THE BELL SYSTEM TECHNICAL JOURNAL

DEVOTED TO THE SCIENTIFIC AND ENGINEERING
ASPECTS OF ELECTRICAL COMMUNICATION

Volume 58 | December 1979 | Number 10

The Packing Problem for Twisted Pairs

By E. N. GILBERT

(Manuscript received June 26, 1979)

When wires are packed together in a bundle, as in a cable or on a shelf of a main distribution frame, the packing fraction f is the fraction of cross-sectional area of the bundle occupied by wire. With wires all the same radius, packing fractions as high as 0.90690 can be achieved. However, when the wires are pairs that have been twisted to avoid crosstalk, the packing fraction is much smaller. The largest obtainable packing fraction depends on other properties of the packing. For example, with pairs twisted by machine, all pairs twist at the same rate, and that influences the packing fraction. Several packing problems are considered, but most attention is given to a particularly regular kind of packing in which pairs twist about straight parallel axes located in a lattice arrangement. The densest lattice packing has packing fraction 0.56767. The densest lattice is a complicated one in which each wire touches 10 wires belonging to 6 other pairs. The numbers 10 and 6 cannot be increased even with nonlattice packings of pairs with straight parallel axes. These other packings are also conjectured to have packing fractions less than 0.56767, although only $f < 0.62240$ is proved.

I. INTRODUCTION

Pairs of telephone wires are often packed closely together in large numbers. These wires may belong to a cable or lie together on a shelf as jumper wires of a main distribution frame. To avoid inductive coupling, which produces crosstalk, the wire pairs are always twisted.

A twisted pair packing problem arose with a proposal for monitoring the accumulation of inoperative jumper pairs on a main distribution frame. Robert Graham of Western Electric has developed a technique for measuring the cross-sectional area of a bundle of jumper wires. Telephone company records determine the number of working jumper pairs in the bundle. The total number of pairs, working or inoperative, could be estimated from the measured area if the density of pairs in the bundle were known.

If each wire has radius r, a twisted pair has cross-sectional area $A_P = 2\pi r^2$. The number N of pairs in a bundle of area A_B is then

$$N = fA_B/A_P, \tag{1}$$

where f is the *packing fraction* (or *density*) of the bundle, the fraction of cross-sectional area filled by wire. Graham's measurements on spools of twisted pair wire suggest a value of f near 0.5. That is a much smaller packing fraction than could be achieved with single wires or untwisted pairs. To show that twisting the pairs reduces the packing fraction, this paper looks for packings that are as dense as possible. The problem takes several forms, depending on what regularities the packing may be assumed to have. For instance, do the pairs all twist around parallel, straight-line axes? If so, do they all twist at the same rate (in turns per foot)? The most regular packings are the "lattice" packings described in Section IV. The densest lattice packing will be found to have $f = 0.56767$.

The same mathematical problems arise in a different setting as follows. Suppose each dancer on a ballroom floor occupies a circular region. Dancing partners form pairs of circles in contact, and each pair rotates as the dance progresses. How densely can the floor be packed without causing couples to collide with one another?

II. PACKINGS

The pairs on the main frame are randomly packed, but there are several reasons for studying deterministic packings that maximize the fraction f. One reason is that the simplest mathematical models of random packing[1] produce low packing fractions. A more complicated random model will necessarily use some other special random process. But the random process that truly describes the main frame packing is not well understood; no special random model can be trusted completely. A packing that maximizes f, although special, has the virtue of giving a firm bound on the packing fraction actually achieved.

Another argument for maximizing f recognizes the tendency of gravity forces to pack the wires tightly. Indeed, the gravitational potential energy of a bundle of wires is minimized when the wires are packed as densely as possible. Of course, the bundle will usually have a different gravitationally stable configuration, but each time the bundle is disturbed, it tends to assume a new configuration of lower potential energy. This phenomenon may be illustrated by filling a jar with beans by pouring them in gently; then, shaking the jar will settle the beans and make room for more. Routine main frame maintenance includes "feathering" the wires, which probably helps to make the packing more dense.

Wires are assumed to be so nearly parallel to one another that they all appear as circles in any plane transverse section through the bundle. The two wires of one pair will always be represented as circles that touch. Figure 1 shows the well-known densest packing of circles in the plane.[2–4] The circles in this packing occupy a fraction $f = \pi/12^{1/2} = 0.90690$ of the plane. The circles within each horizontal row in Fig. 1 can be grouped into pairs of circles that touch. Thus, Fig. 1 can represent one cross section through a bundle of pairs of wires if the pairs are not twisted.

Arrangements like Fig. 1 often appear when circular disks are squeezed together on a flat tray. For twisted pairs, Fig. 1 would be very unlikely. The pairs must somehow twist so that they do not penetrate one another in moving from Fig. 1 to other cross sections farther along the wires.

Strictly speaking, it is possible to achieve $f = 0.90690$ in all cross sections, even with twisted pairs. Let Fig. 1 rotate bodily about some fixed center 0 to represent other cross sections. One full rotation of Fig. 1 then gives each pair one full twist. Of course, each twisted pair then forms a helix spiraling around an axis through 0, and so the pairs intertwine each other. Indeed, this intertwining cancels the crosstalk reduction that twisting the pairs tried to achieve.

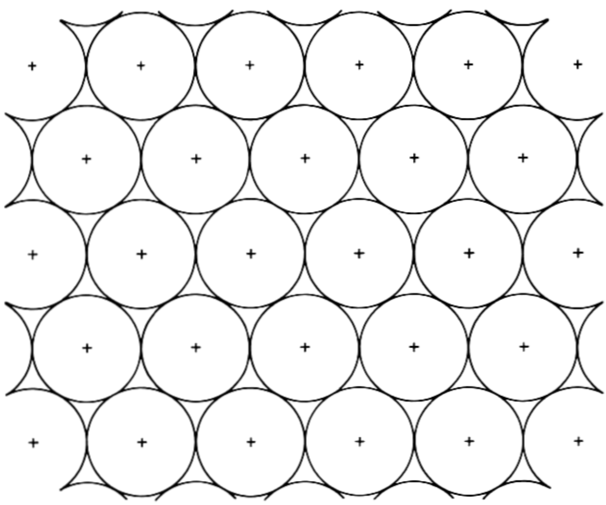

Fig. 1—The densest packing of nonoverlapping circles in the plane, $f = 0.90690$.

Very little intertwining occurs between different jumper pairs on the main frame. One way to prevent intertwining is to assume that the pairs twist about parallel straight-line axes. In a cross section, each axis appears as the point where the two circles of the pair touch. Requiring twisted pairs to have straight axes is a severe restriction. For example, it rules out Fig. 1 as a possible cross section; any rotations of pairs about the axes of Fig. 1 will cause some wires to intersect.

