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The Field of a Line Charge Near the Tip of a Dielectric Wedge

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We calculate the potential of a line charge embedded in a dielectric medium of permittivity ϵ_2 in the presence of a dielectric wedge of permittivity ϵ_1. The potential is calculated with the aid of the Mellin transform, and the answer is given as a definite integral which is then transformed into an infinite series. We show that, for all wedge angles and all ratios ϵ_2/ϵ_1, $\nabla\varphi$ is singular at the tip of the wedge, and we give the strength of the singularity. The results have relevance to the design of contacts on semiconductor devices.

I. INTRODUCTION

Lewis and Wasserstrom[1] have calculated the strength of the field singularity at the tip of a dielectric wedge in the configuration shown in Fig. 1. In particular, with a wedge permittivity ϵ_1 greater than the permittivity ϵ_2 of the surrounding medium and a conductor angle $\beta = \pi$ (the "overhanging electrode"), they found that the tip field was singular for all wedge angles α greater than $\pi/2$. From this analysis, it was concluded that semiconductor devices with undercut edges ($\alpha < \pi/2$) would be advantageous in reducing local field strength and thus preventing breakdown.

Because the analysis of Ref. 1 was strictly local, based on an expansion of the potential in positive powers of the distance from the wedge vertex, multiplied by trigonometric functions of the polar angle, it was felt by some that the results were suspect, since they were not based on the solution of a complete boundary value problem. Here we lay that suspicion to rest by presenting the solution of such a problem, namely the field due to a line charge near a dielectric wedge, as shown in Fig. 2. The solution of this problem, previously treated by Smythe[2] in a somewhat involved fashion, gives Green's function for the composite region. Here we use the Mellin transform, obtaining an expansion of the potential near the wedge tip in terms of the poles of the transform. Based on this analysis, we conclude for the charge-wedge configuration of Fig. 2 that, for *arbitrary* ratios ϵ_2/ϵ_1, the wedge tip field is singular for *all* values of the half-angle α. We show that,

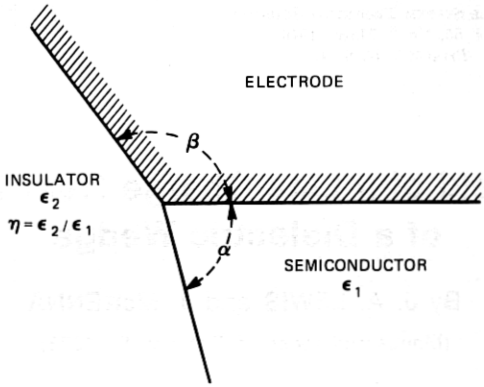

Fig. 1—Electrode, insulator, semiconductor configuration.

when the plane $y = 0$ is replaced by a perfectly conducting sheet, the field singularity due to the line charge is *exactly* as described by Lewis and Wasserstrom.[1] In general, we can conclude that, for any charge distribution for which the resulting potential is neither purely even nor purely odd, the field at the tip of the wedge will be singular for *all* ratios ϵ_2/ϵ_1 and *all* half-angles α.

II. THE PROBLEM

We consider the electrostatic potential due to a line charge of strength q in the presence of a dielectric wedge, as shown in Fig. 2. The charge lies at a distance a from the wedge tip in a dielectric medium with permittivity ϵ_2, while the wedge, with permittivity ϵ_1, occupies the region $-\alpha < \theta < \alpha$. We shall always assume that

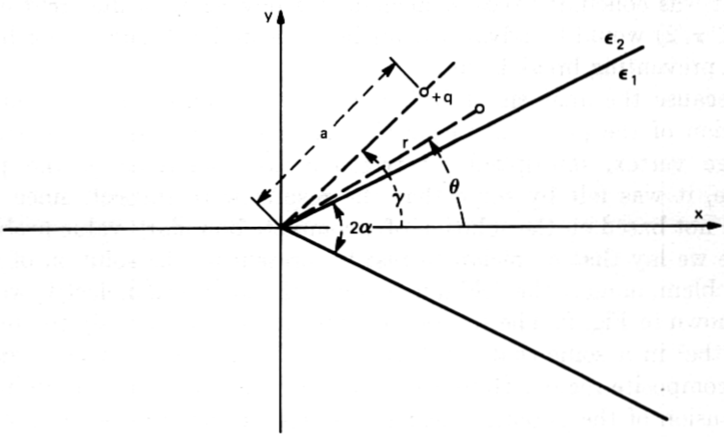

Fig. 2—The dielectric wedge and line charge.

