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On the Required Tap-Weight Precision for Digitally Implemented, Adaptive, Mean-Squared Equalizers

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An analysis is made of the degree of precision required in a digitally implemented adaptive equalizer to achieve a satisfactory level of performance. Considering both the conventional synchronously spaced equalizer and the newer fractionally spaced equalizer, insight is provided into the relationship between the tap-weight precision and the steady-state, mean-squared error. It is demonstrated why the number of adaptive tap weights should be kept to a minimum (consistent with acceptable steady-state performance), both from convergence and precision requirements. A simple formula is given that displays the tradeoff among the equalizer mean-squared error, the number of taps, the channel characteristics, and digital resolution. For typical basic-conditioned voiceband channels operating at 9.6 kb/s, and neglecting the effects that limited resolution might have on timing and carrier phase tracking, analysis and simulation both indicate that the required tap-weight resolution is of the order of 11 or 12 bits. Moreover, the minimum precision is only weakly dependent on the quality of the channel.

I. INTRODUCTION

State-of-the art adaptive equalizers for voiceband modems are digitally implemented and strive to minimize the equalized mean-squared error.[1] An important consideration in assessing the complexity of such an adaptive digital equalizer is the number of bits required to represent the stored signal samples and the equalizer tap weights. Gitlin, Mazo, and Taylor[2] have shown that the precision required for successful adaptive operation, via the estimated-gradient algorithm,[3] can be significantly greater than that required for static or fixed equalization. The purpose of this paper is to determine the precision required in the tap-updating circuitry so that the equalizer mean-squared error can attain an acceptable level.

In the well-known estimated-gradient tap adjustment algorithm,[3] the tap weights are incremented by a term proportional to the product of the instantaneous output error and the voltage stored in the corresponding delay element. When this correction term is less than half a tap-weight quantization interval, the algorithm ceases to make any further substantive adjustment. To determine the minimum number of bits needed to achieve an acceptable performance level (mean-squared error), an appropriate proportionality constant, or step size, must be determined for use in the algorithm. From pure analog, or infinite precision considerations, a relatively large step size is desirable to accelerate initial convergence,[2–5] while a small step size is needed to reduce the residual mean-squared error (that part of the error in excess of the minimum attainable mean-squared error). If the channel is stationary, then in the converged mode the analog algorithm should use a vanishingly small step size to provide almost no fluctuation about the minimum obtainable mean-squared error. However, in a digitally implemented algorithm, a decrease in the step size can actually degrade performance unless there is a compensating increase in the precision of the tap weights. This occurs when the error is so small that an increased number of bits are needed in order that the proportionately smaller corrections be "seen" by the equalizer.

A useful compromise is to choose a step size that provides a slight increase in the steady-state mean-squared error to a level which can be attained by a digital equalizer of reasonable precision. This precludes the choice of an unrealistically small step size—with its concomitant requirement of excessive precision—and provides a mechanism for the analytical determination of the necessary level of precision. Our objective, then, is to be able to directly calculate the precision required to achieve an acceptable performance level.

In Section II, assuming parameters with infinite precision, we determine the step size which produces a specified increase in the mean-squared error. This result is combined in Section III with digital considerations to determine the precision required in the equalizer coefficients. Simulation results are presented in Section IV to illustrate our results for both the conventional synchronous and the newer fractionally spaced equalizers.[6–7]

II. ANALOG CONSIDERATIONS

In this section, we review the basic equalized data communications system in an analog, or infinite precision, environment and determine the steady-state step size associated with a fractional increase in the residual mean-squared error above the minimum attainable error.

2.1 System model

For simplicity, we consider the baseband-equivalent data transmission system of Fig. 1 with received samples

$$r(nT') \triangleq r_n = \sum_m a_m x(nT' - mT) + \nu(nT'), \tag{1}$$

where $\{a_m\}$ are the discrete-valued data symbols, $x(\cdot)$ is the pulse shape at the receiver input, $\{\nu(nT')\}$ are independent noise samples, $1/T$ is the symbol rate, and $1/T'$ is the receiver sampling rate. For the conventional synchronous equalizer, $T' = T$, while for fractionally spaced equalizers[6,7] (FSEs), $1/T'$ will exceed twice the highest frequency component in $x(t)$. The equalizer output is computed only every T seconds and is given by

$$u(nT) = \sum_{m=-N}^{N} c_m r(nT - mT'), \tag{2}$$

where the equalizer has $(2N + 1)$ tap weights. The standard performance measure is the mean-squared error (MSE) at the equalizer output,

$$\mathscr{E} = \langle (u(nT) - a_n)^2 \rangle = \langle (\mathbf{c}'\mathbf{r}_n - a_n)^2 \rangle, \tag{3}$$

where $\langle \cdot \rangle$ denotes the ensemble average, $\mathbf{c}$ and $\mathbf{r}_n$ are, respectively,

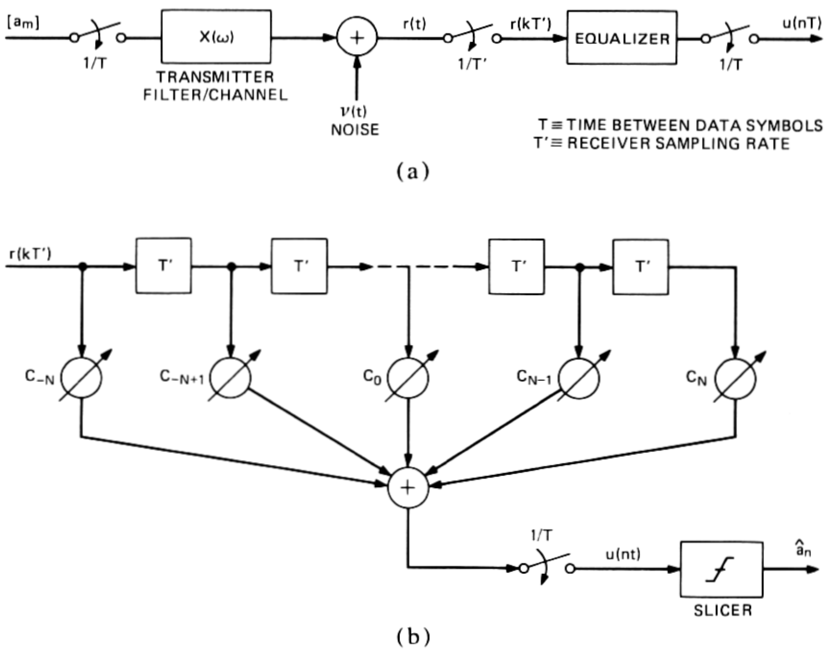

Fig. 1—(a) Simplified baseband-equivalent PAM data transmission system. (b) Tapped delay line equalizer.

the vector of equalizer tap weights and the vector of samples stored in the equalizer at the nth output sampling instant, and the prime denotes vector transpose. Carrying out the indicated expectation in (3) for binary-valued independent data symbols gives

$$\mathscr{E} = \mathbf{c}'A\mathbf{c} - 2\mathbf{c}'\,\mathbf{x} + 1, \tag{4}$$

where $A = \langle \mathbf{r}_n \mathbf{r}_n' \rangle$ is the channel correlation matrix and $\mathbf{x} = \langle a_n r_n \rangle$ is the truncated channel-sample vector; minimizing $\mathscr{E}$ with respect to the tap weights gives the familiar optimum quantities:[1-2]

$$\mathbf{c}_{\text{opt}} = A^{-1}\mathbf{x} \tag{5}$$

$$\mathscr{E}_{\text{opt}} = 1 - \mathbf{x}'A^{-1}\mathbf{x}. \tag{6}$$

2.2 Estimated-gradient algorithm

A well-known, and frequently implemented, algorithm for the iterative adaptive determination of the optimal tap weights is

$$\mathbf{c}_{n+M} = \mathbf{c}_n - \Delta_n e_n \mathbf{r}_n, \qquad n = 0, M, 2M, \cdots, \tag{7}$$

where $e_n = u_n - a_n$ is the error signal,* Δ_n is the step size, and $\mathbf{c}_{n+M}$ is the vector of tap weights at time $(n + M)\,T$. The algorithm is obtained from the (gradient) steepest-descent algorithm by replacing the gradient of $\mathscr{E}$, with respect to $\mathbf{c}$, by the convenient unbiased estimate, $e_n\mathbf{r}_n$. The scaling of the correction term is provided by the step size Δ_n.

