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Multidimensional Polynomial Algebra for Bubble Circuits

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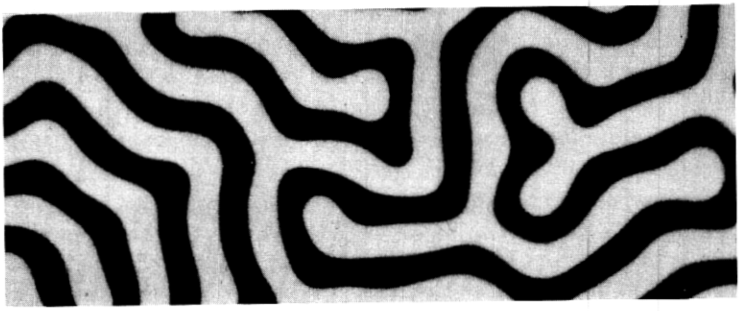
The paper expands the basic concepts of the coding theorists in the representation of data strings by algebraic polynomials, and develops the representation of both time and location of individual binary positions in the same polynomial. Further, it advances a set of algebraic operations on such polynomials to correspond to the various subfunctions that are accomplished in the actual domain circuits. The specific applications of the techniques proposed in this paper for the design and synthesis of such circuits is presented in a companion paper.

"Choose a set of symbols, endow them with certain properties and postulate certain relationships between them. Next, . . . deduce further relationships between them . . . . We can apply this theory *if* we know the "exact physical significance" of the symbols. . . . The applied mathematician always has the problem of deciding what is the exact physical significance of the symbols. *If* this is known, then at any stage in the theory we know the physical significance of our theorems. But the weakest link of physical significance is extremely fragile." The original source of this principium is J. E. Kerrick in *An Experimental Introduction to the Theory of Probability*, Belgisk Import Company, Copenhagen. It is also quoted in a slightly different form by F. M. Reza in *An Introduction to Information Theory*, McGraw-Hill Book Co., New York, 1961.

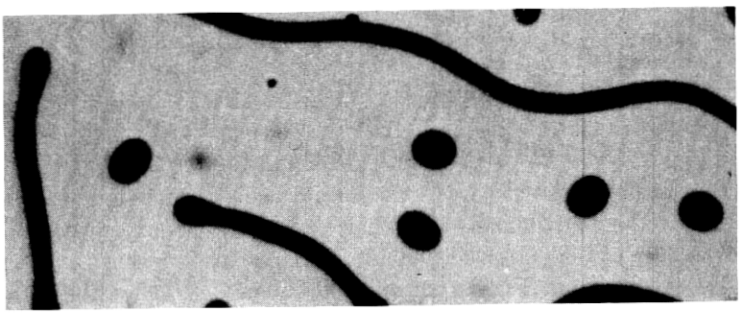
I. INTRODUCTION

Magnetic domains exist freely in thin platelets of orthoferrite crystals obtained by slicing them so that their crystalline axis is perpendicular to the surface of the platelets. Such domains are also present in very thin epitaxial garnet films (Fig. 1a) on suitable substrates. When the platelets or films are subjected to bias fields, these domains assume cylindrical shape and their diameter shrinks to microscopic sizes (Fig. 1b). Such domains (also called "bubbles") are stable under an appropriate bias field condition and they may be manipulated to perform[1,2] storage, gating, looping and also certain elementary logic functions.[3]

The domains are generally propagated from one location in the circuit to the next by subjecting them to the local bias field gradient. Basically there are two methods of providing such a field gradient to propagate the bubbles. In the "field access propagation,"[4] an alternating magnetization is imposed in a patterned soft magnetic overlay by an in-plane rotating magnetic field generated by a pair of coils carrying an alternating current. The coils completely surround the platelet with their axis in its plane. Two of the most commonly used overlay patterns are shown in Figs. 2a and b. During one cycle of the alternating current in the coils, the domains in the platelet move from one point in a pattern to the corresponding point in the adjoining pattern. This finite distance that the domain traverses during one cycle is defined as a "period." In



(a)



(b)

Fig. 1—(a) Magnetic domains as they are observed by Faraday effect in a typical epitaxial film 5 to 8 microns deep, deposited on Gadolinium-Gallium-Garnet (GGG) substrate 20 to 40 mils thick. Magnification 340. (b) Formation of "bubbles" from magnetic domains at a bias field of 30 Oe in same material used in Fig. 1a. Magnification 340.

