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Higher-Order Scattering Losses in Dielectric Waveguides

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This paper discusses the scattering losses of dielectric slab waveguides that are caused by higher-order grating lobes of the sinusoidally distorted core-cladding interface. The results of this paper are used in a companion paper to evaluate the radiation losses of multimode guides with intentional mode coupling. An exact system of equations is derived for the amplitudes of all grating orders. This system is used to derive first- and second-order approximations that hold for small amplitudes of the sinusoidal interface distortion. The theory is used to derive formulas for the average power loss coefficient for first- and second-order scattering processes.

I. INTRODUCTION

Scattering losses in dielectric waveguide caused by core-cladding interface irregularities have been studied extensively by means of first-order perturbation theory. The principle result of this theory can be stated as follows:[1] Two modes with propagation constants β_ν and β_μ are coupled only if a Fourier component of the core-cladding interface function exists the mechanical frequency of which, ϕ, satisfies the relation

$$\phi = | \beta_\nu - \beta_\mu | . \tag{1}$$

The propagation constants β_ν and β_μ may both belong to guided modes or one may belong to a guided mode while the other belongs to the continuum of radiation modes. Coupling of a guided mode to radiation modes results in power loss of the guided mode. This first-order coupling process is very strong and leads to high radiation losses if suitable Fourier components of the mechanical core-cladding interface irregularity function exist.

The result of first-order perturbation theory can be understood by viewing the core-cladding interface as a diffraction grating.[2] Since it modulates the phase of the incident wave passing through it, the dielectric interface acts as a phase grating. The guided modes of dielectric waveguides can be decomposed into plane waves.[3] This decomposition is particularly simple in the case of the modes of a slab waveguide. The following discussion is thus applied to this structure. Two plane waves are superimposed to form a traveling wave in the z direction–the direction of the waveguide axis–and a standing wave in the direction transverse to the z axis. The coupling coefficients for guided mode coupling and the radiation loss coefficients can be calculated by solving the plane wave scattering problem at the dielectric interface.[4] The geometry of the problem is shown in Fig. 1. For clarity of discussion, it was assumed that the incident plane wave approaches the interface at right angles. The actual plane waves making up the guided mode of the slab waveguide approach the interfaces at grazing angles. Figure 1 is drawn with a sinusoidally distorted core-cladding interface. In this case, the incident plane wave decomposes into a wave that continues to travel in the original direction after passing the interface and into a reflected wave plus a series of side lobes that are labeled by positive

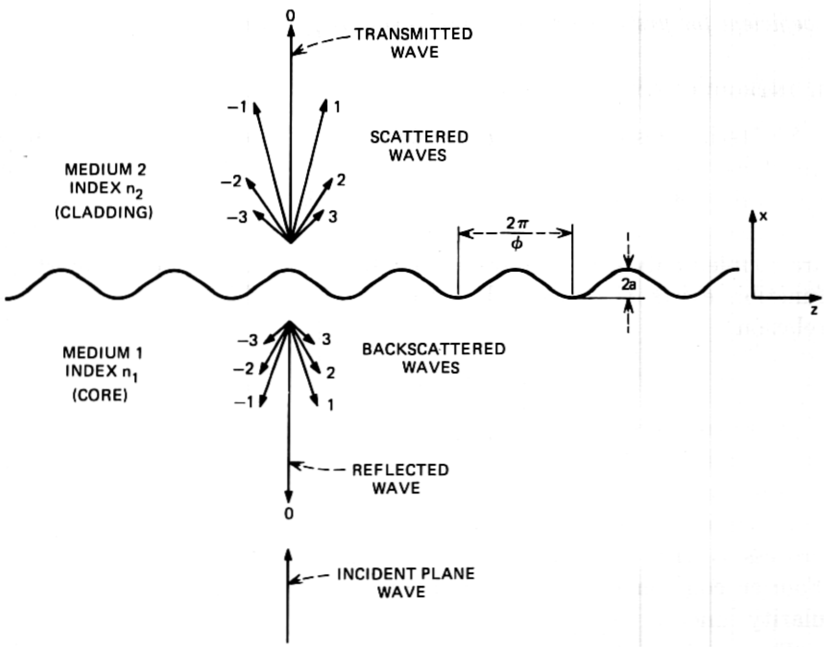

Fig. 1—A sinusoidally deformed dielectric interface functions as a phase grating. The figure shows the propagation vector of an incident plane wave (labeled 0) and reflected as well as transmitted plane waves of the higher-order grating lobes.

and negative integer numbers. These are the lobes of first, second, and higher order of the phase grating. If the incident plane wave make as grazing angle with the interface, the situation is essentially the same; the only difference being that some of the grating lobes form imaginary angles and are consequently evanescent instead of traveling waves. The zero-order lobe does not pass if the incident wave meets the interface at less than the critical angle for total internal reflection. Its angle is thus imaginary and only an evanescent wave exists on the far side of the interface while the incident wave is strongly reflected. The transmitted first-order grating lobes may both have imaginary angles. In this case, there are only scattered reflected waves in the core of the waveguide that can combine to a new guided mode provided that their angles correspond to one of the allowed directions for guided modes. If one of the transmitted first-order side lobes emerges on the far side of the interface with a real angle, it causes power to radiate into the space outside of the waveguide core. This radiation is lost to the guided wave and must be counted as power loss.