$\gamma > \alpha$, taking into account the case where the charge lies within the dielectric wedge by interchanging ϵ_1 and ϵ_2, replacing α by $\pi - \alpha$ and γ by $\pi - \gamma$ where $\gamma < \alpha$. Finally, instead of working with the dimensional potential $\varphi(x, y)$ and distances (x, y), we introduce the dimensionless potential $u(r, \theta) = (\epsilon_2/q)\varphi(x, y)$, and the dimensionless distance $r = (x^2 + y^2)^{\frac{1}{2}}/a$. Thus, we will calculate the dimensionless potential due to a unit line charge at unit distance from the origin. Although we assume no trapped surface charge on the surface of the wedge, our analysis could be extended to cover this case also. It should be noted that, in these units, a unit line charge located at the origin of a homogeneous medium ($\epsilon_1 = \epsilon_2$) gives the potential

$$u = \frac{1}{2\pi} \ln \frac{1}{r}.$$

In the composite medium, u satisfies Laplace's equation

$$\nabla^2 u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0, \tag{1}$$

in the wedge $|\theta| < \alpha$, and the inhomogeneous equation

$$\nabla^2 u = -(2\pi r)^{-1}\delta(r - 1)\delta(\theta - \gamma), \tag{2}$$

where δ is the Dirac delta function, giving the effect of the charge at $(r, \theta) = (1, \gamma)$, for $\alpha < \theta < 2\pi - \alpha$. The problem is completed by the requirement that u and ϵu_θ be continuous across $\theta = \pm\alpha$.

To facilitate further calculations, we split u into the sum of an odd function in y and an even function in y, setting

$$u = \tfrac{1}{2}(v + w),$$

where v and w satisfy eqs. (1) and (2), the continuity conditions, and the boundary conditions

$$v(r, 0) = v(r, \pi) = w_\theta(r, 0) = w_\theta(r, \pi) = 0. \tag{3}$$

Obviously, the pair of problems for v and w are equivalent to the original problem for u. It should be noted, though, that v alone is the potential due to a positive unit line charge at $(1, \gamma)$ and a negative unit line charge at $(1, 2\pi - \gamma)$, in the presence of the dielectric wedge. Alternatively, of course, it can be interpreted as the potential of the unit line charge at $(1, \gamma)$ in the presence of the wedge, when the plane $y = 0$ is replaced by a perfectly conducting sheet. This corresponds to the model of the overhanging electrode used by Lewis and Wasserstrom.[1] Further, w alone is the potential due to positive unit line charges at $(1, \gamma)$ and $(1, 2\pi - \gamma)$ in the presence of the wedge.

We now proceed to calculate v and w, or rather their Mellin transforms, the form of eq. (2) having been chosen to facilitate the application of the transform.

III. THE MELLIN TRANSFORM

The Mellin transform $\bar{v}(\theta, s)$ of $v(r, \theta)$ is given by[3]

$$\bar{v}(\theta, s) = \int_0^\infty r^{s-1} v(r, \theta) dr. \tag{4}$$

If eq. (2) is multiplied by r^{s+1} and integrated from 0 to ∞, after several integrations by parts there results the ordinary differential equation

$$\bar{v}'' + s^2 \bar{v} = -\frac{1}{2\pi} \delta(\theta - \gamma), \tag{5}$$

provided that s lies in the strip $\sigma_1 < \text{Re}\, s < \sigma_2$, where

$$r^{s+1} v_r \to 0, \qquad r^s v \to 0 \tag{6}$$

for both $r \to 0$ and $r \to \infty$. These terms arise from the integration by parts of $r^{s+1}(v_{rr} + r^{-1} v_r)$. We will determine appropriate values of σ_1 and σ_2 later.