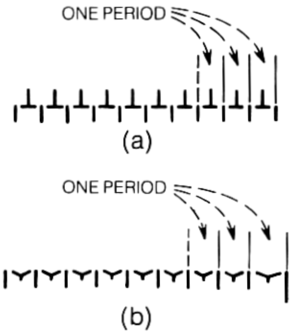


Fig. 2—(a) T-bar type of overlay used for field access propagation. (b) Y-bar type of overlay used for field access propagation.

the "conductor propagation," the local field gradient to move the bubbles is supplied by a current in a conductor (Fig. 3). A single phase current is generally used to periodically shift a bubble from one stable position to the next. Such stable positions are derived by a soft magnetic overlay also embedded on the platelet, and the periodicity of movement of the bubble from one position to the next depends on the frequency of the single phase excitation of the conductor. The distance which the bubble moves during one cycle of the single phase current is also defined as one "period."

Typical orthoferrites ($YbFeO_3$, $YFeO_3$, etc.) sustain 40 to 50 micron diameter bubbles, and the period is approximately 200 microns. Typical garnets ($Er_2Tb_1Al_{1.1}Fe_{3.9}O_{12}$ and $Gd_{2.3}Tb_{0.7}Fe_3O_{12}$) can support 4 to 8 micron diameter bubbles and the period is about 25 microns. The orthoferrites require about one micro-second to shift a bubble position by one period. The newer garnet materials also require about the same time, thus yielding a data rate of about one megacycle. It is customary to employ "bubble-no-bubble coding" with field access propagation and "lateral displacement coding" (LDC) with conductor propagation. In the former type of coding, the presence or absence of a bubble at an appropriate location denotes one or zero. In the latter type of coding, the bubble positions are coded as one or zero by laterally displacing them from one coding position to the other coding position (see Fig. 3).

All the bits of information are propagated by one period in one clock cycle in the field access propagation. In conductor drive circuits with lateral displacement coding, the average velocity during propagation is generally limited to one finite value, even though information bits are sometimes held stationary. When one finite velocity of propagation is

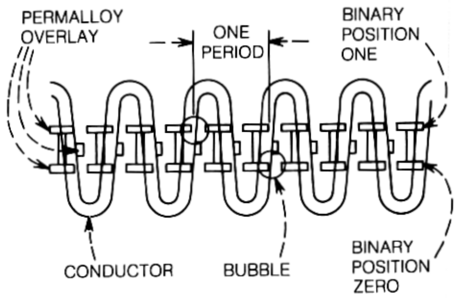


Fig. 3—Conductor propagation of bubbles.

assumed, it is possible to extend the capabilities of the conventional polynomial algebra used by coding theorists[5,6] to encompass the space-time relationships in bubble circuits. Bubble circuits perform in the time dimension and in the space dimension. The space dimension is further divided into subdimensions; the different sections (or elements) of the circuit which perform independently. Hence, any algebraic representation should encompass the representation of time and the representation of space dimensions which constitute the circuit. It is proposed that the time dimension be associated with X and the space dimension be associated with Y in the algebra.

II. DECOMPOSITION OF TIME, SPACE (CIRCUITS) AND FUNCTIONS

Consider the subclassification of time and space dimensions as follows:

The total time for a circuit to perform a function consists of a series of individual time intervals neccessary for the submodular functions. These individual time intervals can each be represented as a certain known number of clock cycles. Thus the unit of time is one clock cycle at the excitation frequency of the main field in field access propagation, or is one clock cycle at the drive circuit frequency in conductor propagation.

The physical layout of the circuit can be classified into various sections (elements such as paths, loops, functional modules, etc.). Each element further consists of a certain predefined number of periods. This leads to the unit of physical (or spatial) dimension as one period corresponding to one pole pitch in the T-bar, Y-bar or chevron pattern in field access propagation, or to one pole pitch of the driving conductor in conductor propagation.