For small amplitudes of the sinusoidal core-cladding interface irregularity, the amplitudes of the grating lobes decrease rapidly with increasing grating order. The first-order grating lobe corresponds to the contribution of first-order perturbation theory and is indeed the only contribution of real interest if the core-cladding interface irregularity has a small amplitude. With increasing amplitude of the interface irregularity, the higher-order grating lobes become increasingly important. Their importance is enhanced by the fact that the angles of the second-order grating lobes may be real even when the transmitted first-order lobes both have only imaginary angles. This means that, to first-order of perturbation theory, no scattering loss exists. It is thus necessary to study the higher order grating responses in order to obtain information about scattering losses in case the first-order theory predicts no scattering loss at all.

The study of these higher-order grating lobes and the derivation of power loss coefficients for the higher-order processes is the object of this paper. It is possible that the grating problem has been solved before to the accuracy that is attempted here. Because of the enormous volume of literature that exists on scattering problems, relevant papers may have escaped the author's attention. However, the application of the grating theory to the waveguide loss problem is probably new.

We derive coupled equation systems for the amplitudes of the grating lobes and use these exact equations to obtain first- and second-order approximations in a straightforward way. Perturbation solutions of the exact equation system are particularly appropriate, since each higher-order solution can be computed from the known lower-order solutions with no need to recompute the lower-order solutions each time the order of perturbation theory is increased by one. Formulae for the first- and second-order scattering loss process from a sinusoidal core cladding interface irregularity are derived. The results of this paper will be used in a companion paper[5] to calculate the loss penalty for intentional mode mixing in multimode waveguides.

II. PLANE WAVE SCATTERING AT A SINUSOIDAL INTERFACE

We consider the problem of a plane wave that impinges on the interface between two dielectric media. The interface is described by the function

$$f(z) = a \sin \phi z. \tag{2}$$

If only first-order scattering is considered, more general shapes of $f(z)$ can be synthesized by superposition of sinusoidal functions. For higher-order processes, mixing of the sinusoidal terms occurs so that the description of scattering from more general interfaces becomes complicated. The incident plane wave is given by (the time dependence is $e^{i\omega t}$)

$$E_y = Ae^{-i(\kappa_i x + \beta_i z)} \tag{3}$$

$$H_x = -\frac{\beta_i}{\omega\mu_0} Ae^{-i(\kappa_i x + \beta_i z)} \tag{4}$$

$$H_z = \frac{\kappa_i}{\omega\mu_0} Ae^{-i(\kappa_i x + \beta_i z)}. \tag{5}$$

The coordinate system is shown in Fig. 1. It is assumed that no variation of either the field components or the material parameters exists in y direction so that we can symbolically write

$$\frac{\partial}{\partial y} = 0. \tag{6}$$

The remaining three field components E_x, E_z, and H_y vanish. The parameters κ_i and β_i are connected by the following equation

$$\kappa_i = (n_1^2 k^2 - \beta_i^2)^{1/2}. \tag{7}$$

The refractive index n_1 belongs to the medium from which the plane wave approaches the interface, and k is the free space propagation constant.

The reflected and scattered waves are expressed as superpositions of plane waves. We thus have in medium 1

$$E_y = \int_{-\infty}^{\infty} B(\beta) e^{i(\sigma x - \beta z)} \, d\beta \tag{8}$$

$$H_x = -\frac{1}{\omega\mu} \int_{-\infty}^{\infty} \beta B(\beta) e^{i(\sigma x - \beta z)} \, d\beta \tag{9}$$

$$H_z = -\frac{1}{\omega\mu} \int_{-\infty}^{\infty} \sigma B(\beta) e^{i(\sigma x - \beta z)} \, d\beta, \tag{10}$$

with

$$\sigma = (n_1^2 k^2 - \beta^2)^{1/2}, \tag{11}$$

and similarly in medium 2

$$E_y = \int_{-\infty}^{\infty} C(\beta) e^{-i(\rho x + \beta z)} \, d\beta \tag{12}$$

$$H_x = -\frac{1}{\omega\mu} \int_{-\infty}^{\infty} \beta C(\beta) e^{-i(\rho x + \beta z)} \, d\beta \tag{13}$$

$$H_z = \frac{1}{\omega\mu} \int_{-\infty}^{\infty} \rho C(\beta) e^{-i(\rho x + \beta z)} \, d\beta, \tag{14}$$

