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Three-Stage Multiconnection Networks Which Are Nonblocking in the Wide Sense

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A multiconnection network deals with the connections of pairs $\{(X, Y)\}$ *where* X *is a subset of the input terminals and* Y *is a subset of the output terminals. We study the conditions under which a three-stage Clos network is nonblocking for such connections. We show that the number of middle switches needed for nonblocking depends on the routing strategy. Therefore the networks satisfying the conditions are networks nonblocking in the wide sense. We also derive formulas for computing the minimum numbers of crosspoints required by such networks.*

I. INTRODUCTION

A *three-stage Clos network*, denoted by $\nu(m, n_1, r_1, n_2, r_2)$, consists of r_1 rectangular $(n_1 \times m)$ input switches, m rectangular $(r_1 \times r_2)$ middle switches and r_2 rectangular $(m \times n_2)$ output switches. There is exactly one link connecting each input switch to each middle switch and one link connecting each middle switch to each output switch. The $n_1 r_1$ inlets of the input switches are called *input terminals* and the $n_2 r_2$ outlets of the output switches are called *output terminals*.

Let I denote the set of input terminals and O the set of output terminals. A connecting pair in the classical sense is a pair (x, y): $x \in I$, $y \in O$ requesting to be connected. Masson and Jordan[1] generalized the definition of a connecting pair to be a pair (x, Y): $x \in I$, $Y \subseteq O$ such that x is to be connected to every output terminal in Y. This definition was further generalized in Ref. 2 so that a connecting pair is a pair (X, Y): $X \subseteq I$, $Y \subseteq O$ such that each terminal in X is to be connected to every terminal in Y. A network dealing with this type of connecting pairs is called a multiconnection network.[2] In practice, we often need only consider X and Y with limited cardinalities. Let $|S|$ denote the cardinality of a set S. Then in a (q_1, q_2) multiconnection network, we are only concerned with connecting pairs $(X, Y), |X| \leq q_1, |Y| \leq q_2$.

A multiconnection network is *nonblocking* if, regardless of what state the network is currently in, a connecting pair involving only idle terminals can always be connected by a subgraph of the network which is link-disjoint to all subgraphs connecting previous pairs. A multiconnection network is *nonblocking in the wide sense*, following Beneš' definition[3] for the classical single-connection case, if it is nonblocking when a particular routing (connection) strategy is followed. Practical networks which are nonblocking in the wide sense for classical assignments rarely exist. In this paper, we show that such networks exist for the multiconnection case.

II. FOUR ROUTING STRATEGIES

Suppose (X, Y) is the current pair to be connected. It is commonly assumed[1,2] that the rectangular switches have the fan-in, fan-out property, i.e., any subset of inlets can be connected simultaneously to any subset of outlets. Therefore, it suffices to consider the pair (X, Y) consisting of at most one terminal from each input switch and at most one terminal from each output switch. For, if we can connect one input (output) terminal to $Y(X)$, then all terminals in the same input (output) switch can be connected to $Y(X)$. Therefore we may assume $r_1 \geq q_1$ and $r_2 \geq q_2$ without loss of generality. We now discuss four possible routing strategies.

Strategy 1: Find $|X||Y|$ middle switches each connecting a distinct pair (x, y), $x \in X, y \in Y$.

Strategy 2: Find $|X|$ middle switches each connecting a distinct pair (x, Y), $x \in X$.

Strategy 3: Find $|Y|$ middle switches each connecting a distinct pair (X, y), $y \in Y$.

Strategy 4: Find one middle switch connecting the pair (X, Y).

We now compute the number of middle switches needed under each strategy so that the pair (X, Y) can always be connected. To avoid discussions of uninteresting modifications, we assume $r_1 \geq q_1 q_2 n_1$ and $r_2 \geq q_1 q_2 n_2$.

Theorem 1: *$\nu(m, n_1, r_1, n_2, r_2)$ is nonblocking as a (q_1, q_2) multiconnection network under Strategy 1 if and only if $m \geq q_2 n_1 + q_1 n_2 - 1$.*

Proof: Consider the connection of the pair (x, y), $x \in X, y \in Y$. The input switch which contains x can be already connected to at most $n_1 q_2 - 1$ distinct middle switches under Strategy 1. This is because there are only n_1 inlets in the switch and each inlet has at most q_2 connections except that x can have at most $q_2 - 1$ connections. Similarly, the output switch which contains y can be already connected to at most $n_2q_1 - 1$ distinct middle switches under Strategy 1. In the worst case, the $q_2n_1 - 1$ middle switches and the $q_1n_2 - 1$ middle switches are disjoint. However, if we have $(q_2n_1 - 1) + (q_1n_2 - 1) + 1$ middle switches, then one middle switch must be available to connect the pair (x, y). Since it is also clear that the worst case can happen, the "only if" part of Theorem 1 is also proved.

Theorem 2: *$\nu(m, n_1, r_1, n_2, r_2)$ is nonblocking as a (q_1, q_2) multiconnection network under Strategy 2 if and only if $m \geq n_1 + q_1q_2n_2 - q_2$.*

Proof: Consider the connection of the pair (x, Y). The input switch which contains x can be already connected to at most $n_1 - 1$ distinct middle switches under Strategy 2. Each output switch in Y can be already connected to at most $(n_2q_1 - 1)$ distinct middle switches under Strategy 2. Since $|Y| \leq q_2$, Theorem 2 follows from a worst-case argument similar to the one given in Theorem 1.

Theorem 3: *$\nu(m, n_1, r_1, n_2, r_2)$ is nonblocking as a (q_1, q_2) multiconnection network under Strategy 3 if and only if $m \geq q_1q_2n_1 + n_2 - q_1$.*

Proof: Analogous to the proof of Theorem 2.