After assuming straight axes, one must make further assumptions about how pairs twist. One possibility is that all pairs twist at the same rate, in turns per foot. That assumption is reasonable if the pairs are cut from a reel of wire that has been twisted automatically by machine. Packings of wires that twist at the same rate will be considered in later sections. An opposite extreme is to assume that different pairs twist at rates that are not only unequal but incommensurable. Under that assumption, no two pairs can have axes lying within distance $4r$ of each other because two pairs with closer axes would overlap in some cross section. When axes are separated by at least $4r$, circles of radius $2r$ and centered at the axes do not intersect. The densest packing of such circles is again Fig. 1, now with circles of radius $2r$. In Fig. 2, these are the larger circles. Each contains two circles of radius r which

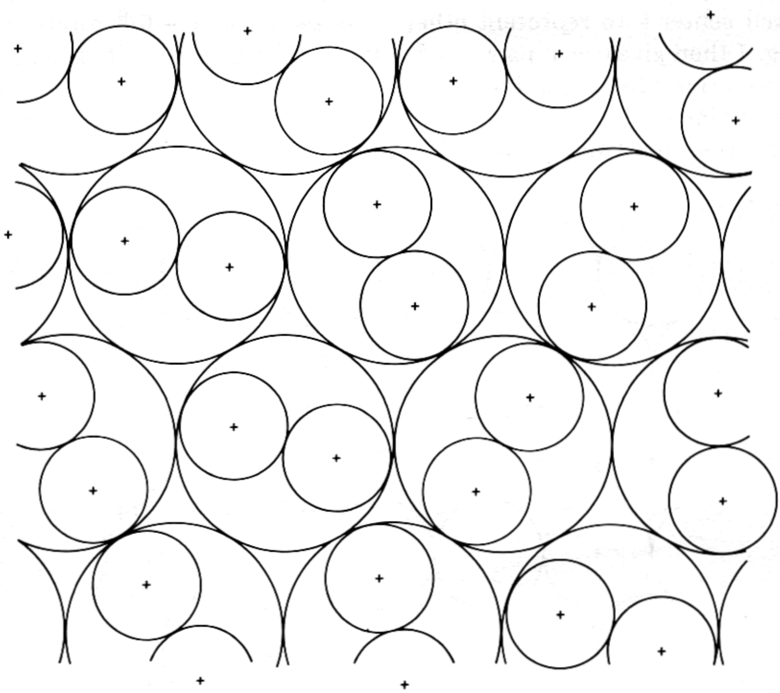

Fig. 2—The densest packing of twisted pairs with straight parallel axes and incommensurable twisting rates, $f = 0.45345$.

represent individual wires. The pair of circles of radius r fills exactly half the area of a circle of radius $2r$. Then Fig. 2 is exactly half as dense as Fig. 1; $f = \pi/(48)^{1/2} = 0.45345$ is the greatest packing fraction obtainable with incommensurable twist rates and straight parallel axes.

In this paper, pairs are always assumed to twist in the same sense, say, as a right-handed screw. Packings with pairs twisting in both senses can achieve other packing fractions. One can achieve $f = \pi/4 = 0.78540$ with half the pairs twisting in a right-handed sense and half twisting at the same rate in a left-handed sense.

III. EQUAL TWIST RATES

In a given cross section, each pair can be assigned a *phase angle* θ measured from the horizontal to a line between centers of the two wires. Figure 2 shows pairs of different phases. If all pairs twist at the same rate, the difference in phase between two pairs remains constant as one moves along the wires. It is no longer necessary to separate pair axes by distance $4r$. Theorem 1 below shows that the minimum allowed distance depends on the phase difference ϕ between the two pairs.

Figure 3 shows two pairs with axes at distance a apart. The constant phase difference for the two pairs is ϕ; one pair has a phase θ and the other has phase $\theta + \phi$.

Theorem 1: Suppose two pairs, with phase difference ϕ as shown in Fig. 3, twist about their straight parallel axes at the same rate. The smallest distance achieved between centers of wires in different pairs is $a - 2rM(\phi)$, where

$$M(\phi) = Max\{|\sin \tfrac{1}{2}\phi|, |\cos \tfrac{1}{2}\phi|\},$$

r is the radius of the wires, and a is the distance between the axes of the pairs.

Proof: The theorem is proved simply if Fig. 3 is regarded as the complex plane. Take the origin to be one pair axis. The other pair axis is at $a \exp(i\psi)$, where ψ is an angle from the horizontal to the line between axes. The two wires of the first pair have centers P_+ and P_- with

$$P_\pm = \pm r \exp(i\theta).$$

The centers Q_+ and Q_- of wires of the second pair are

$$Q_\pm = a \exp(i\psi) \pm r \exp\{i(\theta + \phi)\}.$$

One of the four distances to consider is $|Q_+ - P_+| = |a \exp(i\psi) + r \exp i\theta(\exp i\phi - 1)|$. Write

$$\xi = \exp\{i(\theta - \psi + \tfrac{1}{2}\phi + \tfrac{1}{2}\pi)\}$$

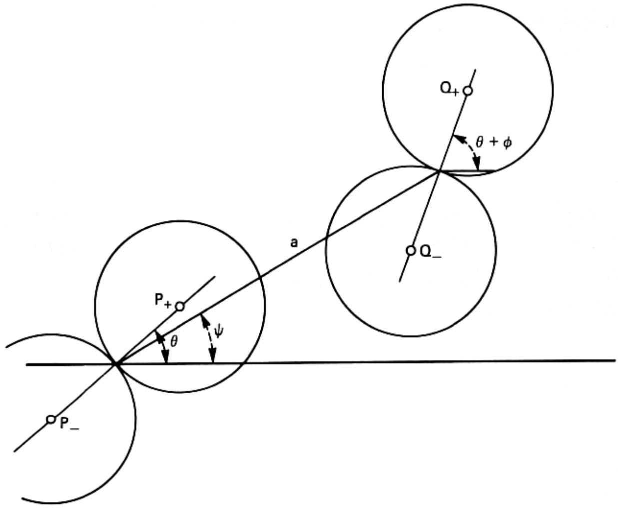

Fig. 3—Two twisted pairs with phase difference ϕ.

so that

$$|Q_+ - P_+| = |a + 2r\xi \sin \tfrac{1}{2}\phi|, \quad (2)$$

and likewise

$$|Q_- - P_+| = |a - 2r\xi \cos \tfrac{1}{2}\phi|, \quad (3)$$

$$|Q_+ - P_-| = |a + 2r\xi \cos \tfrac{1}{2}\phi|, \quad (4)$$

$$|Q_- - P_-| = |a - 2r\xi \sin \tfrac{1}{2}\phi|. \quad (5)$$

As θ varies, ξ moves on the unit circle $|\xi| = 1$. The four distances (2), $\cdots$, (5) have their maxima and minima at $\xi = \pm 1$. The smallest value attained by any of the four distances is either $a - 2r|\sin \tfrac{1}{2}\phi|$ or $a - 2r|\cos \tfrac{1}{2}\phi|$, as the theorem states.