with

$$\rho = (n_2^2 k^2 - \beta^2)^{1/2}. \tag{15}$$

The boundary conditions at the dielectric interface require the tangential component of the electric field E_y and the tangential component of the magnetic field

$$H_t = \frac{1}{(1 + f'^2)^{\frac{1}{2}}} H_z + \frac{f'}{(1 + f'^2)^{\frac{1}{2}}} H_x \tag{16}$$

to be continuous [f' is the z derivative of the interface function (2)]. The boundary conditions thus lead to the two equations:

$$A e^{-i(\kappa_i f + \beta_i z)} + \int_{-\infty}^{\infty} B(\beta) e^{i(\sigma f - \beta z)} \, d\beta = \int_{-\infty}^{\infty} C(\beta) e^{-i(\rho f + \beta z)} \, d\beta \tag{17}$$

and

$$(\kappa_i - \beta_i f') A e^{-i(\kappa_i f + \beta_i z)} - \int_{-\infty}^{\infty} (\sigma + \beta f') B(\beta) e^{i(\sigma f - \beta z)} \, d\beta$$
$$= \int_{-\infty}^{\infty} (\rho - \beta f') C(\beta) e^{-i(\rho f + \beta z)} \, d\beta. \tag{18}$$

In order to remove the z dependence from the equation, we multiply with exp $(-i\beta' z)$ and integrate over z from $-\infty$ to ∞. This procedure transforms the equations to the following form:

$$AF(\kappa_i, \beta_i - \beta') + \int_{-\infty}^{\infty} B(\beta)F(-\sigma, \beta - \beta')\, d\beta = \int_{-\infty}^{\infty} C(\beta)F(\rho, \beta - \beta')\, d\beta, \tag{19}$$

and

$$AG(\kappa_i, \beta_i, \beta') + \int_{-\infty}^{\infty} B(\beta)G(-\sigma, \beta, \beta')\, d\beta = \int_{-\infty}^{\infty} C(\beta)G(\rho, \beta, \beta')\, d\beta \tag{20}$$

with

$$F(\eta, \beta - \beta') = \int_{-\infty}^{\infty} e^{-i[\eta f(z) + (\beta - \beta')z]}\, dz, \tag{21}$$

and

$$G(\eta, \beta, \beta') = \int_{-\infty}^{\infty} [\eta - \beta f'(z)] e^{-i[\eta f(z) + (\beta - \beta')z]}\, dz. \tag{22}$$

It is shown in the Appendix that F and G are related in the following way:

$$G(\eta, \beta, \beta') = \frac{\eta^2 + \beta^2 - \beta\beta'}{\eta} F(\eta, \beta - \beta'). \tag{23}$$

Substitution of (2) into (21) yields

$$F(\eta, \beta - \beta') = \int_{-\infty}^{\infty} e^{-ia\eta \sin \phi z} e^{-i(\beta - \beta')z}\, dz. \tag{24}$$

The first exponential function under the integral sign can be expressed as a series in terms of Bessel functions with the help of the generating function of the Bessel functions. The remaining integration yields delta functions so that we obtain

$$F(\eta, \beta - \beta') = 2\pi \sum_{\nu=-\infty}^{\infty} J_\nu(\eta a)\, \delta[\beta - \beta' + \nu\phi]. \tag{25}$$

Substitution of (23) and (25) transforms the equation systems (19) and (20) to the form

$$\sum_{\nu=-\infty}^{\infty} \{-B(\beta_\nu)J_\nu(\sigma_\nu a) + C(\beta_{-\nu})J_\nu(\rho_{-\nu} a)\} = A \sum_{\nu=-\infty}^{\infty} J_\nu(\kappa_i a)\, \delta(\beta_i - \beta' + \nu\phi), \tag{26}$$

and

$$\sum_{\nu=-\infty}^{\infty} \left\{ \frac{n_1^2k^2 - \beta_\nu\beta'}{\sigma_\nu} B(\beta_\nu) J_\nu(\sigma_\nu a) + \frac{n_2^2k^2 - \beta_{-\nu}\beta'}{\rho_{-\nu}} C(\beta_{-\nu}) J_\nu(\rho_{-\nu} a) \right\}$$

$$= A \sum_{\nu=-\infty}^{\infty} \frac{n_1^2k^2 - \beta_i\beta'}{\kappa_i} J_\nu(\kappa_i a)\ \delta(\beta_i - \beta' + \nu\phi). \quad (27)$$