Next consider the space-time relation. In field access propagation, the speed of the bubbles is one period per clock cycle, and is influenced by the angular velocity of the main rotating field; and stationary bubbles are rarely encountered.* In conductor propagation, the frequency of the exciting circuit determines the velocity;* and stationary bubbles are commonly encountered. However, in every circuit, over a limited duration and within a preselected element of the circuit, one can express the space-time relation with absolute certainty.

Finally, consider the overall algebraic representation. The entire circuit function is modeled by a series of algebraic operations, each one of which corresponds to a subfunction in the circuit. Each subfunction is carried out in the time dimension and in the space dimension. Hence, if we can resolve the function into its subfunctions, the circuit into its elements, and time into sets of clock cycles, and identify the individual subfunction with the circuit element and the appropriate set of clock cycles, then we can analyze and predict the functioning of the circuit with great accuracy. The representations of individual subfunctions by corresponding algebraic operations are developed in Section IV.

III. REPRESENTATION OF TIME, LOCATIONS AND BINARY VALUES OF A BIT POSITION

The origin of time may be chosen to be at any desired instant. However, a certain amount of flexibility and ease of representation results if the origin of time is chosen to coincide with a definite function in the circuit. Generally, bubble circuits perform repetitive functions and it may be convenient to choose the origin of time at the start of a repetitive cycle. When circuits perform a wide variety of nonrepetitive functions, then the analysis should be attempted for each function independently to ascertain the correct operation of each one of the functions. In the algebraic analysis of bubble circuits, it is proposed that the exponent of X (the carrier of time dimension as introduced earlier) be used to represent the number of clock cycles that have elapsed between a prechosen origin of time and the instant under consideration.

Further, it is proposed that the location of any given binary bubble position be represented by two components: (*i*) the element in which the binary position is presently located and (*ii*) the exact period in that

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* The momentary variations of bubble speed at crossovers and compressors are ignored, and the entire distance is considered as one period. The effect of corners is dealt with separately.

element at which the binary position is currently present. This leads to designating a subscript to Y (the carrier of space dimension as introduced earlier) to indicate the element number and an exponent of Y to indicate the exact location within that element.

Finally, it is proposed that the binary value of a bubble position at a given instant of time be denoted by a. The binary value changes as a bubble position passes through the known transition points in the circuit. However, at a given time, which is a certain number of clock cycles past a preselected origin of time (a known exponent of X), and at a given location (known subscript and known exponent of Y), the binary value of a bubble position is either known, or it can be determined with absolute certainty from other circuit considerations.

3.1 *Representation of an Isolated Bubble*

Examine a single binary bubble position (Fig. 4a) the binary value of which is 'a' at an instant of time j clock cycles from a prechosen origin of time within the kth element of the circuit located at the lth location. Then it may be represented as

$$u = aX^jY_k^l.$$

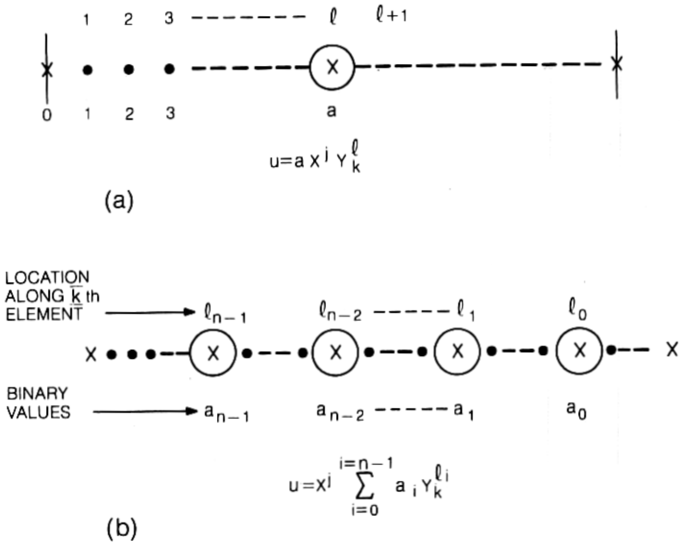

Fig. 4—(a) Representation of a single bubble. See text for explanation of j, k, and l. (b) Representation of a bubble stream j clock cycles after a prechosen origin of time.