Theorem 1 shows that the distance a between axes of two pairs with given phase difference ϕ can be only as small as

$$a = a(\phi) = 2r\{1 + M(\phi)\}. \quad (6)$$

This separation can be less than the $4r$ used in Fig. 2. The smallest allowed separation is obtained with $\phi = \pm 90°$;

$$a(\pm 90°) = (2 \pm 2^{1/2})r = 3.4142r.$$

For a given wire, say the one with center P_+, the closest approach to another wire center Q_+ or Q_- occurs when $\xi = \pm 1$, i.e., when

$$\theta - \psi + \tfrac{1}{2}\phi = 90° \text{ or } 270°.$$

Ordinarily $|Q_+ - P_+|$ and $|Q_- - P_+|$ have different minima and then the minimum distance $a - 2rM(\phi)$ is attained at only one of the two angles θ. If $a = a(\phi)$, each wire at 0 touches only one wire of the other pair. If $\phi = \pm 90°$, $|\sin \tfrac{1}{2}\phi| = |\cos \tfrac{1}{2}\phi|$ and $|Q_+ - P_+|$ has the same minimum value as $|Q_- - P_+|$. If $a = a(\phi) = a(90°)$, each wire at 0 touches both wires of the other pair.

Corollary: If pairs have straight parallel axes and twist at the same rate, the packing fraction cannot exceed

$$2\pi/\{27^{1/2} + 24^{1/2}\} = 0.62240.$$

Proof: Let $2R$ denote $(2 + 2^{1/2})r$. In any packing of twisted pairs, the distance between two pair axes is at least $2R$. Then circles of radius R, centered at the pair axes of a packing, do not overlap. The number ρ of circles per unit area, in any packing of nonoverlapping circles of radius R, satisfies

$$\rho \leq \rho_0 = 1/(12^{1/2}R^2).$$

The maximum ρ_0 would be attained with Fig. 1 again, now with circles of radius R. Each circle of radius R represents one twisted pair of area $2\pi r.^2$ Then the packing fraction is $f = 2\pi r^2\rho \leq 2\pi r^2\rho_0$. The bound simplifies to the number stated by the corollary.

One might try to achieve density 0.62240 by arranging pair axes in the same pattern as the centers in Fig. 1. Each pair would then be required to differ in phase by ±90° from each neighbor at distance $2R$. But that arrangement contains triples of pairs, each pair a neighbor of the other two. There is no way to assign phases to the pairs of such a triple. Packing fractions near 0.62240 are probably not obtainable. The more special packings in the next section have maximum packing fraction 0.56767.

IV. LATTICES

A *point lattice* is a discrete set of points forming a group under vector addition. Thus a point lattice must contain the origin 0 and the sum $P \pm Q$ of each pair of lattice points P, Q. Two-dimensional point lattices, the ones of interest here, can all be generated from pairs u, v of linearly independent vectors. Lattice points are then linear combinations

$$P_{ij} = iu \pm jv \tag{7}$$

of the generator vectors u, v, the coefficients i, j ranging over all integers. For example, circle centers in Fig. 1 form a point lattice with generators u, v both of length $2r$ and 60° apart.

A point lattice in a plane π can be used to construct a *lattice of twisted pairs*. Arrange pairs, all twisting at the same rate and having parallel straight axes normal to π, with a pair axis passing through each point of the point lattice. Let $\theta(P)$ denote the phase of the pair with axis at point P of π. The phases will be required to satisfy

$$\theta(P + Q) = \theta(P) + \theta(Q).$$

Then $\theta(0) = 0$ and all phases can be expressed in terms of two parameters $\sigma = \theta(u)$, $\tau = \theta(v)$. At P_{ij} in (7), the phase $\theta_{ij} = \theta(P_{ij})$ is

$$\theta_{ij} = \theta(iu) + \theta(jv) = i\sigma + j\tau. \quad (8)$$

In a point lattice, each point P is like every other point Q. Adding $P - Q$ to every point merely translates the point lattice rigidly onto itself and carries Q to P. The lattice of twisted pairs has symmetries which are only slightly more complicated. The translation that carries Q to P need not leave the lattice of twisted pairs fixed because the phases $\theta(P)$ and $\theta(Q)$ may differ. However, (8) shows that the lattice of twisted pairs regains its original appearance if this translation is followed by turning every pair through a constant angle $\theta(P) - \theta(Q)$. With pairs that all twist at the same rate, rotating pairs through a fixed angle is equivalent to taking a different cross section through the wires. Thus the twisted pairs do have translation symmetries, although the translations have axial components.

A point lattice determines a tessellation of the plane into congruent parallelogram cells (for a detailed explanation, see Ref. 4). Each lattice point P determines a parallelogram cell with vertices P, $P + u$, $P + v$, $P + u + v$. A cell has area

$$A = |u|\,|v|\,|\sin \alpha|, \quad (9)$$

where α is the angle between u and v. The lattice points have density $\rho = 1/A$ points per unit area. Then a lattice of twisted pairs has packing fraction

$$f = 2\pi r^2/A. \quad (10)$$

To find a lattice of twisted pairs that maximizes f, one must find parameters $|u|$, $|v|$, α, σ, τ that minimize A. These parameters are allowed only values that keep wires from intersecting. For each pair P, P' of lattice points, with phases θ, θ', the separation $|P - P'|$ must be at least $a(\theta - \theta')$ as given by (6). That optimization problem has the following solution.

Theorem 2: The maximum packing fraction of lattices of twisted pairs is

$$f = \tfrac{1}{2}\pi/(2 + 32^{1/2})^{1/2} = 0.56766836 \cdots .$$

It is obtained with $\sigma = 0°$, $\tau = 90°$, $|u| = 4r$, $|v| = |v - u| = (2 + 2^{1/2})r$, *and* $\cos \alpha = 1/(1 + 2^{-1/2})$, *i.e.*, $\alpha = 54.14143°$.

The proof is long and will be deferred to the appendix. Figure 4 shows one cross section through the maximizing lattice of twisted pairs given by Theorem 2. Figure 4 also shows the parallelogram cells, determined by the generator vectors u and v. The vertices of parallelograms form the point lattice, representing the axes of the twisted pairs. The generators used in Fig. 4 are $u = (4r, 0)$, $v = (2, (2 + 32^{1/2})^{1/2})r = (2r, 2.7671r)$. The cell area is $A = 4(2 + 32^{1/2})^{1/2}r^2 = 11.06841r^2$.