The following abbreviations were used

$$\beta_\nu = \beta' + \nu\phi \quad (28)$$

$$\sigma_\nu = (n_1^2k^2 - \beta_\nu^2)^{\frac{1}{2}} \quad (29)$$

and

$$\rho_\nu = (n_2^2k^2 - \beta_\nu^2)^{\frac{1}{2}}. \quad (30)$$

We know from the theory of phase gratings[2] that only discrete plane waves appear in the reflected and transmitted beams. This and the appearance of the delta functions on the right-hand side of (26) and (27) suggest that the solutions should be of the form

$$B(\beta) = \sum_{\mu=-\infty}^{\infty} b_\mu\ \delta(\beta - \beta_i - \mu\phi), \quad (31)$$

and

$$C(\beta) = \sum_{\mu=-\infty}^{\infty} c_\mu\ \delta(\beta - \beta_i - \mu\phi). \quad (32)$$

Substitution of (31) and (32) into (26) and (27) and comparison of the coefficients of the delta functions of equal arguments leads to two infinite equation systems for the unknown coefficients b_μ and c_μ.

$$\sum_{\nu=-\infty}^{\infty} \{-b_{n+\nu} J_\nu(\sigma_{n+\nu} a) + c_{n-\nu} J_\nu(\rho_{n-\nu} a)\} = A J_n(\kappa_i a), \quad (33)$$

and

$$\sum_{\nu=-\infty}^{\infty} \left\{ \frac{\sigma_n^2 - \nu\phi(\beta_i + n\phi)}{\sigma_{n+\nu}} b_{n+\nu} J_\nu(\sigma_{n+\nu} a) \right.$$

$$\left. + \frac{\rho_n^2 + \nu\phi(\beta_i + n\phi)}{\rho_{n-\nu}} c_{n-\nu} J_\nu(\rho_{n-\nu} a) \right\} = \frac{\kappa_i^2 - n\phi\beta_i}{\kappa_i} A J_n(\kappa_i a). \quad (34)$$

The equation system (33) and (34) is exact. An exact solution appears impossible to obtain. However, the equation system is very convenient for obtaining perturbation solutions of arbitrary order. It is also possible to obtain a solution that is exact in the limit $\phi \rightarrow 0$.

The reflected and transmitted fields follow directly from (8), (12), (31), and (32). In medium 1 we obtain for the reflected field

$$E_y = \sum_{\mu=-\infty}^{\infty} b_\mu e^{i(\sigma_\mu x - \beta_\mu z)}. \tag{35}$$

The field in medium 2 is

$$E_y = \sum_{\mu=-\infty}^{\infty} c_\mu e^{-i(\rho_\mu x + \beta_\mu z)}. \tag{36}$$

The parameters σ_μ and ρ_μ are defined by (29) and (30). However, now we must use these equations with $\beta_\mu = \beta_i + \mu\phi$.

III. SOLUTION FOR $\phi \rightarrow 0$

In the limit $\phi \rightarrow 0$, an exact solution of the equation systems (33) and (34) can be obtained. If $\phi = 0$ is assumed, we use the facts that

$$\sigma_\nu = \kappa_i \tag{37}$$

$$\rho_\nu = \rho_0 \tag{38}$$

to write (33) and (34) in the form

$$\sum_{\nu=-\infty}^{\infty} \{-b_{n+\nu} J_\nu(\kappa_i a) + c_{n-\nu} J_\nu(\rho_0 a)\} = A J_n(\kappa_i a) \tag{39}$$

$$\sum_{\nu=-\infty}^{\infty} \{\kappa_i b_{n+\nu} J_\nu(\kappa_i a) + \rho_0 c_{n-\nu} J_\nu(\rho_0 a)\} = \kappa_i A J_n(\kappa_i a). \tag{40}$$

It is now possible to eliminate c_ν from the equations and obtain an equation system for b_ν alone,

$$\sum_{\nu=-\infty}^{\infty} b_{n+\nu} J_\nu(\kappa_i a) = \frac{\kappa_i - \rho_0}{\kappa_i + \rho_0} A J_n(\kappa_i a). \tag{41}$$

Similarly, we obtain by eliminating b_ν

$$\sum_{\nu=-\infty}^{\infty} c_{n-\nu} J_\nu(\rho_0 a) = \frac{2\kappa_i}{\kappa_i + \rho_0} A J_n(\kappa_i a). \tag{42}$$

These equations can be solved with the help of the addition theorem for Bessel functions

$$\sum_{\nu=-\infty}^{\infty} J_{n+\nu}(x) J_\nu(y) e^{i\nu\theta} = J_n(R) e^{in\theta}, \tag{43}$$

with

$$R = (x^2 + y^2 - 2xy \cos \theta)^{1/2}. \tag{44}$$

With $\theta = 0$, we obtain $R = x - y$ so that we see immediately that

$$b_\mu = \frac{\kappa_i - \rho_0}{\kappa_i + \rho_0} A J_\mu(2\kappa_i a) \tag{45}$$

is the solution of (41). Since $J_{-\nu}(x) = J_\nu(-x)$ (for integer values of ν) it is also apparent that

$$c_\mu = \frac{2\kappa_i}{\kappa_i + \rho_0} A J_\mu[(\kappa_i - \rho_0)a] \tag{46}$$

is the solution of (42).