3.2 *Representation of a Bubble Stream*

Consider a string of n binary bubble positions (Fig. 4b). Let the binary values of these positions be $a_0, a_1, \cdots, a_{n-1}$ during a clock cycle which is j clock cycles from a prechosen origin or time. If these binary positions are located in the kth element at location $l_0, l_1, \cdots, l_{n-1}$, respectively, then the string of data can be represented as

$$u = a_0 X^j Y_k^{l_0} + \cdots a_{n-1} X^j Y_k^{l_{n-1}} = X^j \sum_{i=0}^{i=n-1} a_i Y_k^{l_i}. \tag{1}$$

The sign of individual terms does not carry any significance. When the binary positions are adjacent to one another, then $l_0, l_1, \cdots, l_{n-1}$ are consecutive numbers.

Example 1: Consider four bubble positions the binary value of which is a_0, a_1, a_2 and a_3. If they occupy the 3, 2, 1, and 0 location of a sixth element after 28 cycles from a prechosen origin of time, then they may be represented as

$$u = X^{28}(a_0 Y_6^3 + \cdots a_3 Y_6^0) = X^{28} \sum_{i=0}^{i=3} a_i Y_6^{(3-i)}. \tag{2}$$

3.3 *Explanation of the Algebraic Representation*

From the point of view of circuit analysis, the algebraic representation leads to the following propositions.

(*i*) A series of bubble streams in a circuit are represented by a series of polynomials.

(*ii*) Any one bubble stream is represented by a particular polynomial (the sum of individual terms).

(*iii*) Each bit within a bubble stream is represented by a term (the product of components).

(*iv*) The binary value of the bubble position is represented by a.

(*v*) The number of clock cycles between a prechosen origin of time and the end of the cycle under consideration is the exponent of X.

(*vi*) The location of the bit of information is represented in two sections: the element within the circuit (the subscript of Y) and the location within the element (the exponent of Y).

3.4 *Implications of Representation*

The implications of the prechosen representation are:

(*i*) That a "snapshot" (i.e., a complete description of binary values of bubble positions and their respective locations) may be extracted from the algebra after a predetermined interval of time from a prechosen origin of time (a known exponent of X).

(*ii*) That there is a unique polynomial for every bubble stream in a circuit during any one cycle.

(*iii*) That all the pertinent information regarding all binary bits of information in a stream is available within the algebraic polynomial representing it.

IV. REPRESENTATION OF FUNCTIONS

4.1 *Generation of Bubble Streams*

Choose an origin of time at the end of the generation cycle of the first bubble position (i.e., as it is leaving the generator). If the generator is going to generate n binary positions the values of which are a_0, a_1, $\cdots$, and a_{n-1}, then at the end of $(n - 1)$ clock cycles, the binary string just generated may be written as

$$u = X^{n-1}(a_0 Y^{n-1} + \cdots a_{n-1} Y^0) = X^{n-1} \sum_{i=0}^{i=n-1} a_i Y^{(n-1-i)} \qquad (3a)$$

at an instant when the a_{n-1} position is just leaving the generator. The alternate representation of the string after n cycles is

$$u = X^n \sum_{i=0}^{i=n-1} a_i Y^{(n-i)}. \qquad (3b)$$

In (3a) and (3b), the exponents of Y indicate the locations along the bubble path which lead out of the generator. When the coefficients a_0, a_1, $\cdots$, a_{n-1} are consistently one, the action of an unconditional generator is represented (see Fig. 4 of Ref. 4, and Fig. 3 of Ref. 3 for T-bar and Y-bar configurations).

4.2 *Annihilation of Bubble Streams in Bubble-No-Bubble Coding*

The function of annihilation of any bubble stream u may be simply represented as

$$u = 0.$$

It is important to note that it is not identical to a polynomial in which all the binary bit positions are zero. When a conditional generator generates a string of binary bits which are all zero, it is still necessary to represent the binary string (3a) or (3b), since this string of data may interact with other strings at a later point in the circuit.

The function of a conditional annihilator may be represented as the action of a conditional gate (to be discussed later) and the action of an unconditional annihilator.

