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B. S. T. J. BRIEF

The Accuracy of the Equivalent Random Method With Renewal Inputs*

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I. INTRODUCTION

The equivalent random method[1] (aso see Ref. 2) is widely used to approximate the blocking probabilities for non-Poisson traffic streams. Although much numerical experience and some analysis (e.g., Ref. 3) suggests that the method is usually reliable for superpositions of overflows, the reason for its accuracy (or errors) deserves further attention.

The equivalent random method first determines the mean M and variance V of the number of the trunks that would be occupied if the traffic were offered to an infinite trunk group. Then an overflow process with the same M and V is offered to the finite trunk group and its blocking calculated.† This blocking is taken as the approximation for the blocking seen by the original traffic.

In this Brief, we derive the range of the blocking probabilities which may be experienced by renewal streams characterized by the same M and V. Since this range may be rather wide, it follows that the success of equivalent random method cannot be explained solely by the constraints put on blockings by fixing M and V. Rather, one should factor in the special structure of the processes. Furthermore, it is seen that one cannot use an arbitrary renewal process to represent another process with the same mean and variance.

* A version of this Brief was presented at the Seventh International Teletraffic Congress, Stockholm, June 1973.

† That is, the blocking is calculated for the specific renewal process which is the overflow process from a Poisson input. Conceivably, other types of renewal processes could be used.

II. IMPLICATIONS OF THE EQUIVALENT RANDOM METHOD

Consider a nonlattice renewal process, with distribution function $F(t)$ for the interarrival times, offered to a group of N trunks. The holding times are mutually independent exponential random variables with unity mean (or the mean is the time unit). Blocked calls are cleared and the system is in equilibrium. Define

$$m = \int_0^\infty t dF(t), \tag{1}$$

$$\phi(x) = \int_0^\infty e^{-xt} dF(t). \tag{2}$$

Then it is known that the blocking probability is

$$B = \left\{1 + \binom{N}{1} \frac{1 - \phi(1)}{\phi(1)} + \cdots + \binom{N}{N} \frac{[1 - \phi(1)] \cdots [1 - \phi(N)]}{\phi(1)\phi(2) \cdots \phi(N)}\right\}^{-1} \tag{3}$$

(see, e.g., Ref. 4, Chap. 4). Observe that B depends on N values of $\phi(i)$, $i = 1, \cdots, N$, and that it is an increasing function of these $\phi(i)$. We shall show how the equivalent random method constrains these $\phi(i)$ by obtaining upper and lower bounds on them which, in turn, give upper and lower bounds on B.

The description of the equivalent random method in the Introduction leads to the question of how well M and V characterize a traffic. It turns out that they imply much more than is apparent at first glance. For our renewal input, we have the following relationships:

$$M = m^{-1}, \tag{4}$$

$$V = M\left[\frac{1}{1 - \phi(1)} - M\right] \tag{5}$$

(see, e.g., Ref. 4, Chap. 3*). Thus, (M, V) uniquely determines $(m, \phi(1))$ and vice versa. Specifically,

$$\phi(1) = \frac{V/M - 1 + M}{V/M + M}. \tag{6}$$

Hence, the equivalent random method fixes $\phi(1)$ which is particularly important in (3). Moreover, fixing V and M puts important constraints on the other $\phi(i)$, $i = 2, \cdots, N$, which, in turn, further constrains B.

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* Also, see Ref. 5, p. 331/5, for an interesting characterization of peakedness.

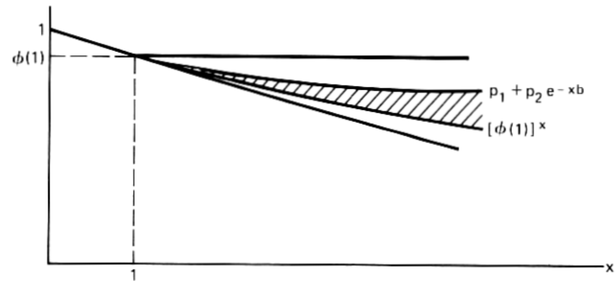

Fig. 1—Constraints on $\phi(x)$ for $x > 1$.

Figure 1 shows how $(m, \phi(1))$ constrains $\phi(x)$ for $x > 1$. The parameters p_1, p_2, b will be given in (16)–(18). All such $\phi(x)$ must lie within the shaded area. The upper bound is a least upper bound and the lower bound, a greatest lower bound. Also shown in Fig. 1 is a wedge for $x > 1$ which respresents simpler, cruder bounds for $\phi(x)$ which follow immediately from the decreasing convex nature of $\phi(x)$.