Under the assumption of tap-weight adjustments infrequent enough (i.e., M large enough) so that the tap-voltage vectors $\{\mathbf{r}_n\}$ are mutually independent, eqs. (4) to (7) can be used to show that

$$\mathscr{E}_n = \mathscr{E}_{\text{opt}} + \langle \boldsymbol{\epsilon}_n' A \boldsymbol{\epsilon}_n \rangle,$$

where

$$\boldsymbol{\epsilon}_n = \mathbf{c}_n - \mathbf{c}_{\text{opt}} \tag{8}$$

is the corresponding tap-weight error. In practice, it is observed that adjustments at the symbol rate ($M = 1$) result in a comparable mean-squared error, even though the independence of the $\{\mathbf{r}_n\}$, used in deriving (8), is clearly not valid.†

If we let

$$q_n \equiv \langle \boldsymbol{\epsilon}_n' A \boldsymbol{\epsilon}_n \rangle \tag{9}$$

denote the excess mean-squared error, then it is known[2] that q_∞ decreases as Δ_∞ decreases, while the rate of convergence (ROC) increases as Δ_n increases up to half the stability limit. Thus, from analog considerations alone, the choice of step size is important in achieving a balance between rate of convergence and the steady-state error.

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* It can be assumed that an initial training data sequence is known to the receiver.
† Current work by J. E. Mazo appears to explain this anomaly.

Previous studies[2,5] have concentrated on finding the best sequence $\{\Delta_n\}$ to maximize the rate of convergence. However, the choice of the final Δ_∞ appropriate for the steady state has not received much attention, perhaps because from analog considerations alone it is clear that Δ_∞ should be as small as possible consistent with a moderate tracking capability. Difficulties can ensue, however, in a digitally implemented equalizer, where too small a value of Δ_∞ can result in an increased steady-state mean-squared error.[2] With this in mind, a reasonable compromise is to accept an $\mathscr{E}_\infty$ which is somewhat larger than $\mathscr{E}_{\text{opt}}$; it will be shown that this implies a finite value of Δ_∞ from which the required tap coefficient precision can be determined.

2.3 An iterative relation for the residual MSE

As a compromise, the step size can be selected such that the residual excess mean-squared error (9) is an acceptable fraction of the minimum attainable steady-state error, (6); i.e., let

$$q_\infty = \gamma \mathscr{E}_{\text{opt}}, \tag{10}$$

where $0 \leq \gamma \leq 1$. With this range for γ there is at most a 3-dB increase in the steady-state MSE, due to a finite value of Δ. To proceed further, we diagonalize the channel-correlation matrix and write

$$A = P \Lambda P', \tag{11}$$

where Λ is a $(2N + 1)$ by $(2N + 1)$ diagonal matrix whose entries are the eigenvalues, λ_i, of A, and P is an orthogonal matrix composed of the eigenvectors, $\mathbf{p}_i$, of A. If we denote the rotated tap-error vector by

$$\mathbf{y}_n = P \boldsymbol{\epsilon}_n, \tag{12}$$

then

$$q_n = \langle \boldsymbol{\epsilon}_n' A \boldsymbol{\epsilon}_n \rangle = \langle \mathbf{y}_n' \Lambda \mathbf{y}_n \rangle = \sum_{i=-N}^{N} \lambda_i \langle y_{ni}^2 \rangle, \tag{13}$$

where y_{ni} is the ith component of $\mathbf{y}_n$.

Using the above definitions, we can investigate the dynamic behavior of the rotated tap-error vector, $\mathbf{y}_n$, by subtracting $\mathbf{c}_{\text{opt}}$ from both sides of (7) to obtain

$$\begin{aligned} \boldsymbol{\epsilon}_{n+M} &= \boldsymbol{\epsilon}_n - \Delta \mathbf{r}_n \, (\boldsymbol{\epsilon}_n' \mathbf{r}_n + \mathbf{c}_{\text{opt}}' \mathbf{r}_n - a_n) \\ &= \boldsymbol{\epsilon}_n - \Delta \mathbf{r}_n \, [\boldsymbol{\epsilon}_n' \mathbf{r}_n + e_n(\text{opt})] \\ &= [I - \Delta \mathbf{r}_n \mathbf{r}_n'] \, \boldsymbol{\epsilon}_n - \Delta \mathbf{r}_n \, e_n(\text{opt}), \end{aligned} \tag{14}$$

where $e_n(\text{opt}) = \mathbf{c}_{\text{opt}}' \mathbf{r}_n - a_n$ is the instantaneous error when the taps are at their optimum settings. From (14) we obtain

$$\mathbf{y}_{n+M} = [I - \Delta P \mathbf{r}_n \mathbf{r}_n' P'] \, \mathbf{y}_n - \Delta e_n(\text{opt}) \, P \mathbf{r}_n, \tag{15}$$

as the iterative equation satisfied by the rotated tap-error vector.

For small M, determination of the behavior of the residual MSE, using (15), remains one of the most difficult and frustrating problems in data transmission. We shall avoid this problem by making the following assumptions:[3]

(i) The interval M (in symbol intervals) between equalizer adjustments is large enough so that the received vectors $\{\mathbf{r}_{n+\ell M}\}$ are mutually independent.

(ii) The minimum error $e_n(\text{opt})$ is effectively statistically independent of all received vectors $\mathbf{r}_m$.

The support for these assumptions is the following: If the channel memory is less than M symbol intervals (MT seconds), then successive received vectors stored in the delay line will be independent, since they depend upon totally disjoint data symbols. Since we are concerned with steady-state equalizer properties, rather than convergence rate, infrequent adjustment does not adversely affect our results. The independence of $e_n(\text{opt})$ and $\mathbf{r}_{n+\ell M}$ is supported by the following observations. First,

$$\begin{aligned}\langle e_n(\text{opt})\mathbf{r}_n\rangle &= \langle[\mathbf{r}_n'\mathbf{c}_{\text{opt}} - a_n]\mathbf{r}_n\rangle \\ &= \langle \mathbf{r}_n\mathbf{r}_n' A^{-1}\mathbf{x}\rangle - \mathbf{x} = \langle \mathbf{r}_n\mathbf{r}_n'\rangle\, A^{-1}\,\mathbf{x} - \mathbf{x} = 0. \qquad (16)\end{aligned}$$

This equation expresses the well-known fact that, at the optimum tap setting, the error signal is uncorrelated with the current received sample vector. Using the assumption of independent sample vectors, and since $\langle \mathbf{r}_n\rangle = 0$, it follows that $\langle e_n(\text{opt})\mathbf{r}_{n+\ell M}\rangle = 0$ for any ℓ. The statistical *independence* of $e_n(\text{opt})$ and $\mathbf{r}_n$ depends on their higher order moments as well but simulation results indicate that the steady-state squared error is a rather insensitive function of the received samples.