First, let us dispose of the singularity by calculating $\bar{v} = \bar{v}_1$ for a homogeneous medium for which $\eta = \epsilon_2/\epsilon_1 = 1$. Then $\bar{v}_1$ satisfies eq. (5) in $0 < \theta < \pi$ and the boundary conditions

$$\bar{v}_1(0, s) = \bar{v}_1(\pi, s) = 0.$$

The expression

$$\bar{v}_1 = A \sin s\theta - \frac{1}{s} \int_0^\theta \delta(\theta' - \gamma) \sin s(\theta - \theta') d\theta'$$

satisfies the equation and the first boundary condition. A is chosen to satisfy the secondary boundary condition. We finally obtain

$$\bar{v}_1(\theta, s) = \begin{cases} \sin s(\pi - \gamma) \sin s\theta / s \sin s\pi, & 0 < \theta < \gamma, \\ \sin s\gamma \sin s(\pi - \theta)/s \sin s\pi, & \gamma < \theta < \pi, \\ -\bar{v}_1(2\pi - \theta, s), & \pi < \theta < 2\pi. \end{cases} \tag{7}$$

Now in this case, $v_1(r, \theta)$ is known, and $v_1 \sim r$ for small r and $v_1 \sim 1/r$ for large r, so for (6) to be satisfied for v_1 it is necessary that $-1 < \text{Re}\, s < 1$.

An analogous calculation yields $\bar{w}$ in the homogeneous medium, viz,

$$\bar{w}_1 = \begin{cases} -\cos s(\pi - \gamma) \sin s\theta / s \sin s\pi, & 0 < \theta < \gamma, \\ -\cos s\gamma \cos s(\pi - \theta)/s \sin s\pi, & \gamma < \theta < \pi, \\ \bar{w}_1(2\pi - \theta, r), & \pi < \theta < 2\pi. \end{cases} \tag{8}$$

Again in this case, $w_1(r, \theta)$ is known, $w_1 \sim r$ for small r and $w_1 \sim \ln r$ for large r, so for (6) to be satisfied for w_1 it is necessary that $-1 < \text{Re}\, s < 0$.

We now use these expressions for the potentials due to a line charge in a homogeneous medium to obtain the potentials in the presence of the wedge. Note the way θ and γ are interchanged in eqs. (7) and (8) to make $\bar{v}_1$ and $\bar{w}_1$ continuous. We choose a similar form for $\bar{v}$, setting

$$\bar{v} = \bar{v}_1 + B \begin{cases} \sin s(\pi - \alpha) \sin s\theta, & \text{for} \quad 0 < \theta < \alpha \\ \sin s\alpha \sin s(\pi - \theta), & \text{for} \quad \alpha < \theta < \pi, \end{cases}$$

thus satisfying the differential equations, the boundary conditions at $\theta = 0$, $\theta = \pi$, and the continuity condition

$$\bar{v}(\alpha -, s) - \bar{v}(\alpha +, s) = 0.$$

B is determined from the second continuity condition

$$\bar{v}'(\alpha -, s) - \eta \bar{v}'(\alpha +, s) = 0,$$

where

$$\eta = \epsilon_2/\epsilon_1.$$

We find

$$B = -\frac{(1 - \eta)\bar{v}_1'(\alpha, s)}{s[\eta \sin s\alpha \cos s(\pi - \alpha) + \cos s\alpha \sin s(\pi - \alpha)]}.$$