The solutions (45) and (46) are exact for $\phi = 0$. One might expect that $\phi = 0$ describes a plane dielectric interface so that no side lobes should be expected. Even though it is true that all the sidelobes coincide for $\phi = 0$, the solutions (45) and (46) do hold approximately even if $\phi \neq 0$. The sinusoidal shape of the interface is apparently built into the equation system (41) and (42) even though ϕ does not appear explicitly. The solutions (45) and (46) are approximations that hold if $k\phi \ll 1$. These solutions show that the amplitudes of the side lobes are proportional to Bessel functions. This result is well known from the theory of phase gratings.[2]

IV. PERTURBATION SOLUTIONS

For our purposes, the solution for $k\phi \ll 1$ is of little use. Therefore, we proceed to derive approximate solutions that hold for

$$ka \ll 1. \tag{47}$$

We use the following approximations for the Bessel functions of small argument

$$J_0(x) = 1 - \frac{x^2}{4} \tag{48}$$

$$J_1(x) = -J_{-1}(x) = \frac{x}{2}\left(1 - \frac{x^2}{8}\right) \tag{49}$$

$$J_2(x) = J_{-2}(x) = \frac{x^2}{8}. \tag{50}$$

In addition, we assume that b_o and c_o are zero-order terms, $b_{\pm 1}$ and $c_{\pm 1}$ are of first order, and $b_{\pm 2}$ and $c_{\pm 2}$ are of second order.

By neglecting all but zero-order terms in (33) and (34), we obtain to zero order of approximation for $n = 0$,

$$\left.\begin{aligned} -b_0 + c_0 &= A \\ \sigma_0 b_0 + \rho_0 c_0 &= \kappa_i A \end{aligned}\right\}. \tag{51}$$

Taking $n = 1$, we obtain to first order

$$\left.\begin{aligned} -b_1 + c_1 &= \frac{a}{2}(\kappa_i A - \sigma_0 b_0 - \rho_0 c_0) \\ \sigma_1 b_1 + \rho_1 c_1 &= \frac{a}{2}\{(\kappa_i^2 - \phi\beta_i)A + [\sigma_1^2 + \phi(\beta_i + \phi)]b_0 \\ &\quad - [\rho_1^2 + \phi(\beta_i + \phi)]c_0\} \end{aligned}\right\}. \tag{52}$$

The corresponding equation system for b_{-1} and c_{-1} is obtained by using $n = -1$. It has exactly the same form as the equation system (52) and is obtained by replacing b_1 with $-b_{-1}$, c_1 with $-c_{-1}$ and ϕ with $-\phi$.

Finally, we obtain the second-order approximation by setting $n = 2$ and keeping only terms up to second order

$$\left.\begin{aligned} -b_2 + c_2 &= \frac{a^2}{8}(\kappa_i^2 A + \sigma_0^2 b_0 - \rho_0^2 c_0) - \frac{a}{2}(\sigma_1 b_1 + \rho_1 c_1) \\ \sigma_2 b_2 + \rho_2 c_2 &= \frac{a^2}{8}\{\kappa_i(\kappa_i^2 - 2\phi\beta_i)A - \sigma_0[\sigma_2^2 + 2\phi(\beta_i + 2\phi)]b_0 \\ &\quad - \rho_0[\rho_2^2 + 2\phi(\beta_i + 2\phi)]c_0\} \\ &\quad + \frac{a}{2}\{[\sigma_2^2 + \phi(\beta_i + 2\phi)]b_1 - [\rho_2^2 + \phi(\beta_i + 2\phi)]c_1\} \end{aligned}\right\}. \tag{53}$$

The equations for b_{-2} and c_{-2} are obtained by replacing b_2 with b_{-2}, c_2 with c_{-2}, b_1 with $-b_{-1}$, c_1 with $-c_{-1}$, and finally ϕ with $-\phi$ in (53).

It is immediately apparent that each order of approximation follows from the preceding order. We can thus solve all the equations (51), (52), and (53) in succession. Each time we need solve only two equations with two unknowns. The result of the previous approximation is then used to obtain the next higher order of approximation from the next equation system.