To determine the steady-state step size, we use the above assumptions to derive an iterative relation for $q_n \equiv \langle \mathbf{y}_n'\ \Lambda\, \mathbf{y}_n\rangle$. We first present results for a synchronous equalizer and then modify those results for a fractionally spaced equalizer. From (13) and (15) with the rotated received vector, $\mathbf{s}_n$, defined by

$$\mathbf{s}_n = P\mathbf{r}_n, \qquad (17)$$

we have

$$\begin{aligned}q_{n+M} &= \mathbf{y}_{n+M}'\,\Lambda\,\mathbf{y}_{n+M} \\ &= \langle[\mathbf{y}_n'\,(I - \Delta\mathbf{s}_n\mathbf{s}_n') - \Delta\, e_n(\text{opt})\mathbf{s}_n'] \\ &\quad \Lambda\,[(I - \Delta\mathbf{s}_n\mathbf{s}_n')\,\mathbf{y}_n - \Delta\, e_n(\text{opt})\mathbf{s}_n]\rangle. \qquad (18)\end{aligned}$$

To simplify (18) and to obtain a first-order linear recursion, we note the following:

(iii) By virtue of the definition of $e_n(\text{opt})$, $\langle e_n(\text{opt})\mathbf{r}_n\rangle = 0$, so that

$$\langle e_n(\text{opt})\mathbf{s}_n\,\Lambda\,(I - \Delta\mathbf{s}_n\mathbf{s}_n)\mathbf{y}_n\rangle = 0. \qquad (19a)$$

(iv) By using the familiar eigenvalue bounds* and (i), we have

$$\langle \mathbf{y}_n'\mathbf{s}_n\mathbf{s}_n' \, \Lambda \, \mathbf{s}_n\mathbf{s}_n'\mathbf{y}_n \rangle \le \lambda_M \, \langle \mathbf{y}_n'\mathbf{s}_n\mathbf{s}_n'\mathbf{s}_n\mathbf{s}_n'\mathbf{y}_n \rangle$$
$$= \lambda_M(2N + 1) \, \langle r_n^2 \rangle \, \langle \mathbf{y}_n' \, \Lambda \, \mathbf{y}_n \rangle, \qquad \text{(19b)}$$

where $\langle r_n^2 \rangle$ is the variance of any element in the vector $\mathbf{r}_n$, and the automatic gain control is accurate enough, so that $\mathbf{s}_n' \, \mathbf{s}_n = (2N + 1) \langle r_n^2 \rangle$ is a good approximation for any data sequence.

(v) By virtue of (ii), we have

$$\langle e_n^2(\text{opt}) \, \mathbf{s}_n' \, \Lambda \, \mathbf{s}_n \rangle = \mathscr{E}_{\text{opt}} \, \langle \mathbf{r}_n' A \mathbf{r}_n \rangle \le \lambda_M \, \mathscr{E}_{\text{opt}} \, (2N + 1) \, \langle r_n^2 \rangle. \qquad \text{(19c)}$$

The bounds (19b) and (19c) are relatively tight, since the bulk of the eigenvalues will, in practice, be comparable to λ_M.

(vi) The term in (18) which contributes a negative sign,

$$\langle \mathbf{y}_n'\mathbf{s}_n\mathbf{s}_n' \, \Lambda \, \mathbf{y}_n \rangle = \langle \mathbf{y}_n' \, \Lambda^2 \, \mathbf{y}_n \rangle = \sum_{i=-N}^{N} \lambda_i^2 \, \langle y_{ni}^2 \rangle, \qquad \text{(19d)}$$

has a very significant influence on both convergence and steady-state behavior and must be treated more delicately. What is needed is a good lower bound on (19d); however, the most direct lower bound,

$$\langle \mathbf{y}_n' \, \Lambda^2 \, \mathbf{y}_n \rangle \ge \lambda_m \, \langle \mathbf{y}_n' \, \Lambda \, \mathbf{y}_n \rangle, \qquad \text{(20)}$$

which involves the minimum eigenvalue, λ_m, is (in general) too loose, since just one small eigenvalue will drastically reduce the magnitude of this term.

There is, unfortunately, no tighter lower bound, since if the only significant component $\langle y_{ni}^2 \rangle$ is associated with the smallest eigenvalue, as is possible when there are no restrictions on these components, then the bound (20) can be achieved. In practice, this is an extremely unlikely event (the mean-squared tap errors, $\langle y_{ni}^2 \rangle$, are pretty much equal in value), and (20) is unduly pessimistic in suggesting the choice of a steady-state step size.

We therefore choose to approximate rather than (lower) bound (19d). If, as suggested above, the $\langle y_{ni}^2 \rangle$ are relatively uniform for all i, then a reasonable approximation is

$$\langle \mathbf{y}_n' \, \Lambda^2 \, \mathbf{y}_n \rangle \approx \bar{\lambda} \langle \mathbf{y}_n' \, \Lambda \, \mathbf{y}_n \rangle = \bar{\lambda} q_n \qquad \text{(21)}$$

where $\bar{\lambda}$ is defined as either the average eigenvalue

$$\bar{\lambda} = \frac{1}{2N + 1} \sum_{i=-N}^{N} \lambda_i, \qquad \text{(22a)}$$

* If A is a symmetric matrix, then $\lambda_m \mathbf{z}'\mathbf{z} \le \mathbf{z}' A \mathbf{z} \le \lambda_M \mathbf{z}'\mathbf{z}$, where λ_m and λ_M are the minimum and maximum eigenvalues of A respectively, and $\mathbf{z}$ is any vector.

or the RMS eigenvalue

$$\bar{\lambda} = \left(\frac{1}{2N+1} \sum_{-N}^{N} \lambda_i^2 \right)^{1/2} \tag{22b}$$

Using (19) to (22) in (18), we have the key iterative relation,

$$q_{n+M} \lesssim [1 - 2\Delta\bar{\lambda} + \lambda_M \Delta^2 (2N+1) \langle r_n^2 \rangle] q_n + \lambda_M (2N+1) \langle r_n^2 \rangle \Delta^2 \mathscr{E}_{\text{opt}}, \tag{23}$$

for the excess mean-squared error.

To apply the above equation to systems which use a fractionally spaced equalizer (FSE), some of the terms appearing in (23) must be appropriately interpreted. In systems which use a FSE, the received signal is sampled at the rate $1/T'$, where $1/T'$ is greater than twice the highest frequency component of the baseband signal. Note that if the time span of an FSE is kept constant, the number of tap weights is in inverse proportion to T'. The channel correlation matrix, A, which is Toeplitz for a synchronous equalizer, is no longer Toeplitz for a FSE. It is shown in the appendix that, for $T' = T/2$ and an infinitely long FSE, half the eigenvalues are zero and the other half tend to follow a uniform sampling of the aliased magnitude-squared channel characteristic. The ith eigenvector corresponding to the nonzero eigenvalues is given approximately as a sinusoid of frequency $\omega_i = [2i/(2N+1)](\pi/T)$, $i = 0, 1, \cdots, N$. The eigenvectors corresponding to the zero eigenvalues have most of their spectral energy concentrated near $1/T$ Hz.