In Fig. 4, certain wires touch. These contacts occur at midpoints of half the horizontal sides of cells. At two other places in the interior of each cell, wires almost touch. The very short gap between these wires is not apparent in a small drawing.

One of the parallelograms is shaded. Figure 5 shows what happens in this shaded cell as pairs rotate. The rotation angles, 9.141°, 80.859°, 90°, etc., were chosen to show contacts that occur between wires.

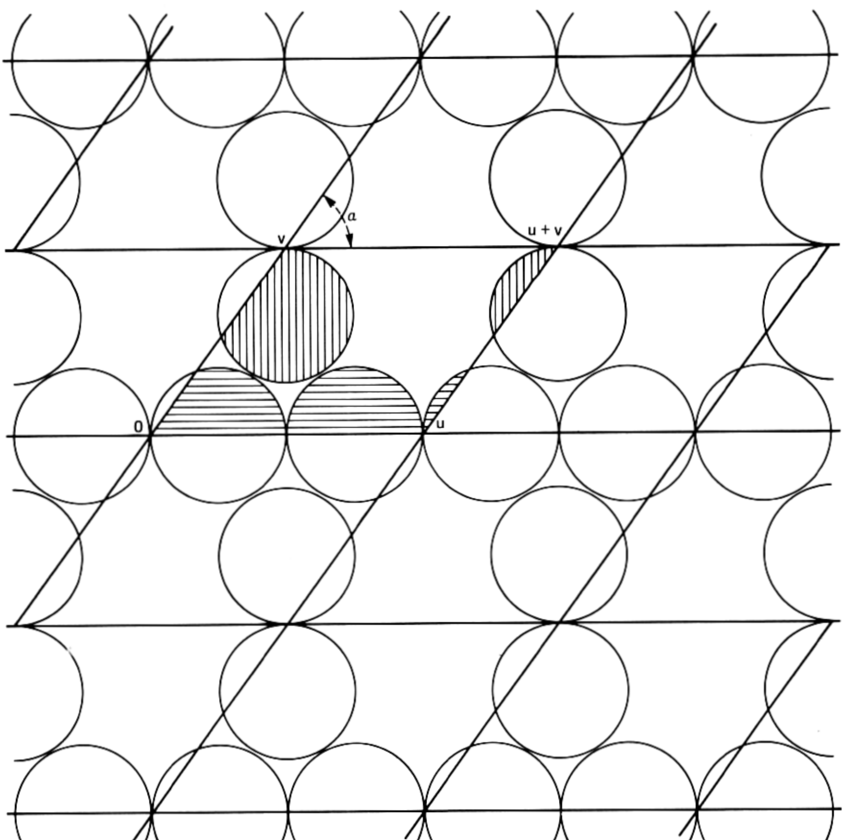

Fig. 4—One cross section through the densest lattice of twisted pairs, $\theta = 0°$.

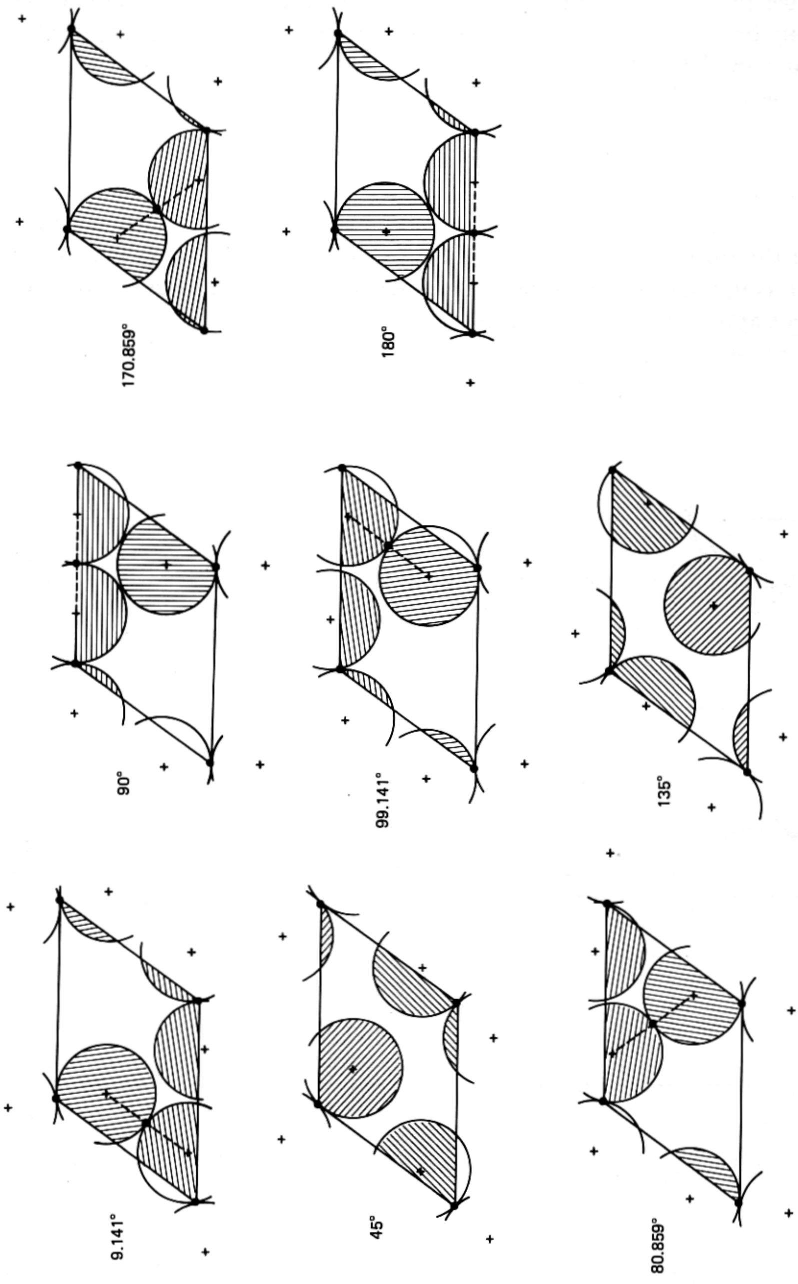

Fig. 5—History of a typical cell of the densest lattice of twisted pairs as θ increases.

Again there are some near misses (at 90°) and so Fig. 5 marks each point of true contact by a dot. A 180° rotation interchanges the wires in each pair and so Fig. 5 does not go beyond 180° rotation. Because of the lattice symmetry, each cell of the lattice goes through the same cycle as any other cell, but perhaps with different phase. In this special lattice, half the cells are in phase with the shaded cell and the rest are 90° out of phase.