The solutions of these equations are listed below.

$$b_0 = \frac{\kappa_i - \rho_0}{\kappa_i + \rho_0} A \qquad c_0 = \frac{2\kappa_i}{\kappa_i + \rho_0} A \tag{54}$$

$$b_1 = \frac{\kappa_i(\kappa_i - \rho_0)}{\sigma_1 + \rho_1} aA \qquad c_1 = b_1 \tag{55}$$

$$\left.\begin{aligned} b_2 &= \frac{a^2A}{4}\kappa_i\frac{\kappa_i-\rho_0}{\sigma_2+\rho_2}\left[\rho_0+\rho_2+2(\kappa_i-\rho_0)\frac{\kappa_i+\rho_0}{\sigma_1+\rho_1}\right] \\ c_2 &= \frac{a^2A}{4}\kappa_i\frac{\kappa_i-\rho_0}{\sigma_2+\rho_2}\left[\rho_0-\sigma_2+2(\kappa_i-\rho_0)\frac{\kappa_i+\rho_0}{\sigma_1+\rho_1}\right] \end{aligned}\right\}. \tag{56}$$

The coefficients b_{-1} and c_{-1} are obtained from (55) by changing the sign in front of the terms and changing the subscripts 1 to -1 on the right-hand side of the equation. The signs of b_{-2} and c_{-2} are the same as those of the coefficients in (56). We obtain these coefficients by changing the signs of the subscripts on the right-hand side of the equations. For $ka \ll 1$ and $k\phi \ll 1$ the equations (45) and (46) can be shown to be identical with (54), (55), and (56).

The amplitudes may belong to plane traveling waves or to evanescent waves. Whether a wave is of the propagating or evanescent type depends on whether the parameters σ_μ and ρ_μ are real or imaginary. For real values we obtain traveling waves while imaginary values indicate that the field decays exponentially with increasing distance from the interface indicating an evanescent wave. The propagation constant in z direction, β_μ, is obtained from (28) by replacing β' with β_i. We thus have for traveling as well as for evanescent waves

$$\beta_\mu = \beta_i + \mu\phi \qquad \mu = 0, \pm 1, \pm 2, \cdots . \tag{57}$$

V. CALCULATION OF SLAB WAVEGUIDE LOSSES

Since the guided modes of the slab waveguide can be expressed as the superposition of two plane waves, whose propagation vectors form equal but opposite angles with the z axis, we can use our present results immediately to calculate the radiation losses suffered by the guided slab waveguide modes.[4] Our calculation applies to TE modes. However, for slight index differences the losses of TE modes and TM modes are nearly identical. The slab waveguide geometry is shown in Fig. 2.

The radiation losses of slab waveguide modes have a somewhat complicated dependence on either frequency or slab width, since the interference of the waves scattered at one of the two dielectric interfaces with the radiation from the other interface–and also the interference with radiation that is reflected at the opposite interface–has to be taken into account. However, these interference effects cause only fluctuations about an average value. If we content ourselves with establishing only the average loss value, disregarding the fluctuations, the description of radiation losses is greatly simplified. We also gain the advantage of obtaining simpler mathematical expressions. In the spirit of this sim-

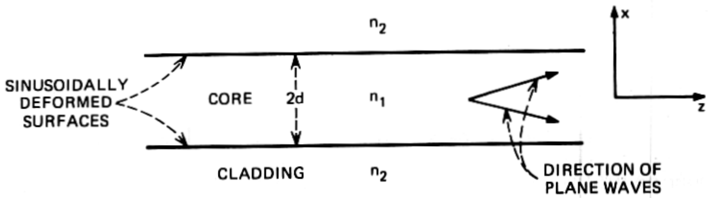

Fig. 2—Cross section of the slab waveguide.

plification, we consider all scattered power as lost–with the exception of those waves that are captured in the core–disregarding incomplete reflection from the other dielectric interface and interference with the directly scattered radiation. The amount of power scattered per unit length of the waveguide is

$$S_x = \tfrac{1}{2} \mid (\mathbf{E}_1 \times \mathbf{H}_1^*)_x \mid + \tfrac{1}{2} \mid (\mathbf{E}_2 \times \mathbf{H}_2^*)_x \mid \cdot \tag{58}$$

The electric and magnetic fields in this expression are only the scattered, untrapped part of the field, exclusive of the incident field. The subscripts 1 and 2 refer to the fields in mediums 1 and 2. With the assumption that only one side lobe is instrumental in dissipating power by radiation we can write

$$S_x = \frac{1}{2\omega\mu} (\sigma_{-\nu} \mid b_{-\nu} \mid^2 + \rho_{-\nu} \mid c_{-\nu} \mid^2). \tag{59}$$

The subscript ν assumes the values 1 and 2 for first- and second-order light scattering. The negative values must be used (ν is now assumed to be positive) because $\sigma_{-\nu}$ and $\rho_{-\nu}$ must be real since evanescent waves do not carry power.