The transform of the odd part of the potential u is then given by

$$\bar{v}(\theta, s) = M(\theta, s)/sP(s, \alpha, \pi), \tag{9}$$

where

$$M(\theta, s) = \begin{cases} P(s, \theta, \theta) \sin s(\pi - \gamma), & 0 < \theta < \alpha \\ P(s, \alpha, \theta) \sin s(\pi - \gamma), & \alpha < \theta < \gamma \\ P(s, \alpha, \gamma) \sin s(\pi - \theta), & \gamma < \theta < \pi \\ -M(2\pi - \theta, s), & \pi < \theta < 2\pi, \end{cases} \tag{10}$$

and

$$P(s, \alpha, \theta) = (1 + \eta) \sin s\theta - (1 - \eta) \sin s(2\alpha - \theta). \tag{11}$$

A similar calculation yields the transform of the even part of the potential, viz,

$$\bar{w}(\theta, s) = N(\theta, s)/sQ(s, \alpha, \pi), \tag{12}$$

where

$$N(\theta, s) = \begin{cases} -R(s, \theta, \theta) \cos s(\pi - \gamma), & 0 < \theta < \alpha \\ -R(s, \alpha, \theta) \cos s(\pi - \gamma), & \alpha < \theta < \gamma \\ -R(s, \alpha, \gamma) \cos s(\pi - \theta), & \gamma < \theta < \pi \\ N(2\pi - \theta, s), & \pi < \theta < 2\pi, \end{cases} \tag{13}$$

and

$$\begin{aligned} Q(s, \alpha, \theta) &= (1 + \eta) \sin s\theta + (1 - \eta) \sin s(2\alpha - \theta), \\ R(s, \alpha, \theta) &= (1 + \eta) \cos s\pi - (1 - \eta) \cos s(2\alpha - \theta). \end{aligned} \tag{14}$$

Next, we must invert $\bar{v}(\theta, s)$, $\bar{w}(\theta, s)$ to obtain $v(r, \theta)$ and $w(r, \theta)$, or rather their forms for small r, since we are primarily interested in the behavior of the potential near the wedge tip.

IV. THE INVERSION INTEGRAL

If the integral (4) defining $\bar{u}(\theta, s)$ converges absolutely for all s in the strip $\sigma_1 < \text{Re} s < \sigma_2$, then $u(r, \theta)$ is given by the inversion integral[3]

$$u(r, \theta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{u}(\theta, s) r^{-s} ds, \tag{15}$$

where the integration contour in the complex s plane is any vertical straight line $\text{Re}\, s = c$ with $\sigma_1 < c < \sigma_2$. We have already seen from the derivation of $\bar{v}_1$ and $\bar{w}_1$ that $-1 < \sigma_2 < \sigma_1 < 0$. An examination of (9) to (14) shows that, while $\bar{v}(\theta, s)$ is regular at $s = 0$, $\bar{w}(\theta, s)$ has double pole there. Further, as we shortly show, both $\bar{v}$ and $\bar{w}$ have a countably infinite number of poles. They are all real, and the nonzero poles are all simple. The largest of the negative poles, at $s = s_0$, satisfies $-1 < -s_0 < 0$. Since the strip $\sigma_1 < \text{Re}\, s < \sigma_2$ can contain no singularities of $\bar{u}(\theta, s)$, it follows that $\sigma_1 = s_0$, $\sigma_2 = 0$. Assuming s_0 is known, since $\bar{u}(\theta, s) = \frac{1}{2}[\bar{v}(\theta, s) + \bar{w}(\theta, s)]$, eqs. (9) to (15) provide an explicit integral representation of the desired potential $u(r, \theta)$. This expression for u seems much more suitable than the expression given by Smythe[2] for determining the small r behavior of u.