4.2.1 *Resetting of Streams in Lateral Displacement Coding*

In a functional sense, the function of resetting in Lateral Displacement Coding (LDC) is analogous to the function of annihilating in bubble-no-bubble coding. However, there is an important representational difference. The location of a bubble position which is reset in LDC is still identifiable, whereas the bubble position in field access circuits loses its identity and location upon entering the annihilator. Upon resetting in LDC, the bubble positions enter a "dead interval", or a sort of coma from the activity of the circuit, yet the algebra has to account for the elapsed time during which the bubble positions are in the reset state. Hence, a stream of n reset bubble positions in locations 0, 1, 2, $\cdots$, $n - 1$ may be represented in the usual way as

$$u = X^{j} \sum_{i=0}^{i=n-1} a_i Y^{n-1-i}, \tag{4}$$

where j represents the number of clock cycles from a prechosen origin of time, and where a_0, a_1, $\cdots$, a_{n-1} invariably represent the reset status. *Example 2:* The representation of 3 reset bubble positions in location 9, 5 and 1 after 20 clock cycles past a given time origin is

$$u = X^{20}(a_0 Y^9 + a_1 Y^5 + a_2 Y^1) = X^{20} \sum_{i=0}^{i=2} a_i Y^{(9-4i)}. \tag{4a}$$

4.3 *Functions of One Stream Resulting in One Stream*

Let u_p be the initial polynomial about to undergo the function, and let u_q be the resulting polynomial after m clock cycles. In a categorical sense, the operation may be represented as

$$u_q = F_m(u_p); \quad \text{or,} \quad F_m(u_p) \rightarrow u_q ,$$

where F may represent: temporary freezing, translation in the forward direction, translation in the reverse direction, looping, special looping used in memory operations for dynamic data allocation, etc.

4.3.1 *Temporary Freezing of Bubble Streams in their Locations*

This function is generally encountered in conductor pattern and rail propagation with lateral displacement coding. Consider an n bit data stream j clock cycles from a prechosen origin of time, and represented as

$$u_p = X^{j} \sum_{i=0}^{i=n-1} a Y_k^{l} , \tag{5}$$

where a and l are a_i and l_i respectively. After freezing the movement of the stream for m clock cycles, we have the stream represented as

$$u_q = X^m \cdot u_p = X^{i+m} \sum_{i=0}^{i=n-1} aY_k^l . \tag{6}$$

If the bubble positions are all located at adjoining locations then

$$u_q = X^{i+m} \sum_{i=0}^{i=n-1} aY_k^{\langle l \rangle - i}, \tag{7}$$

where $\langle l \rangle$ is the location l_0 of the bubble position a_0 .

4.3.2 *Movement of Bubble Streams*

This is the most common bubble function and it is necessary to establish a sense of directionality in the movement. If the bubble stream is moving so that a_{n-1} bubble position of eq. (1) occupies position l_0 after a certain number of clock cycles, then a_0 would be the leading bubble. It is easier to work with a_0 as the leading bubble,* and this implies that $l_0 > l_1 > l_2 \cdots > l_{n-1}$. If the positions are in adjoining locations, then l_{i-1} is $l_i + 1$. Consider a stream of bubbles represented by u_p in eq. (5), which has moved in the forward direction for m clock cycles, then the resulting polynomial u_q is

$$u_q = X^m Y^m \cdot u_p = X^{i+m} \sum_{i=0}^{i=n-1} aY_k^{l+m}, \tag{8}$$

where a and l as defined earlier are a_i and l_i . If the data positions are in adjoining locations with a_0 in $\langle l \rangle$, then

$$u_q = X^{i+m} \sum_{i=0}^{i=n-1} aY_k^{\langle l \rangle - i + m}. \tag{9}$$

Generally, the bubble stream crosses elemental boundaries when it moves. If this stream shifts from element k to element t during the movement, and if their intersection is located at Y_k^z and Y_t^o (Fig. 5), then

$$u_q = X^{i+m} \sum_{i=0}^{i=n-1} aY_t^{l+m-z} \tag{10}$$

* This notation helps the circuit designer to comprehend the location and movement of the leading bubble position first, rather than comprehending the location and movement of the last bubble position first.