The power attenuation is

$$\alpha = \frac{S_x}{2S_z d}. \tag{60}$$

S_z is the power pe runit length (unit area in the three dimensional case) that is carried by the plane wave in z direction. The total power carried by this one plane wave component in z direction inside of the waveguide core of width $2d$ is thus $2S_z d$. The ratio of the power lost (per unit length along the waveguide axis) divided by the power carried by the wave is the power loss per unit length. Actually, two plane waves are needed to describe the guided mode in the slab waveguide. However, thus far we have considered only the scattering loss from one of the two interfaces. It is sufficient to consider that each of the two plane wave components scatters from one interface. Inclusion of the scattering from both interfaces introduces a factor of two in the numerator of (60). However, another factor of two is introduced in the denominator by adding the power of the other plane wave to the total power carried by the guided mode. The expression (60) thus holds for the scattering loss of the guided modes provided that both interfaces contribute an equal amount to the power loss. Using the following expression for the power flow density in z-direction,

$$S_z = \frac{\beta_i}{2\omega\mu_0} \mid A \mid^2, \tag{61}$$

we obtain the general expression for scattering losses from a dielectric slab waveguide with sinusoidally deformed core-cladding interfaces

$$\alpha_\nu = \frac{1}{2\beta_i \mid A \mid^2 d} (\sigma_{-\nu} \mid b_{-\nu} \mid^2 + \rho_{-\nu} \mid c_{-\nu} \mid^2). \tag{62}$$

The index ν indicates the order of the scattering process. Taking $\nu = 1$, we obtain with the help of (55) the scattering loss contribution from the first-order grating lobes

$$\alpha_1 = \frac{\kappa_i^2(n_1^2 - n_2^2)k^2}{2\beta_i(\sigma_{-1} + \rho_{-1})d} a^2. \tag{63}$$

We used the fact that κ_i is real, while ρ_o is imaginary, so that we have $\mid \kappa_i + \rho_o \mid^2 = \kappa_i^2 - \rho_o^2 = (n_1^2 - n_2^2)k^2$.

The loss contribution of the second-order side lobes follows from (56) and (62) with $\nu = 2$.

$$\alpha_2 = \frac{a^4\kappa_i^2(n_1^2 - n_2^2)k^2}{32\beta_i(\sigma_{-2} + \rho_{-2})d} [4\sigma_{-1}^2 + \sigma_{-2}\rho_{-2} + (\gamma_i - 2\gamma_{-1})^2]. \tag{64}$$

The γ parameters are defined by the equations

$$\gamma_i = i\rho_o = (\beta_i^2 - n_2^2k^2)^{1/2} \tag{65}$$

and

$$\gamma_{-1} = i\rho_{-1} = [(\beta_i - \phi)^2 - n_2^2k^2]^{1/2}. \tag{66}$$

We have tacitly assumed that the first-order side lobe belongs to an evanescent wave if the loss contribution of the second-order side lobe is being considered. The parameters γ_i and γ_{-1} of (65) and (66) are consequently assumed to be real quantities. If the first-order side lobe propagates in medium 2 as a traveling wave, the second-order side lobe also gives rise to a traveling wave. However, since the contribution from the grating lobe of first order is much stronger than that from the second-order lobe, we can neglect the loss contribution (64). If first-order scattering does occur, it is the predominant effect. Only if first-order scattering contributes only an evanescent wave, and does not cause radiation loss, must the second-order loss process (64) be considered.

VI. DISCUSSION

We have calculated the loss contributions that result from the grating lobes of first and second order when a plane wave impinges on the sinusoidally deformed interface between two different dielectric media. Not all grating orders belong to traveling waves. Some grating orders cause evanescent waves in medium 2 and guided waves in the core so that they do not contribute to radiation loss of a guided wave in medium 1. The first-order radiation loss coefficient (63) is proportional to $(ak)^2$ while the second-order radiation loss coefficient (64) is proportional to $(ak)^4$. If both processes are effective simultaneously, the lower order process is dominant. Whether the second-order process is the only cause of radiation loss, or whether both first- and second-order processes are acting simultaneously depends on the magnitude of the mechanical frequency ϕ of the sinusoidal interface distortion. If γ_{-1} of (66) is real, the first-order side lobe belongs to an evanescent wave, and only the second-order side lobe is causing radiation losses. When γ_{-1} is imaginary, both first- and second-order side lobes carry away real power. Equation (64) is not applicable in this case since it was derived under the assumption that (65) and (66) are both real. However, if first-order radiation losses are possible, the second-order loss coefficient gives only a small contribution to the total radiation loss.

It is interesting to compare the result of our present theory with earlier results obtained from a modal analysis of the slab waveguide problem. From equation (79) of Ref. 1, we obtain in our present notation

$$\alpha = \frac{a^2k^2(n_1^2 - n_2^2)\kappa_i^2}{4\beta_i d\left(1 + \dfrac{1}{\gamma_i d}\right)} \left[\frac{\rho_{-1}\cos^2\sigma_{-1}d}{\rho_{-1}^2\cos^2\sigma_{-1}d + \sigma_{-1}^2\sin^2\sigma_{-1}d} + \frac{\rho_{-1}\sin^2\sigma_{-1}d}{\rho_{-1}^2\sin^2\sigma_{-1}d + \sigma_{-1}^2\cos^2\sigma_{-1}d}\right]. \tag{67}$$

Equation (67) is modified for the case that both interfaces are sinusoidally distorted, but with a random phase relationship between the two sinusoidal functions. The radiation loss (67) is considered to be an ensemble average over slab waveguides with all possible phase relationships between the sinusoidal distortions of the two interfaces. In addition, we used the expression

$$\cos^2 \kappa_i d = \frac{\kappa_i^2}{(n_1^2 - n_2^2)k^2} \tag{68}$$

which represents the eigenvalue equation of the even TE modes of the slab waveguide.