Another view of the packing follows a single wire as it makes one full turn about its pair axis. Figure 6 shows the successive positions a, b, c, $\cdots$, j of the center of one wire as it comes into contact with other wires. Thus, a, b, c, $\cdots$ lie on a circle of radius r at the angles 9.141°, 80.859°, 99.141°, etc. The wire itself is not drawn but the wires it touches appear in their positions at contact. Each point a, b, c, $\cdots$ is connected by a line to the center of the contacted circle. The second wire of the chosen pair makes another contact whenever the first wire does but from a point 180° away. Thus the second wire goes through the cycle f, g, h, i, j, a, $\cdots$, e.

The contacts shown in Figs. 4, 5, and 6 occur because each pair has neighboring pairs at exactly the minimum allowed distance (6). In particular, $|u| = a(0°)$, $|v| = a(90°)$, $|v - u| = a(90°)$ and so the pair at P makes contact with the six pairs at $P \pm u$, $P \pm v$, and $P \pm (v - u)$. The pairs at $P \pm v$ and $P \pm (v - u)$ differ in phase from the one at P by 90°. Thus, as mentioned following eq. (6), a wire at P will touch

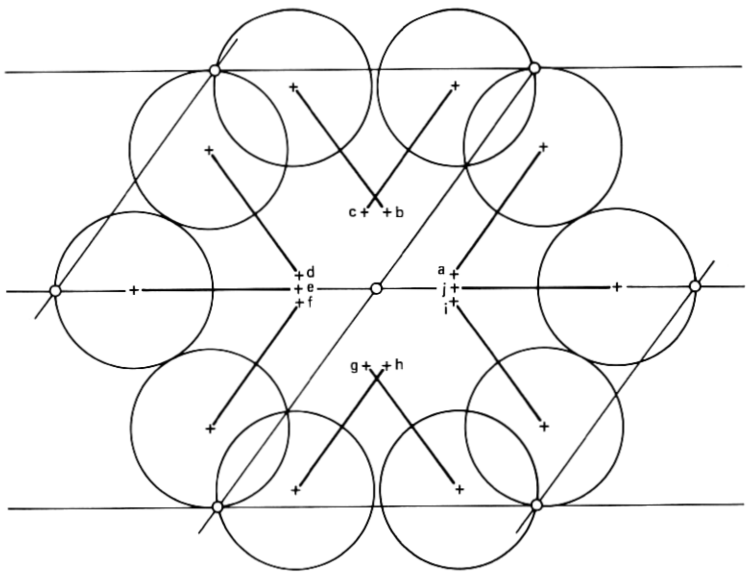

Fig. 6—The wires touched by a given wire of the densest lattice packing during one complete turn. Circular dots mark the pair axes.

both wires of pairs at $P \pm v$ and $P \pm (v - u)$. That accounts for the 8 contacts at positions a, c; b, d; f, h; g, i. At $P \pm u$, with phase difference 0°, only one wire is touched (at j and e). Thus the wire in Fig. 6 touches 10 wires belonging to 6 pairs.

The fact that each wire touches 10 wires belonging to 6 other pairs is an indication that the packing is very tight. The appendix proves the following theorem.

Theorem 3: Suppose all pairs have straight parallel axes and twist at the same rate. Then no wire can touch more than 6 other pairs nor more than 10 wires belonging to other pairs.

Note that the hypotheses of the theorem apply to packings more general than lattice packings.

Three twisted pairs will be said to form a *triplet* if each pair touches the other two. Pairs with axes P_1, P_2, P_3, and phases θ_1, θ_2, θ_3 form a triplet if $|P_i - P_j| = a(\theta_i - \theta_j)$ for the 3 choices of distinct subscripts i, j. One might expect many triplets in a dense packing, lattice or otherwise. Theorem 3 shows that no pair can belong to more than 6 triplets; the lattice packing of Theorem 2 achieves that number. In fact, each parallelogram cell in Fig. 4 is formed from two triangles, having vertices at axes of a triplet. Thus triplet triangles cover the entire plane of Fig. 4. Moreover, these triplet triangles have the least area possible.

Theorem 4: Suppose three twisted pairs with parallel straight axes and the same twisting rate form a triplet. The triangle with vertices at the axes of the three pairs has area at least

$$(2 + 32^{1/2})^{1/2} r^2 = 5.53420 r^2.$$

This minimum area is achieved if the three phase differences are 0°, ±90°, *and* ±90°.

This theorem is proved in the appendix.

V. CONCLUSION

Theorem 2 gives the packing fraction $f = 0.56767$ of the densest packing of twisted pairs having

(*i*) Pairs with parallel straight axes.
(*ii*) The same twisting rate for all pairs.
(*iii*) A lattice arrangement of pair axes and phases.

It may be that assumption (*iii*) can be dropped. Without assuming (*iii*), Theorems 3 and 4 show that pairs in the packing of Theorem 2 are as "close together" as possible in two senses that are not directly connected with f. But at present, without assuming (*iii*) one can only guarantee $f < 0.62240$ (Corollary to Theorem 1). To extend Theorem 2 without assuming (*iii*) may be difficult. Even for circles, the density maximizing property of Fig. 1 has only recently been proved without a lattice assumption.[3] For spheres in three dimensions, the astronomer J. Kepler conjectured that the face-centered cubic lattice packing is densest possible. Two centuries later, Gauss proved the conjecture for lattice packings, but there has been no proof without a lattice assumption.

Assumption (*i*) is too strong for most applications. It would be desirable to drop (*i*) and assume only that different pairs fail to intertwine. Relaxing (*i*) probably permits some increase in packing fraction. Figures 4 and 5 show that no pair ever touches more than two wires at a time in any cross section. Then, in any cross section, every pair is free to move slightly. If a bundle of pairs, packed as in Figs. 4 and 5, were surrounded by a cord and tied tightly, the axes would bend and the pairs would assume a denser packing in the plane of the cord. It should be possible to bend axes in Figs. 4 and 5 to obtain a packing fraction $f > 0.56767$ in all cross sections simultaneously.

In regard to assumption (*ii*), Section III mentioned a denser packing with pairs twisting in opposite senses, although still with the same absolute twist rates.

APPENDIX

Proofs of Theorems 2, 3, and 4

A1. Theorem 2, Part 1

Theorem 2 will be proved in two parts. The first part subjects a given lattice of twisted pairs to deformations that increase f and leave the lattice with generators u, v such that twisted pairs at 0, u, and v form a triplet. The second part of the proof is then also a proof of Theorem 4.

