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THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 51, No. 8, October, 1972
Printed in U.S.A.

Fluctuations of the Power of Coupled Modes

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(Manuscript received May 2, 1972)

Using perturbation theory, an expression for the variance of the power of each mode of a multimode waveguide with randomly-coupled modes is derived. The variance builds up from zero to a constant value as a function of z (length along the waveguide). For most cases of interest, the variance is equal to the square of the average power. This means that the power of each mode of a system of randomly-coupled modes of a multimode waveguide fluctuates like the short-term time averaged power of a narrowband electrical signal the voltage of which is a random variable with Gaussian probability distribution.

I. INTRODUCTION

The behaviour of waves propagating in multimode waveguides can be described by coupled equations for the amplitudes of each mode.[1] This description is rigorous, but has the disadvantage that the coupled wave equations usually cannot be solved. It has been shown that a much simpler description is possible if we limit our interest to knowledge about the average power carried by each mode.[2–4] Coupled equations for the average mode power have been derived and applied to the problem of wave propagation in multimode dielectric waveguides.[4,5] However, the description of multimode waveguides in terms of average power is incomplete unless some information is available about the fluctuations of the actual power about the average value. With the help of the same perturbation approach that was used to derive the coupled power equations,[4] we derive in this paper a differential equation for the variance of the power.

The result of our perturbation theory is expressed in terms of the cross-correlation and the average power of the modes. In order to evaluate this expression, we need to make several assumptions. It has been shown in an earlier paper[5] that the average power settles down to a steady-state distribution of power versus mode number that is independent of the initial excitation of the waveguide. Using this concept of the steady-state distribution and the further assumption that the cross-correlation between the modes is small, we can solve the differential equation for the variance. We find that the variance builds up from zero values at $z = 0$, to a constant value which is equal to the square of the average of the mode power. The relative fluctuation of the power of each mode is thus 100 percent. This means that the power in each of the randomly-coupled modes behaves like the short-term time-averaged power of a narrowband electrical signal the voltage of which is a Gaussian random variable.

II. DERIVATION OF THE DIFFERENTIAL EQUATION FOR THE VARIANCE

Our starting point is the set of coupled wave equations for the slowly varying wave amplitudes (envelops) A which are defined by

$$a_\nu = A_\nu e^{-i\beta_\nu z} \tag{1}$$

with a_ν being the rapidly oscillating mode amplitude. The coupled wave equations can be expressed in the form[4]

$$\frac{dA_\nu}{dz} = \sum_{\substack{\mu=1 \\ \mu\neq\nu}}^{N} c_{\nu\mu} A_\mu e^{i(\beta_\nu-\beta_\mu)(z-z')} \tag{2}$$

z' is used as a convenient reference point. The parameters β_ν are the propagation constants of the modes. The coupling coefficient can be expressed as a product of a constant term times a function of z.

$$c_{\nu\mu} = K_{\nu\mu} f(z). \tag{3}$$

If we define

$$K_{\nu\nu} = 0, \tag{4}$$

we can drop the restriction $\mu \neq \nu$ in (2). Conservation of power leads to the relation[4] (the asterisk indicates complex conjugation)

$$K_{\mu\nu} = -K_{\nu\mu}^* . \tag{5}$$

The perturbation theory uses the approximate solution of (2)

$$A_\nu(z) = A_\nu(z') + \sum_{\mu=1}^{N} K_{\nu\mu} A_\mu(z') \int_{z'}^{z} f(x) e^{i(\beta_\nu-\beta_\mu)x}\, dx. \tag{6}$$

The power of mode ν is

$$P_\nu(z) = |\, a_\nu \,|^2 = e^{-\alpha_\nu(z-z')} \,|\, A_\nu \,|^2. \tag{7}$$

$\alpha_\nu = -2Im\beta_\nu$ is the power attenuation coefficient of mode ν in the absence of coupling. Throughout our derivation we assume that the losses are so slight that we can approximate $\exp(-\alpha_\nu(z - z'))$ by unity.

We use the loss term in (7) only to modify our equations for the lossy case.