It was shown in equations (39) and (40) of Ref. 6 that the average value of the expression in brackets of eq. (67) is given by

$$2/(\sigma_{-1} + \rho_{-1}). \tag{69}$$

Combining (69) with (67) makes it apparent that the average value of the slab waveguide radiation loss is identical to the loss coefficient (63) that was derived from the plane wave model. The only remaining difference can be explained as stemming from the fact that the actual width $2d$ of the slab must be replaced by the effective slab width

$$2d\left(1 + \frac{1}{\gamma_i d}\right) \tag{70}$$

which is caused by the exponential field tail reaching out into the cladding.

The plane wave model used in this paper does not include the interference effects of waves originating from the two interfaces, and from the reflection of the scattered light from the opposite interface. But the averaging processes that were involved in converting the precise scattering loss coefficient of the slab waveguide theory into the loss coefficient derived from the plane wave model may be expected to be effective in a real waveguide. If the phase of the sinusoidal interface distortion varies slowly and randomly along the waveguide, the phase average that is already incorporated in (67) may actually occur. The average over the expression in brackets in (67) would occur either with randomly changing slab half width d, or with randomly varying mechanical frequency ϕ of the sinusoidal interface changes. The loss formula (63) is much simpler than (67), and is of actual practical value for the indicated reasons. The same averaging process involved in the first-order loss coefficient (63) is also implicit in the second-order loss coefficient (64).

Second-order scattering couples two modes ν and μ, if their propagation constants satisfy the condition

$$|\beta_\nu - \beta_\mu| = 2\phi. \tag{71}$$

Mode ν is a guided mode, while mode μ can be either a guided or a radiation mode. Equation (71) replaces eq. (1) for first-order scattering. Higher-order processes have similar coupling laws with the order of the process multiplying the mechanical frequency ϕ.

Second-order scattering losses are of importance for intentionally coupled multimode operation. Mode coupling reduces the pulse delay distortion that is caused by the different group velocities of the modes. It is possible to design the core-cladding interface irregularities in such a way that all guided modes (with the exception of the last) are coupled to each other without coupling to the continuous spectrum of radiation modes by first-order processes.[7] However, radiation losses via second- and higher-order processes are still possible. The discussion of second-order losses for intentionally coupled multimode operation is the subject of a companion paper.[5]

APPENDIX

Proof of the Relation (23)

We begin by using the fact that a sinusoidal function with an infinite argument can be regarded as zero. This assertion is the basis for the following definition of the delta function:

$$\delta(x) = \lim_{A\to\infty} \frac{1}{\pi} \frac{\sin xA}{x} \tag{72}$$

vanishes everywhere except at the point $x = 0$. The singularity of the delta function at $x = 0$ is caused by the appearance of x in the denominator. Without this denominator, we are justified to define

$$\lim_{A\to\infty} \sin xA = 0. \tag{73}$$

Using eq. (73), we write the following identity

$$\begin{aligned} 0 &= \lim_{A\to\infty} - 2i \sin [\eta f(A) + (\beta - \beta')A] \\ &= \lim_{A\to\infty} \int_{-A}^{A} \frac{d}{dz} e^{-i[\eta f(z) + (\beta-\beta')z]} dz \\ &= \int_{-\infty}^{\infty} - i[\eta f' + (\beta - \beta')]e^{-i[\eta f + (\beta-\beta')z]} dz. \end{aligned} \tag{74}$$

It was assumed that $f(-z) = -f(z)$. From the last line of (74), we obtain immediately

$$\int_{-\infty}^{\infty} f' e^{-i[\eta f + (\beta - \beta')z]}\, dz = -\frac{\beta - \beta'}{\eta} \int_{-\infty}^{\infty} e^{-i[\eta f + (\beta - \beta')z]}\, dz$$

or

$$\int_{-\infty}^{\infty} [\eta - \beta f'(z)] e^{-i[\eta f(z) + (\beta - \beta')z]}\, dz$$

$$= \frac{\eta^2 + \beta(\beta - \beta')}{\eta} \int_{-\infty}^{\infty} e^{-i[\eta f(z) + (\beta - \beta')z]}\, dz. \qquad (75)$$

From the definitions (21) and (22) it follows that (75) is identical with (23).

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