In the light of this information, we wish to determine if the bounds (19b) and (19c) are still reasonably tight for a suitably long FSE. Recall that (19b) was obtained by using the bound

$$\sum_{-N}^{N} \lambda_i s_i^2 = \mathbf{s}' \Lambda \mathbf{s} \le \lambda_M \sum_{-N}^{N} s_i^2.$$

Since half the eigenvalues will be quite small, we have as a tight bound that

$$\sum_{-N}^{N} \lambda_i s_i^2 \approx \sum_{N/2}^{N/2} \lambda_i s_i^2 \le \lambda_M \sum_{N/2}^{N/2} s_i^2,$$

where the indices greater than $N/2$ will be associated with the zero eigenvalues. We can, however, recover the full summation by noting that s_i, a component of $\mathbf{s} = P\mathbf{r}$, is given by the convolution of the input samples and the ith eigenvector. For $|i| > N/2$, this convolution is equivalent to passing the received bandlimited signal through a narrow-band filter centered at $1/T$ Hz, and is thus close to zero. We can conclude that

$$\sum_{-N}^{N} \lambda_i s_i^2 \leq \lambda_M \sum_{N/2}^{N/2} s_i^2 \cong \lambda_M \mathbf{s}'\mathbf{s} = \lambda_M \mathbf{r}'\mathbf{r},$$

and hence (19b) and (19c) remain valid. We now reconsider the discussion which precedes (20) and consider the term

$$\begin{aligned}\langle \mathbf{y}' \Lambda^2 \mathbf{y} \rangle &= \sum_{-N}^{N} \lambda_i^2 \langle y_i^2 \rangle \approx \sum_{N/2}^{N/2} \lambda_i^2 \langle y_i^2 \rangle \\ &\approx \bar{\lambda} \sum_{N/2}^{N/2} \lambda_i \langle y_i^2 \rangle \\ &\approx \bar{\lambda} \sum_{-N}^{N} \lambda_i \langle y_i^2 \rangle = \bar{\lambda} \langle \mathbf{y}' \Lambda \mathbf{y} \rangle = \bar{\lambda} q, \end{aligned} \tag{24}$$

where $\bar{\lambda}$ is an average eigenvalue over the set of *significant* eigenvalues of the channel covariance matrix. In obtaining (24), we have again assumed that the $\langle y_i^2 \rangle$ are fairly uniform (in contrast to the s_i^2, which depend critically on the index i), and we interpret $\bar{\lambda}$ as the average of the "nonzero" eigenvalues of the channel correlation matrix.

In practice, it is not difficult to estimate $\bar{\lambda}$ for a FSE, as the eigenvalues, λ_i, tend to approach zero quite rapidly. A reasonable criterion is the average eigenvalue over the partial set of eigenvalues containing all but a small fraction (perhaps 5 percent) of the eigenvalue mass. With this discussion in mind, we can apply (23) to both synchronous and fractionally spaced equalizers.

2.4 Choice of initial and steady-state step sizes

We first investigate the conditions under which the excess MSE will decrease with time. Now in order for the mean-squared error to decay it is clear from (23) that

$$|1 - 2\Delta\bar{\lambda} + \lambda_M \Delta^2 (2N+1) \langle r_n^2 \rangle| < 1$$

or

$$\Delta \leq \Delta_{\text{MAX}} = \frac{2\bar{\lambda}}{\lambda_M} \frac{1}{(2N+1)} \frac{1}{\langle r_n^2 \rangle}. \tag{25}$$

Even with all the bounds and approximations which have been made in reaching (23), a significant difference is readily apparent in the maximum allowable step size for the known-gradient[2,3] algorithm and the estimated-gradient algorithm. From Refs. 2 and 3, we know that, for the known-gradient algorithm to converge, it is required that $|1 - \Delta\lambda_i| < 1$, or equivalently $0 \leq \Delta \leq 2/\lambda_M$. The fact that the maximum step size for the estimated-gradient algorithm is considerably smaller than that for the known-gradient algorithm is deduced from (22a) and (25):†

$$\Delta_{\mathrm{MAX}} = \frac{2}{\lambda_M} \frac{\bar{\lambda}}{(2N+1)} \frac{1}{\langle r_n^2 \rangle}$$

$$= \frac{2}{\lambda_M} \frac{\bar{\lambda}}{\sum_{i=-N}^{N} \lambda_i} < \frac{2}{\lambda_M (2N+1)}. \tag{26}$$

Thus the maximum permissible step size for the estimated-gradient algorithm is reduced, by a factor on the order of the number of tap weights, from the maximum step size permitted in the steepest descent (known-gradient) algorithm.

By differentiating the right-hand side of (23) with respect to Δ, we obtain the step size, Δ_n^*, which provides the maximum rate of convergence (relative to the bound (23)):

$$\Delta_n^* = \frac{\bar{\lambda} q_n}{\lambda_M (2N+1) \langle r_n^2 \rangle} \cdot \frac{1}{[q_n + \mathscr{E}_{\mathrm{opt}}]}. \tag{27a}$$

Note that Δ_n^* is a function of time, n, and the generally unknown (to the receiver) quantities q_n and $\mathscr{E}_{\mathrm{opt}}$. During the early stages of convergence, $q_n \gg \mathscr{E}_{\mathrm{opt}}$, so that (27) becomes the constant value

$$\Delta_0^* = \frac{\bar{\lambda}}{\lambda_M (2N+1) \langle r_n^2 \rangle} = \frac{1}{2} \Delta_{\mathrm{MAX}}, \tag{27b}$$

and q_n converges exponentially towards a steady-state value. Thus a useful rule is: *The initial step size should be half the maximum permissible step size.* Equations (25) to (27) are similar to those proposed by Ungerboeck,[5] except for the channel-dependent factor, $\bar{\lambda}/\lambda_M$, appearing in our equations. This factor suggests a reduction in the step size for most rapid convergence with highly distorted channels.

As convergence nears completion, the steady-state step size, Δ, resulting in a specified mean-squared error, $\mathscr{E}_{\mathrm{opt}} + q_\infty$, is found by equating the two sides of (23). Substituting the constraint, (10), into this relation gives

$$\Delta = \frac{2\bar{\lambda}}{\lambda_M} \cdot \frac{1}{(2N+1)\langle r_n^2 \rangle} \cdot \frac{q_\infty}{q_\infty + \mathscr{E}_{\mathrm{opt}}}$$

$$= 2 \cdot \frac{\bar{\lambda}}{\lambda_M} \cdot \frac{\gamma}{1+\gamma} \cdot \frac{1}{(2N+1)\langle r_n^2 \rangle} = \frac{\gamma}{1+\gamma} \Delta_{\mathrm{MAX}}, \tag{28}$$

† We use the fact that the trace of $A = (2N+1)\langle r_n^2 \rangle = \sum_{-N}^{N} \lambda_i$.

as the formula for the steady-state step size. Thus the steady-state step size ranges from 0 to $\frac{1}{2}\Delta_{MAX}$ as γ varies from zero to unity. Ideally, the step size should vary between the initial value, $\Delta_0^* = \frac{1}{2}\Delta_{MAX}$, and the steady-state value, $\Delta = [\gamma/(1+\gamma)]\,\Delta_{MAX} < \Delta_0^*$, in accord with (27a). In practice, the step size is generally changed in discrete steps between Δ_0^* and Δ.