Theorem 4: *$\nu(m, n_1, r_1, n_2, r_2)$ is nonblocking as a (q_1, q_2) multiconnection network under Strategy 4 if and only if $m \geq q_1n_1 + q_2n_2 - q_1 - q_2 + 1$.*

Proof: Consider the connection of the pair (X, Y). The input switches in X can be already connected to at most $|X|(n_1 - 1) \leq q_1(n_1 - 1)$ distinct middle switches under Strategy 4. Similarly, the output switches in Y can be already connected to at most $|Y|(n_2 - 1) \leq q_2(n_2 - 1)$ distinct middle switches. Theorem 4 follows immediately from a worst-case argument.

We note that, for $q_1 = q_2 = 1$, Theorems 1, 2, 3, and 4 are all reduced to the famous Clos Nonblocking Theorem.[4]

The existence of networks nonblocking in the wide sense can now be easily shown. For example, assume $q_2n_1 + q_1n_2 - 1 > m \geq q_1n_1 + q_2n_2 - q_1 - q_2 + 1$. Then the network is nonblocking under Strategy 4 but not necessarily nonblocking under any other strategy, for instance, Strategy 1.

III. COMPUTING THE NUMBERS OF CROSSPOINTS

For given numbers of input terminals and output terminals $N_1 = n_1r_1$, $N_2 = n_2r_2$, we would like to determine n_1, n_2 and m such that $\nu(m, n_1, r_1, n_2, r_2)$ is nonblocking in the wide sense for (q_1, q_2) multiconnection networks and has a minimum number of crosspoints. The optimal solutions for n_1, n_2 and m, of course, depend on which routing strategy we adopt. However, we will give a mathematical formulation general enough for all four cases.

Let Q be the number of crosspoints for (m, n_1, r_1, n_2, r_2). Then

$$Q = r_1 n_1 m + m r_1 r_2 + r_2 m n_2$$
$$= m\left(N_1 + \frac{N_1}{n_1}\frac{N_2}{n_2} + N_2\right).$$

Assume

$$m = u n_1 + v n_2 - w,$$

where u, v and w are nonnegative constants. Setting the first partial derivatives of Q with respect to n_1 and n_2 to zero, we obtain

$$\frac{\partial Q}{\partial n_1} = u\left(N_1 + \frac{N_1}{n_1}\frac{N_2}{n_2} + N_2\right) - (u n_1 + v n_2 - w)\frac{N_1 N_2}{n_1^2 n_2} = 0$$

$$\frac{\partial Q}{\partial n_2} = v\left(N_1 + \frac{N_1}{n_1}\frac{N_2}{n_2} + N_2\right) - (u n_1 + v n_2 - w)\frac{N_1 N_2}{n_1 n_2^2} = 0.$$

Solving for n_1 and n_2, we obtain

$$n_1 = n_2 v/u$$

and n_2 is the unique real root (easily verified by standard methods) of the cubic equation

$$v^2(N_1 + N_2)n_2^3 - uvN_1N_2n_2 + uwN_1N_2 = 0.$$

Let Q_i, $i = 1, 2, 3, 4$, denote the minimum Q under Strategy i, namely, m is replaced by $q_2n_1 + q_1n_2 - 1$, $n_1 + q_1q_2n_2 - q_1$, $q_1q_2n_1 + n_2 - q_2$ and $q_1n_1 + q_2n_2 - (q_1 + q_2 - 1)$, respectively. Then we will select Strategy j such that

$$Q = Q_j = \min_{i=1,2,3,4} Q_i.$$

For example, let $Q_1 = \min_{i=1,2,3,4} Q_i$. Approximating m by $q_2n_1 + q_1n_2$, we obtain the solution

$$n_1 = \sqrt{\frac{q_2N_1N_2}{q_1(N_1 + N_2)}}, \qquad n_2 = \sqrt{\frac{q_1N_1N_2}{q_2(N_1 + N_2)}}.$$

Substituting back in Q, we obtain

$$Q \cong \left(q_2\sqrt{\frac{q_1N_1N_2}{q_2(N_1 + N_2)}} + q_1\sqrt{\frac{q_2N_1N_2}{q_1(N_1 + N_2)}}\right)$$
$$\left.N_1 + \frac{N_1N_2}{\sqrt{\frac{q_1N_1N_2}{q_2(N_1 + N_2)}}\sqrt{\frac{q_2N_1N_2}{q_1(N_1 + N_2)}}} + N_2\right)$$
$$= 4\sqrt{q_1q_2N_1N_2(N_1 + N_2)}.$$

IV. SOME CONCLUDING REMARKS

Strategy 1 has been adopted in previous works on multiconnection networks[1,2] and Theorem 1 was proved in Ref. 2 with the special case $q_1 = 1$ and $q_2 = r_2$ first proved in Ref. 1. A $(1, r_2)$ multiconnection network under Strategy 1 was called an expansion network by Masson.[5] In Ref. 6, Masson considered $(1, r_2)$ multiconnection networks under a routing strategy which is a weakened version of Strategy 2, namely to use Strategy 2 whenever possible. Under this strategy, he stated the result that $(n_1, n_1, r_1, n_2, r_2)$ is nonblocking if $r_2 \leq (2n_1/n_2)$ where $[x]$ is the smallest integer not exceeding x. However, the following example shows that this result is incorrect. Consider a network $\nu(3, 3, 2, 2, 3)$. Then

$$r_2 = \left[\frac{2n_2}{n_1} \right] = 3$$

satisfying the condition of Masson's result. However, $\nu(3, 3, 2, 2, 3)$ is not nonblocking even as a classical single connection network, since it does not satisfy the necessary and sufficient condition $m \geq n_1 + n_2 - 1$ of the Clos Nonblocking Theorem.

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