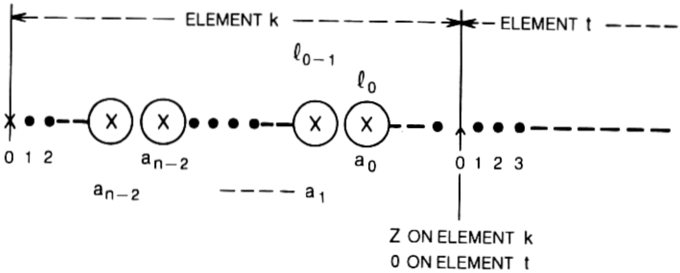

Fig. 5—Forward movement with a shift from element k to element t.

in general.* If the bubble positions are located in adjoining positions $l = l_i = l_0 - i = \langle l \rangle - i$ and

$$u_q = X^{j+m} \sum_{i=0}^{i=n-1} a Y_t^{\langle l \rangle - i + m - z}. \tag{11}$$

Example 3: The representation of 7 bubble positions, the binary positions of which are a_0 through a_6 located at the 15th, 14th, $\cdots$, 9th locations of the 3rd element of a circuit at an instant of time, 36 clock cycles from a prechosen origin of time, is

$$u_p = X^{36} \sum_{i=0}^{i=6} a_i Y_3^{15-i}. \tag{10a}$$

Further, if this stream travels from element 3 to element 4, after traversing for 20 cycles with the boundary between elements 3 and 4 being located at Y_3^{20} and Y_4^0 , then the final polynomial is represented as

$$u_q' = X^{56} \sum_{i=0}^{i=6} a_i Y_3^{35-i},$$

but

$$Y_3^{35-i} = Y_4^{35-i-20} = Y_4^{15-i},$$

* An alternate way to visualize the crossing of boundaries is to write

$$u_q' = X^{j+m} \sum_{i=0}^{i=n-1} a Y_k^{l+m},$$

and then replace $Y_k^{z+\alpha}$ by Y_t^{α}, yielding $\alpha = l + m - z$, which leads to eq. (10). The prime indicates that the polynomial as such does not represent an observable stream but will do so after the next operation (s).

and thus

$$u_q = X^{56} \sum_{i=0}^{i=6} a_i Y_4^{15-i}. \tag{11a}$$

The exponent of Y_4 becomes negative if the stream does not traverse at least 11 cycles when it is 7 bits long, or if the stream traverses for 20 cycles but is 16 bits long. It is thus possible to have the stream segmented as u_p and u_q by an improper choice of the traversing time. Sometimes negative exponents may be carried for a few steps in the analysis without interpreting polynomials as streams during these steps.

Examine a bubble stream represented by u_p in eq. (5) having moved in the backward direction for m clock cycles, then

$$u_q = X^m Y^{-m} \cdot u_p = X^{j+m} \sum_{i=0}^{i=n-1} a_i Y_k^{l-m},$$

where a and l are a_i, and l_i respectively, and a_0 is the last bubble in this case. If the bubble stream moved from element k and entirely shifted in tth element, then

$$u_q = X^{j+m} \sum_{i=0}^{i=n-1} a_i Y_t^{z-(l-m)}, \tag{12}$$

where* the intersection of elements k and t is located at Y_k^0 and Y_t^z. If the bubble positions are in the adjoining locations, then $l_i = l_0 - i$ where l_0 is the location of the bubble a_0 in u_p before its transformation to u_q.

Example 4: The representation of a 4 bubble stream, the binary values of which are a_0, a_1, a_2 and a_3 located in the 7, 5, 3, 1 periods of element 5 at an instant 35 clock cycles after a prechosen origin of time, is

$$u_p = X^{35} \sum_{i=0}^{i=3} a_i Y_5^{7-2i}. \tag{12a}$$

After 9 cycles of backward movement, if the stream is in element 3 with the boundary of elements 5 and 3 located at Y_5^0 and Y_3^{20}, then

$$u_q = X^{44} \sum_{i=0}^{i=3} a_i Y_3^{18-2i}. \tag{12b}$$

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* An alternate way to visualize the crossing of the boundary is to write

$$u_q' = X^{j+m} \sum_{i=0}^{i=n-1} a_i Y_t^{l-m},$$

and then replace $Y_k{}^\alpha = Y_t{}^{z+\alpha}$, when α is ≤ 0, thus yielding $z + \alpha = z + (l - m)$ which leads to eq. (12).