In summary, (28) provides an approximation to the required steady-state step size in terms of the number of taps, the *effective eigenvalue ratio* (which depends implicitly on the number of taps), the power of the received samples, and the acceptable residual mean-squared error. Note that, as $\gamma \rightarrow 0$, we require a vanishingly small step size, and that increasing the number of taps (to achieve the desired level of $\mathscr{E}_{opt}$) also requires a diminished Δ.

III. DIGITAL CONSIDERATION

In this section, we first review[2] the effects of digital implementation on the estimated-gradient tap-adjustment algorithm, and we then combine our analog and digital results to compute the minimum precision necessary to achieve acceptable performance.

3.1 Digital cutoff of the algorithm

In Fig. 2 we sketch the evolution of the mean-squared error in high- and low-precision equalizers. In the low-precision equalizer, the steady-

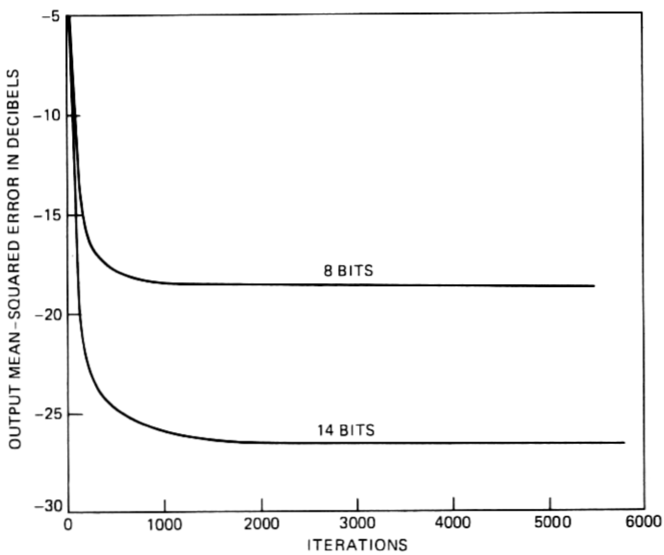

Fig. 2—Evolution of mean-squared error for 32-tap equalizers with tap weights of 8 bits and 14 bits resolution. Steady-state step size = 0.00021.

state, mean-squared error is constrained by the impossibility of changing a tap weight when the correction term in (7) decreases below half a quantization interval. Some corrections will be possible until the peaks of the correction terms fall below the critical level, i.e., until

$$|\Delta e_n r_n| \leq \tfrac{1}{2}\delta, \tag{29}$$

where δ is the interval between quantization levels, or conversely, the algorithm continues to adapt if

$$|\Delta e_n r_n| \geq \tfrac{1}{2}\delta = 2^{-B}\alpha, \tag{30}$$

where B is the number of bits (including sign) used to represent the equalizer tap weights and $(-\alpha, \alpha)$ is the range covered by the (uniform) quantizer.

The above stopping condition can be approximated by replacing the magnitude in (30) by its peak value which is assumed to be $\sqrt{2}$ times its RMS value, i.e., adaptation continues if

$$\sqrt{2}\,\Delta \cdot \sqrt{\langle e_n^2 \rangle} \cdot \sqrt{\langle r_n^2 \rangle} \geq 2^{-B}\alpha; \tag{31}$$

where the MSE which satisfies (31), with equality, will be called the digitally limited MSE. In a passband equalizer for which (30) applies separately to the in-phase and quadrature parts of the tap increment, the condition equivalent to (31) is $\Delta\sqrt{\langle |e_n|^2\rangle\langle |r_n|^2\rangle/2} \geq 2^{-B}\alpha$.

Two important consequences of (31) are:

(*i*) Any attempt to make Δ arbitrarily small, for the purpose of reducing q_∞, will ultimately *increase* the steady-state MSE so that (31) is satisfied.

(*ii*) The ratio of the mean-squared error of an *adaptive* digital equalizer to the MSE due to quantizing the $\{c_n\}$ to within a LSB of their optimum values in a non-adaptive equalizer grows linearly with the number of taps and the "effective" eigenvalue ratio (see Appendices II and III in Ref. 2). In other words, considerably more precision is required for adaptation of the tap weights than for filtering the received signal (performing the equalizer convolution).

Some further observations follow immediately from the above discussions and from numerical substitutions in (31):

(*i*) The number of adaptive parameters should be kept to a minimum consistent with achieving the desired steady-state MSE, since an increase in the number of taps calls for a decreased Δ, which in turn increases the precision required [from (31)].

(*ii*) The excess MSE (associated with a finite step size) evaluated in the last section can be traded against the required precision.

(*iii*) Highly dispersive channels have a larger eigenvalue spread than do good channels and thus require more precision to achieve the same MSE. However, the increased precision will be shown typically to be only 1 bit for channels of moderate distortion.

(*iv*) Digital word size will be restricted to a reasonable value if the digital equalizer is designed to allow an appropriate excess MSE on the order of $\mathscr{E}_{\text{opt}}$.

3.2 Required precision

We now presume that the steady-state step size Δ, determined from (28), is used in a *digitally implemented* equalizer. The analog parameters are assumed such that the steady-state MSE is equal to the digitally limited MSE. Thus performance will be determined by the available precision. Hence the digital word length, B bits, needed in the tap weights to achieve $\mathscr{E}_\infty$ can be determined by substituting (28) in the digital stopping condition (31):

$$2\frac{\bar{\lambda}}{\lambda_M}\frac{\gamma}{1+\gamma}\frac{1}{(2N+1)}\cdot\frac{1}{\langle r_n^2\rangle}\cdot\sqrt{\langle e_n^2\rangle\langle r_n^2\rangle} \geq 2^{-B}\alpha, \tag{32}$$

which reduces to the important relation

$$2^{-B}\alpha \leq 2\frac{\bar{\lambda}}{\lambda_M}\frac{\gamma}{1+\gamma}\cdot\frac{1}{(2N+1)}\cdot\frac{1}{\sqrt{\text{SNR}\cdot\rho}}, \tag{33}$$

where

$$\text{SNR} = \frac{S_{\text{out}}}{\langle e_n^2\rangle} \tag{34}$$

is the (equalized) output signal-to-noise ratio, and where

$$\rho = \frac{\langle r_n^2\rangle}{S_{\text{out}}} \tag{35}$$

is the ratio of input signal power to output (baseband) signal power.† Note that the equalized mean-squared error appearing in (32) can be written as $\langle e_n^2\rangle = \mathscr{E}_{\text{opt}} + q_\infty = (1+\gamma)\mathscr{E}_{\text{opt}} = (1+\gamma)\,[1 - \mathbf{x}'A^{-1}\mathbf{x}]$, where the last equality follows from (6). Thus for a given (or known) channel, all the terms (33) can be readily computed. We now use (33) to estimate the required digital word length under various conditions typical of 9.6-kb/s data transmission.‡

(*i*) *Operation on a Good Channel* (Fig. 3a). A good channel is one for which $\bar{\lambda}/\lambda_M \cong 1$, and the number of equalizer taps can be quite small. In practice, however, a synchronously spaced equalizer will have a fixed number of taps, typically 32. With a passband equalizer[1] having 32 complex tap pairs, the effective eigenvalue ratio is 0.93 for the

† For the complex passband equalizer, the factor on the right-hand side of (33) is $\sqrt{2}$ instead of 2, and $\langle |r_n|^2\rangle$ replaces $\langle r_n^2\rangle$ in (34).