The variance of the power of mode ν is defined as

$$(\Delta P_\nu)^2 = \langle P_\nu^2 \rangle - \hat{P}_\nu^2 , \tag{8}$$

with the simplified notation

$$\hat{P}_\nu = \langle P_\nu \rangle . \tag{9}$$

The derivative of the variance can be written with the help of (7) (replacing the exponential term by unity)

$$\frac{1}{2}\frac{d}{dz}(\Delta P_\nu)^2 = \left\{ \left\langle A_\nu A_\nu^* \frac{dA_\nu}{dz} A_\nu^* \right\rangle + \text{c.c.} \right\} - \hat{P}_\nu \frac{d\hat{P}_\nu}{dz} . \tag{10}$$

The expression c.c. indicates that the complex conjugate of the first term in the bracket must be added. The derivative of the average power has already been evaluated so that we do not need to express it in terms of the wave amplitudes. With the help of (2), (10) can be written as follows:

$$\frac{1}{2}\frac{d}{dz}(\Delta P_\nu)^2 = \{ \sum_\mu \langle A_\nu A_\nu^* A_\mu A_\nu^* f(z) \rangle K_{\nu\mu} e^{i(\beta_\nu - \beta_\mu)z} + \text{c.c.} \} - \hat{P}_\nu \frac{d\hat{P}_\nu}{dz} . \tag{11}$$

We now follow the technique that was developed in Ref. 4. We replace all the amplitudes in (11) with the approximate solution (6), but keep only terms up to second order in $K_{\nu\mu}$. The first-order terms vanish if we assume that $f(z)$ is statistically independent of $A(z')$. This assumption is justified if we let $z - z'$ be much larger than the correlation length of $f(z)$. For the same reason, we write the ensemble average of products of the field amplitudes with terms containing $f(z)$ as a product of an ensemble average containing only amplitude terms, times an ensemble average of a term that contains only $f(z)$. We thus obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dz}(\Delta P_\nu)^2 \\
&= \Bigg\{ \sum_{\mu,\delta} \Big[K_{\nu\mu} K_{\nu\delta} \langle A_\delta A_\nu^* A_\mu A_\nu^* \rangle e^{i(2\beta_\nu - \beta_\mu - \beta_\delta)z} \\
&\quad \cdot \int_{z'}^{z} \langle f(z) f(x) \rangle e^{i(\beta_\nu - \beta_\delta)(x - z)}\, dx + 2K_{\nu\mu} K_{\nu\delta}^* \langle A_\nu A_\delta^* A_\mu A_\nu^* \rangle e^{i(\beta_\delta - \beta_\mu)z} \\
&\quad \cdot \int_{z'}^{z} \langle f(z) f(x) \rangle e^{-i(\beta_\nu - \beta_\delta)(x - z)}\, dx + K_{\nu\mu} K_{\mu\delta} \langle A_\nu A_\nu^* A_\delta A_\nu^* \rangle e^{i(\beta_\nu - \beta_\delta)z} \\
&\quad \cdot \int_{z'}^{z} \langle f(z) f(x) \rangle e^{i(\beta_\mu - \beta_\delta)(x - z)}\, dx \Big] + \text{c.c.} \Bigg\} - \hat{P}_\nu \sum_\mu h_{\nu\mu} (\hat{P}_\mu - \hat{P}_\nu) .
\end{aligned} \tag{12}$$

The amplitudes A_i are understood to have the argument z'. The last term was obtained by using the lossless coupled equations for the average power derived in Ref. 4. The power coupling coefficient is defined as[4]

$$h_{\nu\mu} = | K_{\nu\mu} |^2 F(\beta_\nu - \beta_\mu). \tag{13}$$

The power spectrum $F(\beta_\nu - \beta_\mu)$ is the ensemble average of the absolute square value of the Fourier coefficient of $f(z)$.

The next step in the derivation is based on the realization that only nonoscillatory terms contribute appreciably to the growth of the variance as a function of z. We thus neglect all but the nonoscillating terms in (12). This procedure is reinforced by the fact that the ensemble averages of cross terms of amplitudes are likely to be smaller than the ensemble averages of absolute squares of the amplitudes. The first term in (12) causes some concern since it appears that there may be several combinations of μ and δ in addition to $\mu = \delta = \nu$ that contribute nonoscillatory terms. However, the uneven spacing of the modes along the β axis makes it appear unlikely that combinations of modes can be found for which the exponent of the exponential function in front of the integral vanishes. However, even if a few such combinations could be found, we could still consider the term belonging to such combinations as small because of the lack of correlation between the amplitude coefficients belonging to different modes. Since $K_{\mu\mu} = 0$, we find that the first term in (12) does not contribute appreciably to the derivative of the variance and can be neglected. The integrals can be expressed in terms of the power spectrum of the function $f(z)$ as was shown in Ref. 4. We thus obtain, with the help of (5) and (13), and dropping all oscillatory terms

$$\frac{d}{dz}(\Delta P_\nu)^2 = 2 \sum_{\mu=1}^{N} h_{\nu\mu}[2\langle P_\nu P_\mu \rangle - \hat{P}_\nu \hat{P}_\mu - (\Delta P_\nu)^2]. \tag{14}$$