4.3.3 *Looping of Bubble Streams*

Looping is generally encountered in T-bar or Y-bar propagation when a fixed delay or storage is necessary for bubble streams without actually freezing their movement.

Consider a bubble stream represented by u_p in (5), which is circulating in a loop with g periods* for m clock cycles, then

$$u_q = X^m Y^m \cdot u_p = X^{i+m} \sum_{i=0}^{i=n-1} aY_k^{l+m},$$

but in a loop $(l + m = (l + m) \bmod g)$ thus leading to

$$u_q = X^{i+m} \sum_{i=0}^{i=n-1} aY_k^{(l+m) \bmod g}, \tag{13}$$

where a and l are read as a_i and l_i respectively, and $l_i = (l_0 - i)$ if the bubble positions are in the adjoining location with a_0 as the leading bubble.

Example 5: The representation of 26 bubble positions† $a_0, a_1, \cdots, a_{25}$, occupying locations 28, 27, $\cdots$, 3 in a loop (element 5) at an instant 1214 clock cycles from a prechosen origin of time is

$$u_p = X^{1214} \sum_{i=0}^{i=25} a_i Y_5^{28-i}. \tag{13a}$$

After looping for 980 clock cycles, the final representation is

$$u_q = X^{2194} \sum_{i=0}^{i=25} a_i Y_5^{(1008-i) \bmod 39} = X^{2194} \sum_{i=0}^{i=25} a_i Y_5^{33-i}. \tag{13b}$$

4.3.4 *Movement Around Corners*

Corners‡ in T-bar and Y-bar circuits need special attention since bubble positions lose or gain a quarter period when a 90-degree turn is present. In most cases, their effect may be eliminated by considering a movement to span two or an even number of compensating 90-degree corners. However, when it is necessary to predict the movement of

* The number of T-bar periods should be considered as the number of clock cycles to bring back the leading bubble to its original location in the loop. The orientation of the T-bars and the direction of rotation both play an important part in the determination of the number. In any case, the value of g is the actual number of T-bars ± 1 depending on the orientation of T-bars and the direction of rotation.

† One encounters this bubble stream in (39, 26) the shortened BCH encoder with magnetic domains constructed along the same principles as discussed in Section 3 of Ref. 7.

‡ When the effect of the corners in a loop is being considered, the value of k used should be (k mod 4), and the number of periods in the loop should be measured as indicated in Sec. 4.3.3. Else the value of k should be its real value.

streams which include k number of 90-degree unidirectional corners, then a polynomial u_p in eq. (5) becomes

$$u_q = X^m Y^{m \pm k/4} \cdot u_p , \tag{14}$$

where the sign is determined by the direction of the turn with respect to the direction of rotation of the main driving field and the orientation of the T-bars.

No special algebraic consideration is necessary in conductor pattern and rail propagation.

Example 6: Consider the two bubble positions represented as

$$u_p = X^{20}(a_0 Y_4^7 + a_1 Y_4^6). \tag{14a}$$

If these positions traverse for 13 cycles, and encounter three unidirectional -90-degree turns in the path, then the final polynomial is

$$u_q = X^{33} Y^{(13-\frac{3}{4})} \cdot u_p = X^{33}(a_0 Y_4^{19\frac{1}{4}} + a_1 Y_4^{18\frac{1}{4}}). \tag{14b}$$

The fractional exponent of Y indicates that the binary positions a_0 and a_1 are lagging 270-degrees behind their corresponding positions had there been no turns. If it is necessary to bring them back in phase, then an additional $\frac{3}{4}$ clock cycle is required, and the polynomial u_q would be

$$u_q = X^{33\frac{3}{4}} (a_0 Y_4^{20} + a_1 Y_4^{19}). \tag{14c}$$

4.3.5 *Routing of Bubble Streams*

This function plays a critical role when it is necessary to transfer streams of binary information into a certain branch of a circuit at a node. The algebraic representation after this function is identical to the polynomial u_q in eq. (10), when the polynomial crosses the boundary of one element k at Y_k^z and enters another element t at Y_t^0. In the general case, when the bubble stream is channeled from an element p at Y_p^z, and enters another element t at $Y_t^{(z)}$, u_q may be represented as

$$u_q = X^{i+m} \sum_{i=0}^{i=n-1} a_i Y_t^{l+m-z+(z)}, \tag{15}$$

where a and l as defined earlier are a_i and l_i respectively.