‡ It should be noted that the word lengths derived in the following examples are considerably longer than the precision indicated from using a variance equal to $\sigma^2 = \delta^2/12$ for the quantization error in each tap weight.

"good" channel using a near-optimum sampling epoch. We assume that the maximum quantization level is $\alpha = 1$, that $\gamma = ½$ (corresponding to a 1.1-dB degradation in output s/n ratio), and an output s/n ratio of 25.7 dB is observed (down 1.1 dB from that observed with effectively infinite resolution and vanishing step size). An AGC setting is assumed such that $\rho = 2$. We find from (33) that $B \approx 11$ bits.

(*ii*) *Operation on a Severely Distorted Channel* (Fig. 3b). This severely distorted channel, which just meets the requirements of basic voice-grade line conditioning, has been found to have an effective eigenvalue ratio $\bar{\lambda}/\lambda_M$ (for the best sampling phase) of 0.5. Using this latter value and with the parameters of the previous sample, except for an output s/n ratio of 23.4 dB associated with the distorted channel, we find that $B \approx 11.5$ bits.

(*iii*) *Operation with a Fractionally Spaced Equalizer on the Distorted Channel.* With channel samples taken at $T/2$ intervals, where T is a symbol interval, a 64-tap equalizer is appropriate. The effective eigenvalue ratio† is 0.58, and we find that $B \approx 12$ bits for the same 1.1-dB degradation used in the above examples.

In the next section, the precision predictions of these three examples are compared with simulation results.

IV. SIMULATIONS

The same channels for which eigenvalue ratios were derived for the examples of the last section were used in a simulation program for a QAM data communication system[1] operating at 9600 bps with a baud of 2400/s. Only the tap weights were quantized (rounded to the nearest quantization level); all other variables had the IBM 370 single-precision resolution of roughly 24 bits. The magnitude of the largest real tap weight was about 0.5 in a full quantization range of (−1, 1). Timing and phase references were ideal and not subject to statistical fluctuation. The steady-state step size used in each case was computed from (28). The equalizer was either a 32 (complex) tap synchronous (tap spacing = T = symbol interval) structure, or a 64 (complex) tap $T/2$ structure.

Some simulation runs were made with a gear shifting sequence of adaptation step sizes[4] to reach convergence within a reasonable number of iterations. However, great care was taken to reach the smallest (steady-state) step size well before complete convergence. This is because, with a larger step size, digital equalizer performance can possibly be better for a transient period than that corresponding to the chosen steady-state value. Deterioration of this "good" performance, once it is achieved, depends upon large signal and/or noise

† The "average" tap weight was the average over the 26 tap weights which collectively contained 95 percent of the tap-weight mass.

values, and may not be observed over the short duration of a simulation run.

Curves A in Fig. 4, for operation with the synchronous equalizer on the "good" channel, indicate a degradation of about 1.5 dB for the step size of 0.001 calculated from (28). This is not far from the 1.1 dB ($\gamma = 0.5$) used in that formula; however, the s/n ratio degrades another 0.6 dB when the predicted digital word size of 11 bits is used instead of infinite resolution. The source of this additional degradation has not been investigated, but ordinary quantization noise will make a contribution, and probably also a slowed rate of adaptation, just before the

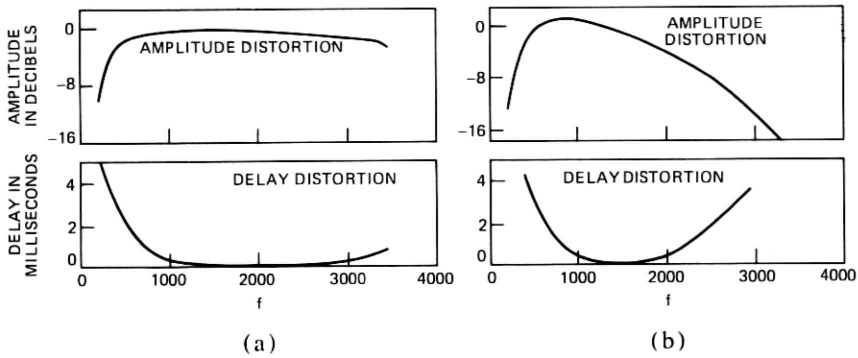

Fig. 3—(a) "Good" and (b) "distorted" channels used in analytical and simulation examples.

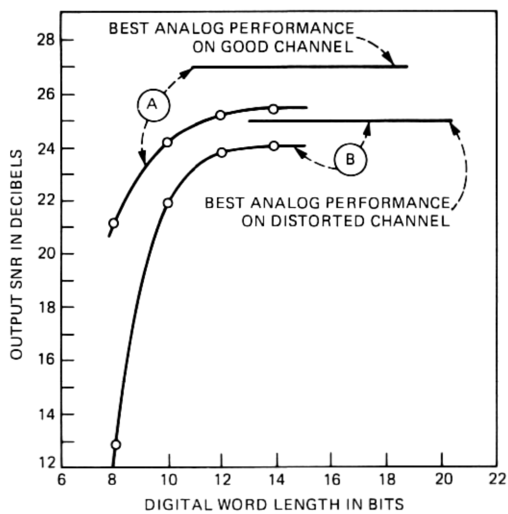

Fig. 4—Measured output s/n ratio vs digital word length (of equalizer tap weights) for (A) a "good" channel (eigenvalue ratio = 0.93), $\Delta = 0.001$; and (B) a distorted channel (eigenvalue ratio = 0.5), $\Delta = 0.0005$. A 32-tap synchronous equalizer was used in both cases.

digital stopping condition prevails. Curve B, for operation on the distorted channel, is down 1.1 dB for a digital word length very close to the predicted 11.5 bits.

Figure 5 illustrates a similar curve for operation on the distorted channel with the 64-tap $T/2$ equalizer, as described in example (*iii*). The digital resolution of about 12 bits predicted from (27) for 1.1-dB degradation is consistent with the simulation results. Figure 6 presents another view of the performance of the $T/2$ equalizer on the distorted channel. This is a curve of output s/n ratio vs adaptation step size for the 12-bit resolution determined from (33). The step size of 0.0003 determined from (28) and used in the experiments represented as points on the curve of Fig. 4, corresponds to a near-peak value of output s/n ratio. This again supports the analysis of Section III as providing a useful formula for deciding on the steady-state step size to be used in a digitally implemented equalizer.