Assuming that $P_\nu(z') \approx P_\nu(z)$, we use z as the argument of P_ν. Finally, we introduce losses into the theory. The example of the coupled power equations serves well to illustrate the procedure. Neglecting losses we obtain[4]

$$\frac{d\hat{P}_\nu}{dz} = \sum_{\mu=1}^{N} h_{\nu\mu}(\hat{P}_\mu - \hat{P}_\nu). \tag{15}$$

According to our derivation, we have used the approximation $\hat{P}_\nu = \langle | A_\nu |^2 \rangle$. Using (7), we obtain

$$\frac{d\langle | A_\nu |^2 \rangle}{dz} = \left(\alpha_\nu \hat{P}_\nu + \frac{d\hat{P}_\nu}{dz}\right) e^{\alpha_\nu (z-z')}. \tag{16}$$

The left-hand sides of (15) and (16) are identical according to our derivation. By substituting (16) and assuming again that $\alpha(z - z') \ll 1$, we obtain

$$\alpha_\nu \hat{P}_\nu + \frac{d\hat{P}_\nu}{dz} = \sum_{\mu=1}^{N} h_{\nu\mu}(\hat{P}_\mu - \hat{P}_\nu). \tag{17}$$

Equation (17) is identical with equation (29) of Ref. 4. There, we introduced the loss simply as a phenomenological parameter. Our present treatment shows how the loss term can be obtained directly from the derivation based on perturbation theory. By applying the same reasoning to (14), we obtain

$$\frac{d}{dz}(\Delta P_\nu)^2 = -\kappa_\nu(\Delta P_\nu)^2 + 2\sum_{\mu=1}^{N} h_{\nu\mu}[2\langle(P_\nu - \hat{P}_\nu)(P_\mu - \hat{P}_\mu)\rangle + \hat{P}_\nu\hat{P}_\mu]. \tag{18}$$

The parameter κ_ν is defined as

$$\kappa_\nu = 2\alpha_\nu + 2\sum_{\mu=1}^{N} h_{\nu\mu}\,. \tag{19}$$

We regrouped the terms under the summation sign in (18) in order to express $\langle P_\nu P_\mu\rangle$ in terms of the cross correlation $\langle(P_\nu - \hat{P}_\nu)(P_\mu - \hat{P}_\mu)\rangle$. Integration of (18) yields, finally, the desired expression for the variance of the mode power

$$(\Delta P_\nu)^2 = (\Delta P_\nu)^2_{z=o} + 2e^{-\kappa_\nu z} \cdot \int_0^z e^{\kappa_\nu x} \sum_{\mu=1}^{N} h_{\nu\mu}[2\langle(P_\nu - \hat{P}_\nu)(P_\mu - \hat{P}_\mu)\rangle + \hat{P}_\nu\hat{P}_\mu]\,dx. \tag{20}$$

Equation (20) is the solution of the variance problem. The expression in brackets under the summation sign can be positive or negative, so that the variance can increase or decrease with increasing z.

III. EVALUATION OF THE VARIANCE FOR SPECIAL CASES

In order to be able to evaluate the general expression (20) for the variance, we would need to know the cross correlation and the average power as functions of z. The average power can be obtained by solving the coupled power equations. However, the cross correlation is not known. It appears reasonable to assume that the cross correlation may be small in many cases of practical interest. One would not expect to obtain small values of the cross correlation for only two modes because as one mode gains power the other must lose an equal amount of power.

However, for large numbers of modes, it appears reasonable to expect that the cross correlation between different modes may be small.

It is known from the theory of coupled power equations that the distribution of power versus mode number settles down to a steady state.[5] Once the steady state is reached, each mode decays with the same attenuation coefficient. The shape of the distribution of average power versus mode number remains unchanged, but its level decreases exponentially with a power attenuation constant α_s. If we launch a power distribution at $z = 0$ that corresponds to the steady-state distribution, we obtain power averages that do not change with z except for a common exponential decay term. Assuming, therefore, that the cross correlation is negligible and that the steady-state power distribution is launched into the guide, allows us to solve (20) immediately. Using $\hat{P}_\nu = \hat{P}_{\nu o} e^{-\alpha_s z}$ we obtain for $(\Delta P_\nu)^2_{z=o} = 0$

$$\frac{\Delta P_\nu}{\hat{P}_\nu(z)} = \left\{ 2 \frac{1 - e^{-(\kappa_\nu - 2\alpha_s)z}}{\kappa_\nu - 2\alpha_s} \sum_{\mu=1}^{N} h_{\nu\mu} \frac{\hat{P}_{\mu o}}{\hat{P}_{\nu o}} \right\}^{\frac{1}{2}} \tag{21}$$