Many choices of u, v, σ, τ produce the same lattice of twisted pairs. For example, given one set of generators, another is obtained by changing v to $u + v$ and τ to $\sigma + \tau$. The pairs remain packed exactly as before although the point now called P_{ij} in (7) is the one formerly called $P_{i,i+j}$. This freedom to choose among different generators is used in the first part to obtain generators u, v with simplifying properties.

The deformations in the first part must be performed without causing wires to intersect. Because the pairs are symmetric to one another as explained in Section IV, it suffices to ensure that the pair at 0 never intersects a pair at any other point P. From (6) one obtains the requirement

$$|P| \geq 2r\{1 + M(\theta(P))\}. \tag{11}$$

Or, if $R(P)$ is defined to be the ratio

$$R(P) = |P|/\{2r[1 + M(\theta(P))]\},$$

the requirement is $R(P) \geq 1$ for all $P \neq 0$.

If a given lattice of twisted pairs has Min $R(P) = R_0 > 1$, then the lattice of P axes can be shrunk, moving each point P to P/R_0. Shrinking the lattice would increase f by a factor R_0^2. Hence a lattice of twisted pairs with maximum density must have points P such that $R(P) = 1$. One of these points will be taken for the generator vector u; $R(u) = 1$. This choice also determines $\sigma = \theta(u)$.

Figure 7 shows the point lattice of twisted pair axes. The points may be grouped in rows parallel to a central row of points $\cdots, -u, 0, u, 2u, \cdots$. These are horizontal rows in Fig. 4. Since $R(u) = 1$, $|u| \leq 2r\{1 + M(0)\} = 4r$. From (11), any point P, except the origin 0, satisfies $|P| \geq 2r\{1 + M(90°)\} = (2 + 2^{1/2})r$. Thus

$$|P| > 0.85\,|u|.$$

Similarly, any point P except ku satisfies $|P - ku| > 0.85\,|u|$. Then it follows that the distance between the horizontal rows of points is at least $0.68\,|u|$. Now start to compress Fig. 7 linearly in a vertical direction. The compression only increases the packing factor. The compression must stop before the vertical separation between rows

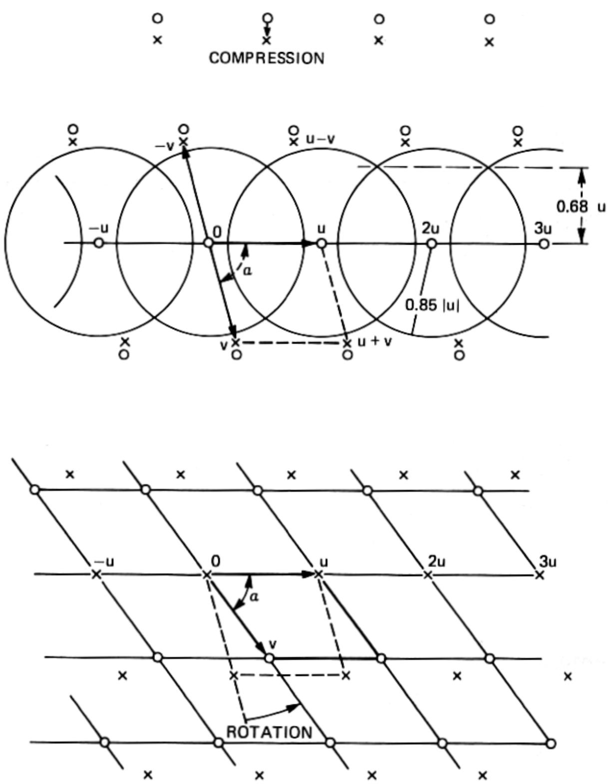

Fig. 7—Deformations of a lattice to increase packing fraction. Above: compression to produce a point v with $R(v) = 1$. Below: rotation of v to make $R(v - u) = 1$.

reaches $0.68|u|$ because, for some P, the ratio $R(P)$ will become equal to 1. A point P linearly independent of u and with $R(P) = 1$ can only lie in a row directly above or below the central row. For in a more remote row,

$$|P| > 2 \times 0.68|u| \geq 1.36(2 + 2^{1/2})r > 4r;$$

then $R(P) > 1$. A point P with $R(P) = 1$ and linearly independent of u will now be taken as the second generating vector v of the lattice. Because $R(-P) = R(P)$ there will be a choice between two vectors for v. Pick the vector that lies within 90° of u, as shown in Fig. 7; then $|\alpha| \leq 90°$ in (9). That determines $\tau = \theta(v)$.

Since $R(u) = R(v) = 1$, $|u|$ and $|v|$ are as small as (11) allows, for the given phases σ and τ. Another deformation of the lattice, changing α in (9) while holding $|u|$, $|v|$, σ, and τ fixed, may increase f. Because $|\alpha| \leq 90°$ the change must be in the direction of decreasing $|\alpha|$. The requirement $R(v-u) \geq 1$ sets a lower limit on the size of $|\alpha|$. Since $|u|$ and $|v|$ lie between $(2 + 2^{1/2})r$ and $4r$, $|v - u|$ would become smaller than $(2 - 2^{1/2})r$ and have $R(v-u) \leq (2-2^{1/2})/(2+2^{1/2}) < 1$ at $\alpha = 0$. Thus, with $R(u) = R(v) = 1$ and fixed σ and τ, A in (10) is always at least as large as the value given by (9) with $|\alpha|$ determined from the condition $R(v-u) = 1$. When $R(v-u) = R(u) = R(v) = 1$, twisted pairs at 0, u, v form a triplet and pairs at u, v, $u+v$ form another (recall the definition of triplet, given following Theorem 3). The two congruent triplet triangles (0, u, v) and (u, v, $u+v$) together constitute the parallelogram cell (0, u, v, $u+v$). Given σ and τ, (10) shows that f is no greater than πr^2 divided by the area of a triplet triangle 0, u, v with $R(u) = R(v) = R(v-u) = 1$; i.e.,

$$|u| = 2r\{1 + M(\sigma)\} \tag{12}$$

$$|v| = 2r\{1 + M(\tau)\} \tag{13}$$

$$|v-u| = 2r\{1 + M(\tau - \sigma)\}. \tag{14}$$

The first part of the proof of Theorem 2 is now finished. It remains to adjust σ and τ to minimize the area of the triplet triangle 0, u, v. That will lead to $\sigma = 0°$, $\tau = 90°$, and prove Theorem 4. Another detail to verify is that the lattice determined by (12), (13), and (14) with $\sigma = 0°$, $\tau = 90°$ actually has $R(P) \geq 1$ for all $P \neq 0$. Since $R(P) \leq 1$ only if $|P| \leq 4r$, there are only a few lattice points to examine. A short calculation shows $R(P) \geq 1$, with equality holding only for $P = \pm u$, $\pm v$, and $\pm(v-u)$. These six vectors locate the axes of the six twisted pairs mentioned in Theorem 3.