The integral can be evaluated by the residue theorem[4] by closing this contour with large semicircles, to the left for small r and to the right for large r. Examination of the forms for $\bar{v}$ and $\bar{w}$, given by eqs. (9) to (14), reveals that the integrand of eq. (14) vanishes so rapidly on the semicircles that, as the semicircle radii tend to infinity, the semicircles make no contribution to the integral around the contour. The sum of the residues enclosed by the left semicircle thus gives the small r behavior of u; those to the right the large r behavior. It is clear from (11) and (14) that, if $p \neq 0$ is a zero of $P(s, \alpha, \pi)$, then so is $-p$, and, similarly, the nonzero roots of $Q(s, \alpha, \pi)$ come in pairs. Let p_n, q_n, $n = 1, 2, \cdots$ denote the positive roots of P and Q, respectively. Then it follows that, for $r < 1$,

$$u(r, \theta) = -\frac{1}{2} \sum_{n=1}^{\infty} \left\{ \frac{M(\theta, p_n) r^{p_n}}{p_n P'(p_n, \alpha, \pi)} + \frac{N(\theta, q_n) r^{q_n}}{q_n Q'(q_n, \alpha, \pi)} \right\}, \tag{16}$$

while, for $r > 1$,

$$u(r, \theta) = \frac{N(\theta, 0)}{2Q'(0, \alpha, \pi)} \ln r - \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \frac{M(\theta, p_n) r^{-p_n}}{p_n P'(p_n, \alpha, \pi)} + \frac{N(\theta, q_n) r^{-q_n}}{q_n Q'(q_n, \alpha, \pi)} \right\}. \tag{17}$$

The poles of $\bar{v}$ and $\bar{w}$ lie at the zeros of $P(s, \alpha, \pi)$ and $Q(s, \alpha, \pi)$ except, of course, when $M(s, \theta)$ and $N(s, \theta)$ also vanish for the same value of s. For example, $\bar{v}$ has a removable singularity at $s = 0$. Since the zeros also depend on η, we emphasize this by writing $P(s, \alpha, \pi)$ and $Q(s, \alpha, \pi)$ as $P(s, \alpha, \pi; \eta)$ and $Q(s, \alpha, \pi; \eta)$. Then it is simple to show that

$$Q(s, \alpha, \pi; \eta) = \eta P\left(s, \alpha, \pi; \frac{1}{\eta}\right), \tag{18}$$

$$Q(s, \alpha, \pi; \eta) = P(s, \pi - \alpha, \pi; \eta). \tag{19}$$

If we set $s = p$, then $P(s, \alpha, \pi; \eta) = 0$ can be written

$$(1 + \eta) \sin p\pi + (1 - \eta) \sin p(\pi - 2\alpha) = 0,$$

which is identical to eq. (15) of Ref. 1 with $\beta = \pi$, the case of an overhanging electrode. The two smallest values of p for various values of η are then given by Fig. 11 of Ref. 1, here reproduced as Fig. 3. From Fig. 3 and eqs. (18) and (19) we see that, if $0 < \eta < 1$, $0 < \alpha < \pi/2$, or $1 < \eta$, $\pi/2 < \alpha < \pi$, then $p_1 > 1$, $q_1 < 1$, while if $0 < \eta < 1$, $\pi/2 < \alpha < \pi$ or $1 < \eta$, $0 < \alpha < \pi/2$, then $p_1 < 1$, $q_1 > 1$. In all cases, $p_2 > 1$, $q_2 > 1$. If $\rho_1 = \min(p_1, q_1)$, we have shown that for $r < 1$, $\nabla u \sim r^{\rho_1 - 1}$ and that for all angles α and ratios η, $\rho_1 < 1$ so the field is always singular at the tip of the wedge. For the case of an overhanging electrode for which the potential is given by v alone, $\nabla v \sim r^{p_1 - 1}$, so we have substantiated the local analysis of Ref. 1 by the solution of a complete boundary value problem.