To derive the lower bound, let $y = e^{-\xi}$ in

$$E|y| \leqq E^{1/x}|y|^x, \qquad x > 1, \tag{7}$$

with ξ the renewal interarrival time.
We obtain

$$E^x(e^{-\xi}) \leqq E(e^{-x\xi}) \tag{8}$$

or, in other words,

$$[\phi(1)]^x \leqq \phi(x), \tag{9}$$

so that $\phi(x)$ must lie above the indicated curve for $x > 1$ in Fig. 1. To show that this is a sharp lower bound, let

$$dF(t) = [p_1\delta(t-a) + p_2\delta(t-b)]dt. \tag{10}$$

(p_1, a, p_2, b) must satisfy

$$p_1 + p_2 = 1, \tag{11}$$

$$p_1 a + p_2 b = m, \tag{12}$$

$$p_1 e^{-a} + p_2 e^{-b} = \phi(1). \tag{13}$$

By letting b get large and $p_1 \to 1$, we can show that

$$\phi(x) = p_1 e^{-xa} + p_2 e^{-xb} \to p_1 e^{-xa} \to [\phi(1)]^x. \tag{14}$$

The sharp lower bound for $\phi(x)$ may also be derived using Theorem 2.1 on p. 472 of Ref. 6 (see Remark 2.3, p. 474). Use of this theorem*

* The problem to which we applied this theorem is to find sharp upper and lower bounds for $\int_0^\infty e^{-xt}dF(t)$ subject to $\int_0^\infty dF(t) = 1$, $\int_0^\infty t\,dF(t) = m$, and $\int_0^\infty e^{-t}dF(t) = \phi(1)$, a number fixed by (6).

also leads to the following sharp upper bound for $\phi(x)$:

$$\phi(x) \leqq \phi_m(x) = p_1 + p_2 e^{-xb} \tag{15}$$

with (p_1, p_2, b) satisfying

$$b = \frac{m(1 - e^{-b})}{1 - \phi(1)}, \tag{16}$$

$$p_2 = \frac{m}{b}, \tag{17}$$

$$p_1 = 1 - p_2. \tag{18}$$

We thus obtain that the true B satisfies

$$B_L \leqq B \leqq B_u, \tag{19}$$

where

$$B_L = \left\{1 + \binom{N}{1}\frac{1 - \phi(1)}{\phi(1)} + \cdots + \binom{N}{N}\frac{[1 - \phi(1)]\cdots[1 - \phi^N(1)]}{[\phi(1)]^{[N(N+1)/2]}}\right\}^{-1}, \tag{20}$$

$$B_u = \left\{1 + \binom{N}{1}\frac{1 - \phi_m(1)}{\phi_m(1)} + \cdots + \binom{N}{N}\frac{[1 - \phi_m(1)]\cdots[1 - \phi_m(N)]}{\phi_m(1)\cdots\phi_m(N)}\right\}^{-1}. \tag{21}$$

The blocking probability obtained by the equivalent random method, B_{er}, also satisfies (19) so that a bound on the error for the method is

$$\max\{B_u - B_{er}, B_{er} - B_L\}.$$

III. INTERPRETATION OF EXTREMAL SOLUTIONS

Some feeling for these bounds can be obtained by considering the maximum and minimum blocking probabilities attainable when only the mean interarrival time m is constrained (the equivalent random V is unspecified). It is shown in Ref. 7 that the minimum blocking is achieved when arrivals are regular with a separation of m. Our inf may be viewed as approaching that of regular arrivals but with a different mean [the impulse at b in (10) keeps the equivalent random M and V satisfied]. Observe that B_L is the blocking probability seen by a renewal input with constant interarrival times with mean m_1 determined from

$$e^{-m_1} = \phi(1). \tag{22}$$

It is shown in Ref. 7 that with a given m, blocking probabilities

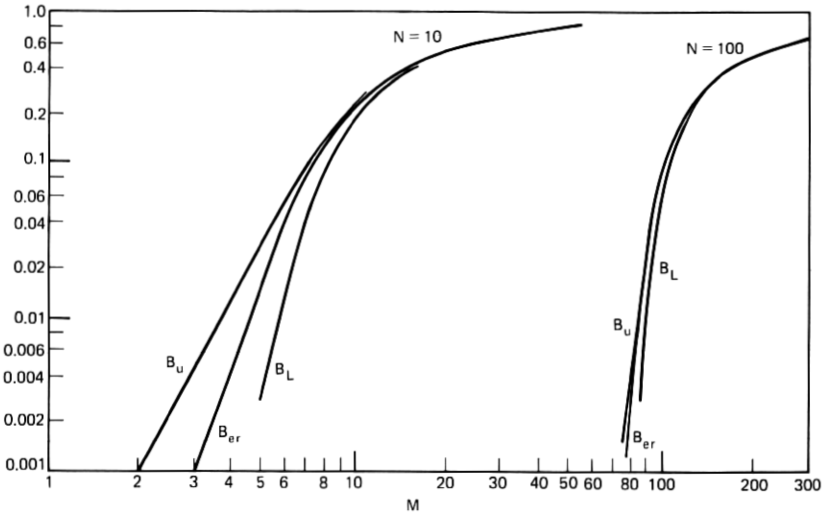

Fig. 2—B_u, B_{er}, B_L for $V/M = 1$.

arbitrarily close to unity may be obtained by having $F(t)$ consist of a step at t sufficiently small and another small step at t sufficiently large. This causes most of the arrivals to come tripping on each other's heels. Our maximum blocking may be viewed as trying to approach this but constrained by V to keep the second step at a finite t.