A2. Theorem 2, Part 2 and Theorem 4

The proof of Theorem 4 will use a formula, of Heron of Alexandria, for the area of a triangle with sides of given lengths a, b, c.[5,6] Here

Table I—Packing fraction f of lattices of twisted pairs

τ \ σ =	0°	10°	20°	30°	40°	50°	60°
0°	0.4534						
10°	0.4546						
20°	0.4581	0.4569					
30°	0.4640	0.4616					
40°	0.4725	0.4689	0.4676				
50°	0.4839	0.4788	0.4763				
60°	0.4985	0.4918	0.4878	0.4865			
70°	0.5168	0.5082	0.5025	0.4997			
80°	0.5396	0.5286	0.5210	0.5165	0.5150		
90°	0.5677	0.5540	0.5439	0.5373	0.5341		
100°			0.5418	0.5332	0.5281	0.5265	
110°					0.5262	0.5227	
120°							0.5209

$a = |v - u|$, $b = |u|$, $c = |v|$, which depend on σ, τ as in (12), (13), (14). Heron's formula converts the cell area A in (9) (twice the area of a triplet triangle) to

$$A = \{s(s-a)(s-b)(s-c)\}^{1/2}, \tag{15}$$

where $s = (a + b + c)/2$. Table I shows how the packing fraction f, obtained from (15) and (10), depends on σ and τ.

Table I shows only values of σ and τ in the range

$$0 \le 2\sigma \le \tau \le 90^\circ + \tfrac{1}{2}\sigma. \tag{16}$$

Values of f for other angles can be obtained by exploiting symmetries in formulas (12), (13), (14), (15). Write $(\sigma', \tau') \approx (\sigma, \tau)$ if substituting σ', τ' for σ, τ leaves the three lengths a, b, c unchanged, except perhaps for a permutation. For example, $(\sigma + 180^\circ, \tau) \approx (\sigma, \tau) \approx (\sigma, \tau + 180^\circ)$ because $M(\theta)$ is a function with period 180°. Then σ and τ can be assumed nonnegative. Second, $(\tau - \sigma, \tau) \approx (\sigma, \tau)$, and so one can assume $\sigma \le \tau - \sigma$, i.e., $2\sigma \le \tau$. Finally $M(\theta)$ has a symmetry $M(180^\circ - \theta) = M(\theta)$, and so $(\sigma, 180^\circ - \tau + \sigma) \approx (\sigma, \tau)$ and $(180^\circ - \tau, 180^\circ - \tau + \sigma) \approx (\sigma, \tau)$. It suffices to require $\tau \le 180^\circ - \tau + \sigma$ or $\tau \le 90^\circ + \tfrac{1}{2}\sigma$.

Table I indicates a maximum of f at $\sigma = 0^\circ$, $\tau = 90^\circ$. However, for the sake of mathematical exactness, an analytical proof follows.

First, note that A is an increasing function of a, b, and c. To prove this, differentiate A^2 with respect to these variables. For example,

$$(8A/a)\frac{\partial A}{\partial a} = 2bc - a^2 \ge 2(2 + 2^{1/2})^2 - 4^2 > 0. \tag{17}$$

It now follows that A cannot have a maximum in the part of the set (16) where $\tau < 90^\circ$. For, in that part $M(\tau) \approx \cos \tau/2$ and, because $\tau - \sigma < 90^\circ$, too, $M(\tau - \sigma) = \cos\{(\tau - \sigma)/2\}$. For fixed σ, $|v|$ and $|v - u|$ are decreasing functions of τ while $|u|$ remains constant. Then (17) shows that A is decreasing and hence f can have no local maximum with $0 \le \tau < 90^\circ$.

The remaining portion of the range (16), with $90° \leq \tau$, can be cut into three parts. These are

$$0 < \sigma \leq 45°, \qquad \tau = 90°, \tag{18}$$

$$0 < \sigma \leq 45°, \qquad 90° < \tau \leq 90° + \tfrac{1}{2}\sigma, \tag{19}$$

and

$$45° \leq \sigma, \qquad 2\sigma \leq \tau \leq 90° + \tfrac{1}{2}\sigma \tag{20}$$

[note that the second inequality of (20) actually implies $45° \leq \sigma \leq 60°$]. In all three parts,

$$M(\sigma) = \cos \sigma/2$$

$$M(\tau) = \sin \tau/2$$

$$M(\tau - \sigma) = \cos\{(\tau - \sigma)/2\}. \tag{21}$$

Consider the range (20) first. Since $45° \leq \sigma \leq 60°$, one has $90° \leq \tau \leq 120°$ and $45° \leq \sigma \leq \tau - \sigma \leq 90° - \frac{1}{2}\sigma \leq 67.5°$. Then

$$|u| \geq (2 + 2^{1/2})r = 3.41421r$$

$$|v| \geq (2 + 3^{1/2})r = 3.73205r$$

$$|v - u| \geq (2 + 2 \cos 33.75°)r = 3.66294r.$$

When these minimum lengths are substituted for a, b, c in Heron's formula, one obtains a lower bound $A > 11.19573r^2$ and hence $f < 0.56121$ throughout (20). Thus these parameters σ, τ can never minimize A nor give a packing fraction as high as $f = 0.567668$, which is obtained with $\sigma = 0°$, $\tau = 90°$.

Next consider (19). Those inequalities imply $0 < \sigma \leq 45°$ and $90° \leq \tau \leq 112.5°$ so that

$$M(\tau) \leq M(112.5°) = M(67.5°) < M(45°) \leq M(\sigma)$$

and

$$|v| < |u|. \tag{22}$$

To show that there is no local maximum of f with σ, τ satisfying (19), consider a small change from σ, τ to $\sigma + x$, $\tau + x$. Changing σ and τ by the same amount keeps $|v - u|$ constant but changes $|v|$ and $|u|$ in opposite directions. The effect on A is determined by differentiating. Equation (15) provides

$$8A \frac{dA}{dx} = |u| \, (2\,|v|\,|v - u| - |u|^2) \frac{d|u|}{dx} + |v| \, (2\,|u|\,|v - u| - |v|^2) \frac{d|v|}{dx}. \tag{23}$$

The derivatives of $|u|$ and $|v|$ are obtainable from (12), (13), and (21):

$$\frac{d|u|}{dx} = -r \sin \sigma/2 < 0, \qquad \frac{d|v|}{dx} = r \cos \tau/2 > 0.$$

Then the inequality (22) can be used to simplify (23) to a bound

$$(8A/r)\frac{dA}{dx} > |u|(2|v||v-u| - |v|^2)(-\sin \sigma/2)$$

$$+ |v|(2|u||v-u| - |u||v|) \cos \tau/2$$

$$= |u||v|(2|v-u| - |v|)(\cos \tau/2 - \sin \sigma/2).$$

The inequalities (19) imply $\sin \sigma/2 < \sin 22.5°$ and $\cos \tau/2 > \cos 56.25° = \sin 33.75° > \sin \sigma/2$. Also, $2|v-u| - |v| \geq 2(2 + 2^{1/2})r - 4r > 0$. Then $dA/dx > 0$ in the range (19) and there can be no local maximum there.

