stage linear graph is *regular* if, for each i, $1 \leq i \leq t - 1$, if e and f are any two edges between stage i and stage $i + 1$, than $\lambda(e) = \lambda(f)$. In this case, let $\lambda_i = \lambda(e)$. Thus a regular linear graph is associated with a unique degree sequence $(\lambda_1, \lambda_2, \cdots, \lambda_{t-1})$.

A degree sequence $(\lambda_1, \lambda_2, \cdots, \lambda_{t-1})$ *majorizes* another degree sequence $(\lambda_1', \lambda_2', \cdots, \lambda_{t-1}')$ if and only if $\lambda_1\lambda_2 \cdots \lambda_i \geq \lambda_1'\lambda_2' \cdots \lambda_i'$ for every i, $1 \leq i \leq t - 1$.

A *series-parallel* regular linear graph is a regular linear graph which is either a series combination or a parallel combination of two smaller series-parallel regular linear graphs with an edge being the smallest such graph.

I. A COUNTEREXAMPLE

In Ref. 2, Chung and Hwang conjectured that one series-parallel regular linear graph is superior to another if the degree sequence of the first majorizes the degree sequence of the second.

The graphs of Fig. 1 are a counterexample to this conjecture. The degree sequence of graph (a) is (2, 1, 1, 1/2), the degree sequence of graph (b) is (2, 1/2, 2, 1/2). Thus, the degree sequence of graph (a) majorizes the degree sequence of graph (b).

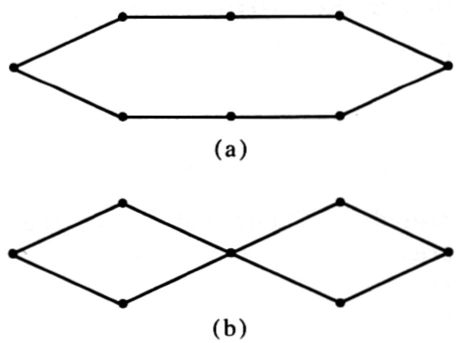

Fig. 1—Counterexample.

Let (p_1, p_2, p_3, p_4) be the link occupancies of the two graphs. Let $q_i = 1 - p_i$, for $1 \leq i \leq 4$. Then the blocking probability of graph (a) is $A = (1 - q_1q_2q_3q_4)^2$. The blocking probability of graph (b) is $B = (p_1 + p_2 - p_1p_2)^2 + (p_3 + p_4 - p_3p_4)^2 - (p_1 + p_2 - p_1p_2)^2(p_3 + p_4 - p_3p_4)^2$. Now let $p_i = 0.1$, for $1 \leq i \leq 4$. Then $A = (1 - (0.9)^4)^2 = (1 - 0.6551)^2 > (1 - 0.7)^2 = 0.09$. But $B = 2(0.19)^2 - (0.19)^4 < 2(0.2)^2 = 0.08$. Thus for this set of p_i's, the blocking probability of graph (b) is less than that of graph (a); so graph (a) is not superior to graph (b), contradicting the conjecture.

II. ACKNOWLEDGMENTS

The author would like to thank the referee for his helpful suggestions and for noting that graphs (a) and (b) permit 2 paths and 4 paths, respectively, between terminals. Because of this, they are not really alternative choices of linking patterns for a fixed network with given switch sites and number of stages. If the restriction "total paths = constant" is maintained, the original conjecture may well be true.

REFERENCES

1. F.R.K. Chung and F.K. Hwang, "A Generalization of Takkagi's Theorem on Optimal Channel Graphs," B.S.T.J., *57,* No. 1 (January 1978), pp. 171–178.
2. F.R.K. Chung and F.K. Hwang, "On Blocking Probabilities for a Class of Linear Graphs," B.S.T.J., *57,* No. 8 (October 1978), pp. 2915–2925.