IV. EXAMPLES AND DISCUSSION

Some bounds are shown in Figs. 2 through 4. These results do not necessarily imply that the equivalent random method is commonly subject to errors of such magnitude. In practice, the method is usually applied to superpositions of overflows and these are a special class of processes, generally not renewal.* Nevertheless, the relatively large differences between the inf and sup blockings suggest that the apparent success of the equivalent random method for superpositions of overflows cannot be explained solely by the constraints put on blockings by fixing M and V. Rather, explanation of this accuracy should factor in the special structure of such processes. (It may be of interest to extend the results of this Brief to take special structures into account.) Furthermore, it is seen that one cannot use an arbitrary re-

* Teletraffic interest need not be confined to simple superposition of overflows from trunk groups; e.g., switching center congestion can alter a traffic.

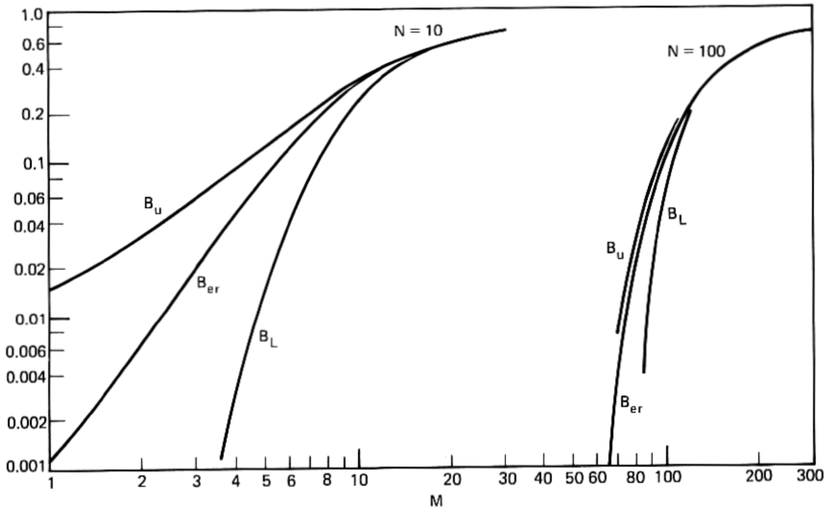

Fig. 3—B_u, B_{er}, B_L for $V/M = 2$.

newal process to represent another process with the same mean and variance.

As an aside, observe that if $V/M > 1$, the blocking B is bounded away from zero no matter how small M is. That is, (6) implies that

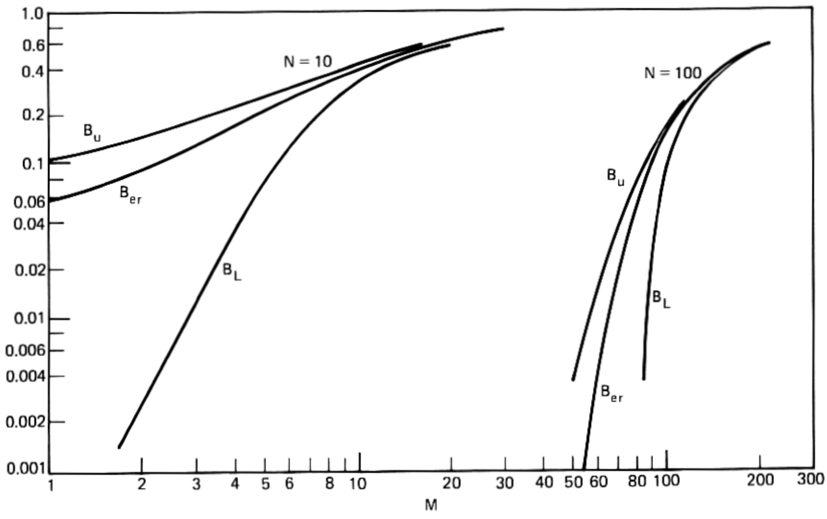

Fig. 4—B_u, B_{er}, B_L for $V/M = 4$.

B_L of (20) for fixed V/M cannot get below B_L evaluated with $\phi(1) = 1 - (M/V)$.

V. ACKNOWLEDGMENTS

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