V. DISCUSSION AND CONCLUSIONS

We have proposed a criterion for determining the number of bits needed to represent the tap weights in a digitally implemented equalizer. For a given steady-state adaptation step size, with its attendant increase in steady-state mean-squared error, this criterion is that the word size used be just large enough to "match" this increase without further degrading performance. The word length is a function of the output s/n ratio (the ratio of output signal power to steady-state mean-squared error), the fractional increase in the mean-squared error over

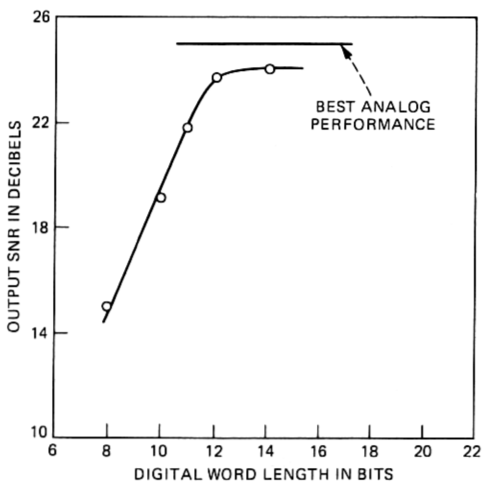

Fig. 5—Output s/n ratio vs digital word length (of equalizer tap weights) for a 64 $T/2$ tap equalizer operating on the severely distorted channel of Fig. 1, with step size = 0.0003 determined from (28).

that of an ideal infinite-resolution receiver, the number of taps, and the effective eigenvalue ratio of the channel-covariance matrix.

For data transmission at 9.6 kb/s, it can be concluded, on the basis of the representative voice-grade channels simulated in this study and neglecting the effects of limited tap-weight resolution on timing recovery and carrier phase tracking as well as the (relatively minor) degradations resulting from limiting the resolution of variables other than the tap weights, that a digital word length of 12 bits is adequate to represent the tap weights for updating purposes in a 32-tap synchronous equalizer or a 64-tap $T/2$ equalizer.

APPENDIX

Asymptotic Eigenvalue Distribution for the Correlation Matrix of Synchronous and Fractionally Spaced Equalizers

In this appendix, we describe the eigenvalues of the correlation matrix for infinitely long synchronous and fractionally spaced equalizers.

A.1 Synchronous equalizer

From the definition $A = \langle \mathbf{r}_n \mathbf{r}'_n \rangle$, we note that A is a Toeplitz matrix, and that the eigenvalue equation is given by

$$\sum_{\ell=-N}^{N} A_{k-\ell} p_\ell = \lambda p_k \qquad -N \le k \le N, \tag{36}$$

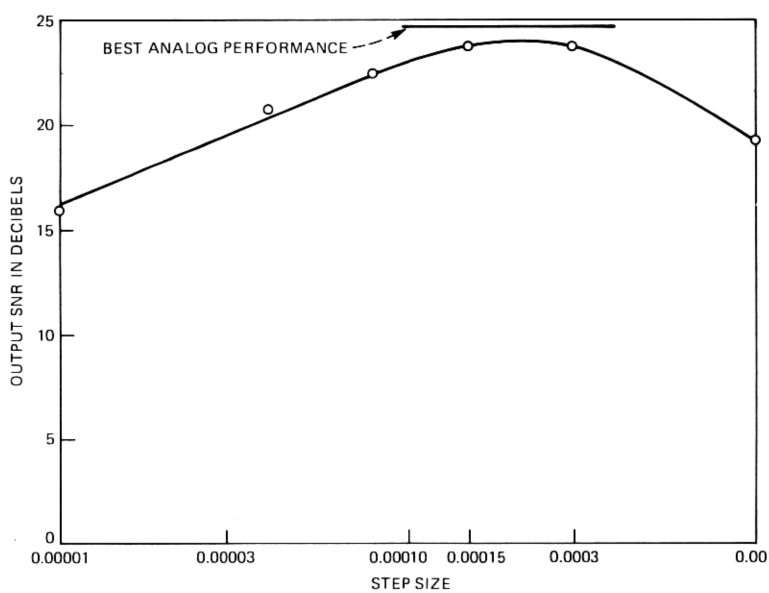

Fig. 6—Output s/n ratio vs adaptation step size for 64 $T/2$ tap equalizer, tap weights quantized to 12 bits, and severely distorted basic-conditioned channel.

where $A_{k-\ell}$ is the $k\ell$th element of A, λ is an eigenvalue, and $\mathbf{p}' = (p_{-N}, \cdots, p_0, \cdots, p_N)$ is the associated eigenvector. As $N \to \infty$, taking the Fourier transform of both sides of (36) yields

$$A(\omega)P(\omega) = \lambda P(\omega), \qquad |\omega| \le \frac{\pi}{T}, \tag{37}$$

where

$$A(\omega) = \left| \sum_k X\left(\omega + k\frac{2\pi}{T}\right) \right|^2 + \sigma^2 = |X_{eq}(\omega)|^2 + \sigma^2, \qquad |\omega| \le \frac{\pi}{T}. \tag{38}†$$

For very large N, the discrete asymptotic approximation to (37) is

$$A(\omega_i)P(\omega_i) = \lambda P(\omega_i), \qquad \omega_i = \frac{2\pi i}{(2N+1)T}, \qquad -N \le i \le N. \tag{39a}$$

Unless $A(\omega_i)$ has the same value for two or more values of the index i, the only way (39a) can be satisfied is for $P(\omega)$ to be concentrated at a single frequency, i.e., be a sinusoid. Repeated values of $A(\omega_i)$ correspond to repeated eigenvalues and an eigenvector subspace which can be spanned either by distinct sinusoids or by combinations of sinusoids. Then the solution to (37) is

$$\left.\begin{aligned} \lambda_i &= A(\omega_i) \\ P_i(\omega) &= \delta(\omega - \omega_i) \pm \delta(\omega + \omega_i) \end{aligned}\right\} \quad -N \le i \le N. \tag{39b}$$

Thus for a synchronous equalizer, the asymptotic ($N \to \infty$) eigenvalues uniformly sample the folded-channel-plus-noise spectrum, and the eigenvectors are the corresponding sinusoids.

A.2 Fractionally spaced equalizers

Here the channel-correlation matrix while symmetric is *not* Toeplitz; thus Fourier transform techniques do not yield the eigenvalues and eigenvectors in the above short order. For convenience, we consider the noiseless situation, and the eigenvalue equation is

$$\sum_{\ell=-N}^{N} A(kT', \ell T')p(\ell T') = \lambda p(kT') \qquad -N \le k \le N, \tag{40}$$

† The Nyquist-equivalent spectrum, $X_{eq}(\omega)$, is defined as $X_{eq}(\omega) = \sum_k X[\omega + k(2\pi/T]$, where the receiver sampling phase is incorporated into $X(\omega)$.