The proof so far has shown that f is too small in range (20) to achieve a maximum there and that, elsewhere with $\tau \neq 90°$, f increases if (σ, τ) moves toward the line $\tau = 90°$. One must consider (18) and show that $dA/d\sigma > 0$ with τ fixed at 90°. Then $0 < \sigma \leq 45° < \tau - \sigma \leq 90° = \tau$, which implies $|v| < |v-u| < |u|$ because

$$|u| = 2(1 + \cos \sigma/2)r$$

$$|v| = (2 + 2^{1/2})r$$

$$|v-u| = 2(1 + \cos\{(90° - \sigma)/2\})r.$$

Now a formula like (23) may be written for $dA/d\sigma$. The proof that $dA/d\sigma > 0$ is similar to the one given for the range (19), here using the inequality $|v-u| < |u|$. That completes the proof of Theorems 2 and 4.

A3. Theorem 3

To prove Theorem 3, consider a wire twisting about an axis at the origin 0. Let $P_1, P_2, \cdots, P_K$ denote axes of neighboring pairs that this wire touches. The names $P_1, P_2, \cdots, P_K$ may be assigned in order of increasing polar angle about 0. Magnitudes $|P_k|$ must satisfy (6) with ϕ the phase difference $\phi_k = \theta(P_k) - \theta(0)$. Thus all $|P_k|$ lie between $(2 + 2^{1/2})r$ and $4r$. Also $|P_k - P_j| \geq (2 + 2^{1/2})r$. Let t_k denote the number of times the wire at 0 touches wires of the pair at P_k. Then $t_k = 2$ if $|P_k| = (2 + 2^{1/2})r$ (i.e., if $\phi_k = \pm 90°$), but otherwise $t_k = 1$. The total number of contacts is

$$T = t_1 + t_2 + \cdots + t_K$$

and the theorem states $K \leq 6$ and $T \leq 10$.

Let α_k denote the angle $P_k 0 P_{k+1}$. The K angles $\alpha_1, \cdots, \alpha_K$ fall into three types.

Type 1: If $t_k = t_{k+1} = 2$, then $|P_k| = |P_{k+1}| = (2 + 2^{1/2})r$ and $\phi_k = \pm\phi_{k+1} = \pm 90°$. Then $|P_{k+1} - P_k| \geq a(0°) = 4r$ and $\alpha_k \geq 71.7°$ follows from the cosine law.

Type 2: If $t_k = 2$ but $t_{k+1} = 1$, then $|P_k| = (2 + 2^{1/2})r$, $|P_{k+1}| \leq 4r$ and $|P_{k+1} - P_k| \geq (2 + 2^{1/2})r$. Then $\alpha_k \geq 54.1°$. The same bound holds if $t_k = 1$ and $t_{k+1} = 2$.

Type 3: If $t_k = t_{k+1} = 1$ then $|P_k|$ and $|P_{k+1}|$ may be as large as $4r$ and $|P_{k+1} - P_k| \geq (2 + 2^{1/2})r$. Then $\alpha_k \geq 50.5°$.

Let N_1, N_2, N_3 be the numbers of angles α_k of types 1, 2, 3. Then

$$71.7°\ N_1 + 54.1°\ N_2 + 50.5°\ N_3 \leq 360°. \tag{24}$$

Moreover,

$$T = \tfrac{1}{2}\{(t_1 + t_2) + (t_2 + t_3) + \cdots + (t_K + t_1)\}$$

and each term $(t_k + t_{k+1})$ has value 4, 3, or 2 according to the type 1, 2, or 3 of α_k. Thus

$$T = \tfrac{1}{2}\{4N_1 + 3N_2 + 2N_3\}. \tag{25}$$

Subject to the constraint (24), nonnegative integers N_1, N_2, N_3 give T a maximum value $T = 10$. That proves half of the theorem.

The other half is more delicate because the constraint (23) allows $N_3 = 7$, $N_1 = N_2 = 0$, $K = N_1 + N_2 + N_3 = 7$. An improved bound on α_k for type 3 is needed to prove $K \leq 6$. The angle $\alpha_k = 50.5°$ is not actually achievable because it would require both ϕ_k and ϕ_{k+1} to be 0° or 180° while $\phi_{k+1} - \phi_k = \pm 90°$. For given ϕ_k and ϕ_{k+1}, the smallest α_k is obtained with $|P_k| = a(\phi_k)$, $|P_{k+1}| = a(\phi_{k+1})$ and $|P_{k+1} - P_k| = a(\phi_{k+1} - \phi_k)$. Then the cosine law determines α_k as a function of ϕ_k and ϕ_{k+1}. The minimum α_k is found to occur at $\phi_{k+1} = 135°$, $\phi_k = 45°$. The details will be omitted. In this way, one finds $52.67° \leq \alpha_k$, $K \leq [360°/52.67°] = 6$, and the theorem is proved.

In (25) there are actually two ways to make $T = 10$. The solution $N_1 = 2$, $N_2 = 4$, $N_3 = 0$ corresponds to Figs. 4 and 5. Another solution $N_1 = 5$, $N_2 = N_3 = 0$ can represent an isolated arrangement of five twisted pairs with phase 90°, surrounding a central pair with phase 0°. That configuration cannot occur as part of a lattice. A lattice would also contain a pair at $P_2 - P_1$ with phase 0°, but that pair would conflict with the one at P_3.

REFERENCES

1. E. N. Gilbert, "Randomly Packed and Solidly Packed Spheres," Canadian J. Math., *16* (1964), pp. 286–298.

2. L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und im Raum*, Berlin: Springer, 1953.
3. L. Fejes Tóth, *Regular Figures*, N.Y.: Macmillan (Pergamon), 1964.
4. D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination*, N.Y.: Chelsea, 1952.
5. Sir Thomas L. Heath, *A Manual of Greek Mathematics*, Oxford: Clarendon Press, 1931.
6. H. S. M. Coxeter, *Introduction to Geometry*, N.Y.: John Wiley, 1961.