with κ_ν given by (19). Equation (21) represents the relative fluctuation of the power of mode ν. It shows clearly that the relative fluctuations build up from zero to a constant value which is reached when the z-dependent exponential function in (21) becomes negligibly small. The shape of the steady-state power distribution depends on the interplay between the coupling between the guided modes and the loss of power to radiation. The loss coefficient α_ν that appears in the coupled power equations (17), depends on the mode number. Usually higher-order modes suffer more losses than lower-order modes. Modes with a large loss coefficient carry only little power once the steady-state power distribution is reached. Modes with small average power are of little interest. Concentrating on those modes that carry appreciable amounts of power allows us to neglect the attenuation coefficient α_ν that appears implicitly in (21) through relation (19). If all the guided modes couple strongly to each other, they are also strongly coupled to the radiation field, thus losing a large amount of power by radiation. Since reasonably low loss operation is of most interest in practical applications, we can limit our discussion to the situation where only neighboring guided modes are coupled to each other. This means that $h_{\nu\mu}$ is small for large values of $| \mu - \nu |$. The sum over $h_{\nu\mu}$ thus extends only over those values of μ which are close to ν. The steady-state power distribution is continuous in the sense that neighboring modes carry nearly equal amounts of power. Neglecting the small steady-state loss coefficient α_s compared to the sum over the coupling coefficients $h_{\nu\mu}$, and using the fact that the power of neighboring modes is nearly equal, allows us to obtain for $z \to \infty$ from (21) the important relation

$$\frac{\Delta P_\nu}{\hat{P}_\nu} \approx 1. \tag{22}$$

IV. DISCUSSION OF THE RESULT

The relative fluctuation of the power of those modes that carry appreciable amounts of power is approximately 100 percent. Such fluctuations are not unusual, however. The short-term time-averaged power carried by a narrowband electrical signal, the voltage of which is a Gaussian random variable, is known to fluctuate in the same way. The probability distribution for P_ν can, in analogy to the electrical case, be assumed to be

$$W(P_\nu) = \frac{1}{\hat{P}_\nu} \exp\left(-\frac{P_\nu}{\hat{P}_\nu}\right). \tag{23}$$

From this analogy we can immediately state that the relative fluctuations of the power of M modes (assumed to be uncorrelated) is equal to $M^{-1/2}$.

The fluctuations that we are considering do not occur in time at the output of any given waveguide. They are fluctuations of random variables in an ensemble sense. If we were to measure the power in a given mode for each of a large number of similar waveguides, we would expect to obtain results that fluctuate according to (22). Equation (22) thus tells us the accuracy of predicting the value of the power in a given mode on the basis of the coupled power equations. Since it is very hard to measure the power carried by one individual mode of a multimode waveguide, we are more likely to observe the power P_M in a fairly large number of M modes simultaneously. In this case, we expect to obtain fluctuations according to the law

$$\frac{\Delta P_M}{\hat{P}_M} = \frac{1}{\sqrt{M}}. \tag{24}$$

It is helpful to remember that the power of all N modes does not fluctuate at all.

V. CONCLUSIONS

We have discussed the problem of the relative fluctuations of the power in individual modes of a multimode waveguide in the case that the modes are coupled by a random coupling function. Our discussion was limited to the c.w. case and does not directly apply to pulsed operation. We derived the general expression (20) for the variance of the power in terms of the cross correlation and the average power carried by the modes. Under the assumption that the modes are approximately uncorrelated among each other, and assuming further that only neighboring modes are coupled, we found that the relative fluctuations are nearly 100 percent. This result is reminiscent of the fluctuations of the short term time averaged power of a narrowband electrical signal the noise voltage of which is a Gaussian random variable.

Cross correlation between the modes can either increase or decrease the fluctuations depending on the sign of the cross correlation term in (20). It is reasonable to assume that the sign would tend to be negative. As stated earlier, coupling between only neighboring modes is necessary for low loss operation. If mode ν should, at a given point on the z axis, carry more than the average amount of power, we conclude that this power has been transferred from the neighboring mode (or modes) μ so that this mode is expected to have less than the average amount of power. The sign of the two factors in the cross correlation term must thus be different so that the term assumes a negative sign. This qualitative discussion indicates that correlations among the modes would tend to reduce the variance $(\Delta P_\nu)^2$. I have observed fluctuations of the mode power as large as those predicted by this theory in numerical solutions of coupled line equations with random, band-limited coupling function. This "experimental" result confirms the assumption that cross correlation between modes does not appreciably reduce the variance of the power fluctuations.

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