where the $k\ell$th element of A is given by

$$A(kT', \ell T') = \sum_m x(mT - kT')x(mT - \ell T'). \tag{41}$$

With $T' = T/2$, we write (40) for even and odd values of k,

$$\sum_{\substack{\ell \text{ even} \\ |\ell| \le N}} A\left(k\frac{T}{2}, \ell\frac{T}{2}\right)p\left(\ell\frac{T}{2}\right) + \sum_{\substack{\ell \text{ odd} \\ |\ell| \le N}} A\left(k\frac{T}{2}, \ell\frac{T}{2}\right)p\left(\ell\frac{T}{2}\right)$$

$$= \lambda p\left(k\frac{T}{2}\right), \quad k \text{ even} \tag{42}$$

$$\sum_{\substack{\ell \text{ even} \\ |\ell| \le N}} A\left(k\frac{T}{2}, \ell\frac{T}{2}\right)p\left(\ell\frac{T}{2}\right) + \sum_{\substack{\ell \text{ odd} \\ |\ell| \le N}} A\left(k\frac{T}{2}, \ell\frac{T}{2}\right)p\left(\ell\frac{T}{2}\right)$$

$$= \lambda p\left(k\frac{T}{2}\right), \quad k \text{ odd.} \tag{43}$$

Now (42) and (43) can be written as

$$\sum_{\ell=-N/2}^{N/2} A(kT, \ell T)p(\ell T) + \sum_{\ell=-N/2}^{N/2} A\left(kT, \ell T + \frac{T}{2}\right)p\left(\ell T + \frac{T}{2}\right)$$

$$= \lambda p(kT), \qquad -N/2 < k < N/2 \tag{44}$$

$$\sum_{\ell=-N/2}^{N/2} A\left(kT + \frac{T}{2}, \ell T\right)p(\ell T) + \sum_{\ell=-N/2}^{N/2}$$

$$\cdot A\left(kT + \frac{T}{2}, \ell T + \frac{T}{2}\right)p\left(\ell T + \frac{T}{2}\right) = \lambda p\left(kT + \frac{T}{2}\right), \tag{45}$$

respectively, where both equations hold for N integer values of k, and more importantly the various component matrices are now all Toeplitz.† If we consider the situation where $X(\omega)$ has less than 100 percent excess bandwidth,[8] then it is useful to introduce the four spectra

$$X_{\text{eq}}(\omega) \triangleq X(\omega) + X\left(\omega - \frac{2\pi}{T}\right) + X\left(\omega + \frac{2\pi}{T}\right)$$

$$\tilde{X}_{\text{eq}}(\omega) \triangleq X(\omega) - X\left(\omega - \frac{2\pi}{T}\right) - X\left(\omega + \frac{2\pi}{T}\right), \qquad |\omega| \le \frac{\pi}{T} \tag{46}$$

$$P_{\text{eq}}(\omega) \triangleq P(\omega) + P\left(\omega - \frac{2\pi}{T}\right) + P\left(\omega + \frac{2\pi}{T}\right)$$

† For example, $A(kT, \ell T + (T/2)) = \sum_m x(mT - kT)x(mT - \ell T + (T/2)) = \sum_n x(nT)x(nT + (k - \ell)T + (T/2))$.

$$\tilde{P}_{eq}(\omega) \triangleq P(\omega) - P\left(\omega - \frac{2\pi}{T}\right) - P\left(\omega + \frac{2\pi}{T}\right), \qquad |\omega| \leq \frac{\pi}{T}, \tag{47}$$

where we note that the discrete Fourier transform of the eigenvector $p(\ell(T/2))$, $P(\omega)$, is given by

$$P(\omega) \triangleq \sum_{\ell=-N}^{N} p\left(\ell \frac{T}{2}\right) e^{-j\omega_j \ell (T/2)}$$
$$= P_{eq}(\omega_j) + e^{-j\omega_j (T/2)} \tilde{P}_{eq}(\omega_j), \qquad \omega_j = \left(\frac{j}{N}\right)\left(\frac{\pi}{T}\right), -N \leq j \leq N. \tag{48}$$

Taking the *synchronous* Fourier transform (i.e., with respect to the T-sec sampling interval) of (44) and (45) gives as an approximation (exact as $N \to \infty$)

$$|X_{eq}(\omega_j)|^2 P_{eq}(\omega_j) + X_{eq}(\omega_j)\tilde{X}^*_{eq}(\omega_j)\tilde{P}_{eq}(\omega_j) = \lambda P_{eq}(\omega_j)$$
$$\omega_j = \left(\frac{2j}{N}\right)\left(\frac{\pi}{T}\right), -N/2 \leq j \leq N/2.$$
$$\tilde{X}_{eq}(\omega_j) X^*_{eq}(\omega_j) P_{eq}(\omega_j) + |\tilde{X}_{eq}(\omega_j)|^2 \tilde{P}_{eq}(\omega_j) = \lambda \tilde{P}_{eq}(\omega_j). \tag{49}$$

Note that $p(kT + (T/2))$ has the discrete Fourier transform exp $(-j\omega_j(T/2)\ \tilde{P}_{eq}(\omega_j))$, where the synchronous transform of $p(kT)$ is $P_{eq}(\omega_j)$. Arguing as we did for the synchronous equalizer, we see that the ith eigenvectors $P_i(\omega_j)$ and $\tilde{P}_i(\omega_j)$ must again be delta functions at $\omega_i = (2i/N)(\pi/T)$. Consequently, the eigenvalues, λ_i, must satisfy

$$\lambda_i^2 - \lambda_i[|X_{eq}(\omega_i)|^2 + |\tilde{X}_{eq}(\omega_i)|^2] = 0, \tag{50}$$

and thus the eigenvalues are

$$\lambda_i^{(1)} = 0$$
$$\lambda_i^{(2)} = |X_{eq}(\omega_i)|^2 + |\tilde{X}_{eq}(\omega_i)|^2 = \sum_k \left| X\left(\omega_i + \frac{k2\pi}{T}\right)\right|^2,$$
$$-\frac{N}{2} \leq i \leq \frac{N}{2}. \tag{51}$$

In contrast to (38), which applies to the synchronous equalizer, half the eigenvalues are exactly zero, while the other half are samples of the aliased magnitude-squared channel transfer function. Not surprisingly, the eigenvalues are independent of both the channel phase characteristics and the receiver sampling phase, and if $|X(\omega)|^2$ is Nyquist, then all the eigenvalues are unity. Since the eigenvalues are determined, we can now solve for the eigenvectors. Since $p(\ell T)$ has the transform $P_{eq}(\omega)$ and $p(\ell T + (T/2))$ has the transform $\tilde{P}_{eq}(\omega)\ e^{-j\omega(T/2)}$, the eigenvectors associated with the zero eigenvalue are constructed as

$$p_i\left(n\,\frac{T}{2}\right) = \begin{cases} \tilde{X}_{eq}(\omega_i)e^{j\omega_i n\,(T/2)}, & n \text{ even} \\ -X_{eq}(\omega_i)e^{j\omega_i n\,(T/2)}, & n \text{ odd,} \end{cases} \tag{52}$$

while the eigenvector associated with the nonzero eigenvalue is

$$p_i\left(n\,\frac{T}{2}\right) = \begin{cases} X_{eq}(\omega_i)e^{j\omega_i n\,(T/2)}, & n \text{ even} \\ \tilde{X}_{eq}(\omega_i)e^{j\omega_i n\,(T/2)}, & n \text{ odd.} \end{cases} \tag{53}$$

At this point, we remark that when ω_i is not in the rolloff region, then $X_{eq}(\omega_j) = \tilde{X}_{eq}(\omega_i)$, and (53) describes a sinusoid of frequency ω_i, since the even and odd portions of $p_i(n(T/2))$ mesh together in a continuous manner (i.e., $p_i(n(T/2)) = X_{eq}(\omega_i)e^{j\omega_i n\,(T/2)}$). However, (52) describes a function which changes sign and oscillates almost a full cycle in T seconds. Consequently, $p_i(n(T/2))$, as given by (52), will have most of its spectral energy concentrated near $1/T$ Hz. When ω_i is in the rolloff region, the frequency content of (52) and (53) will differ somewhat from the above extreme cases, but the general results will still be as above. Numerical evaluations have confirmed the above.

This completes our discussion concerning the nature of the eigenvalues and eigenvectors for a fractionally spaced equalizer.

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