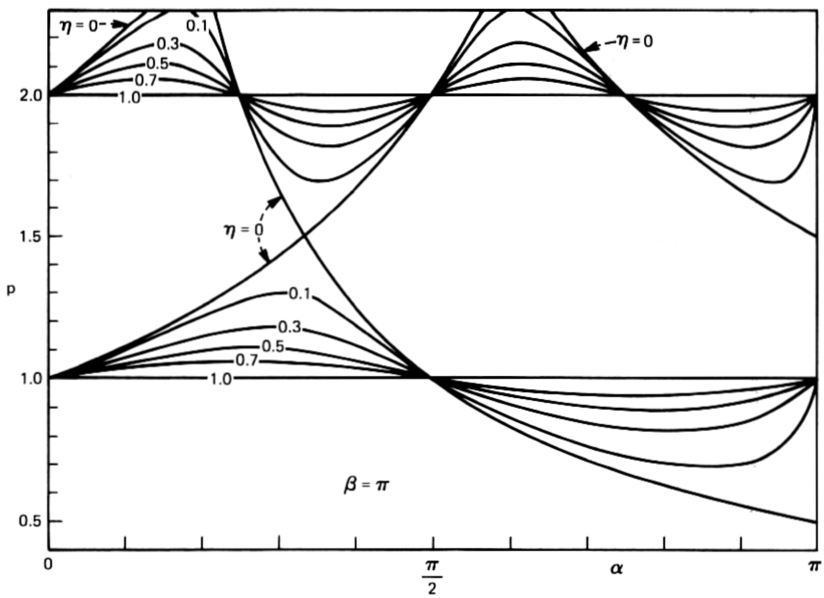

Fig. 3—The zeros of $P(s, \alpha, \pi)$ for various η.

V. REALITY AND SIMPLICITY OF POLES

One minor task remains to complete our analysis. We must show that all the roots of $P(s, \alpha, \pi) = 0$ and $Q(s, \alpha, \pi) = 0$ are real and simple. We write the two equations as

$$\sin \pi s = \pm E \sin (\pi - 2\alpha)s, \tag{20}$$

where $E = (1 - \eta)/(1 + \eta)$. We exclude the case $2\alpha = \pi$, for which the zeros are clearly real and simple. If we set $s = \sigma + i\tau$, the real and imaginary parts of eq. (20) become

$$\sin \pi\sigma \cosh \pi\tau = \pm E \sin (\pi - 2\alpha)\sigma \cosh (\pi - 2\alpha)\tau.$$
$$\cos \pi\sigma \sinh \pi\tau = \pm E \cos (\pi - 2\alpha)\sigma \sinh (\pi - 2\alpha)\tau.$$

Divide the first by $\cosh \pi\tau$, the second by $\sinh \pi\tau$, square, and add to obtain

$$E^2 \left[\sin^2 (\pi - 2\alpha)\sigma \frac{\cosh^2 (\pi - 2\alpha)\tau}{\cosh^2 \pi\tau} + \cos^2 (\pi - 2\alpha)\sigma \cdot \frac{\sinh^2 (\pi - 2\alpha)\tau}{\sinh^2 \pi\tau} \right] = 1. \tag{21}$$

With $2\alpha \neq \pi$, $|\pi - 2\alpha| < \pi$, $\tau \neq 0$, so that $\cosh^2 (\pi - 2\alpha)\tau < \cosh^2 \pi\tau$, $\sinh^2 (\pi - 2\alpha)\tau < \sinh^2 \pi\tau$, eq. (21) implies $E^2 > 1$, which is impossible since $E^2 \leqq 1$ for $0 \leqq \eta < \infty$. By assuming a complex zero, we arrive at a contradiction, so all the zeros of P and Q must be real.

If s is a multiple zero of (20), it must also be a zero of one of the equations obtained by differentiating (20),

$$\cos \pi s = \pm E \left(1 - \frac{2\alpha}{\pi}\right) \cos (2\alpha - \pi)s. \tag{22}$$

If we square and add (21) and (22), we get

$$\frac{1}{E^2} = \sin^2 (\pi - 2\alpha)s + \left(1 - \frac{2\alpha}{\pi}\right)^2 \cos^2 (2\alpha - \pi). \tag{23}$$

Since $(1 - 2\alpha/\pi)^2 < 1$, (23) implies $(1/E^2) < 1$, which is a contradiction. Thus, all the zeros of P and Q must be simple.

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