4.3.6 *Inverting the Binary Content of Bubble Streams*

Consider a polynomial

$$u_p = X^i \sum_{i=0}^{i=n-1} a_{ip} Y_k^l . \tag{16}$$

If the bubble stream so represented has gone through an inverting gate during the following m cycles, then the resulting polynomial may be written as

$$u_q = X^{j+m} \sum_{i=0}^{i=n-1} a_{iq} Y_k^{l+m}, \tag{16a}$$

where l denotes l_i and

$$a_{iq} = \bar{a}_{ip} .$$

Example 7: Consider 3 data positions in an LDC circuit. If a_0, a_1, a_2 denote their status at an instant 49 clock cycles from a time origin, and are located in the seventh element at locations 17, 13 and 9 then,

$$u_p = X^{49} \sum_{i=0}^{i=2} a_{ip} Y_7^{17-4i}. \tag{16b}$$

If these binary positions pass through an inverting gate located at Y_7^{20} and Y_9^3, then the bubble stream after 15 clock cycles is

$$u_q = X^{64} \sum_{i=0}^{i=2} a_{iq} Y_9^{15-4i}. \tag{16c}$$

4.3.7 *Opening and Closing Gaps in Bubble Streams*

This function is quite effectively used in dynamic data reallocation with T-bars. The bubble stream in the loop has two preferred paths, one for each direction of rotation. In one direction, the stream traverses $(n + 1)$ periods in the loop, and in the other, it traverses (n) periods in the loop. (See Fig. 6). If after z clock cycles of clockwise movement of a bubble stream in the loop the direction is reversed for one clock cycle, then the data bit at the nth (Fig. 6) location is at the 0th location, and the data position which was at $((z - 1) \bmod n)$ location is at the nth location. Now z clock cycles of anticlockwise rotation would have effectively included the data position at the nth bit at the $(z \bmod (n + 1))$ location, and all the remaining positions one location behind their original locations. A converse process takes place for excluding a data bit position in the stream.

The algebraic equivalent of this function may be represented as follows:

Let the numbering of the locations in the loop be in the direction (anticlockwise in Fig. 6) which permits the stream to traverse n periods in the loop. Let the contents of the loop (without the nth position) be represented as

$$u_p = X^i \sum_{i=0}^{i=n-1} a_i Y^i. \tag{17}$$

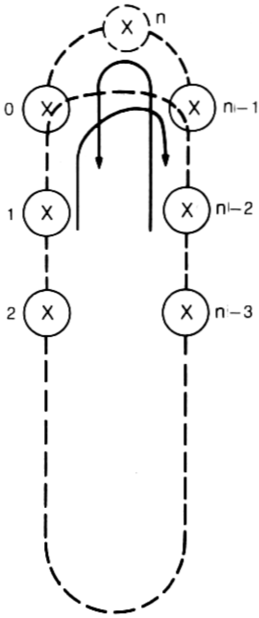

Fig. 6—Schematic representation of the dynamic data reallocation function.

The polynomial u_p^* undergoes a shift in clockwise direction for m (with $m < n - 1$) clock cycles. The resulting polynomial u_q becomes

$$u_q = X^{i+m} \sum_{i=0}^{i=n-1} a_i Y^{(i-m) \bmod n}. \tag{18}$$

As a distinct step from the above representation, let

$$u_p = X^i \sum_{i=0}^{i=n} a_i Y^i \tag{19}$$

represent the contents of the loop including the nth location before a series of counter clockwise shifts for m clock cycles. The resulting polynomial is

$$u_q = X^{i+m} \sum_{0}^{n} a_i Y^{(i+m) \bmod (n+1)}. \tag{20}$$

Example 8: Consider a string of bubbles the binary values of which are a_0, a_1, $\cdots$, a_7 at an instant 92 clock cycles from a time origin located at 0, 1, 2, $\cdots$, 7 in a memory loop (Fig. 6) with 255 (i.e., $n - 1$)

* There is no need for a subscript for Y since only one element (the loop) is being considered.

periods in the clockwise direction and 256 periods in the anticlockwise direction. This can be represented as

$$u_p = X^{92} \sum_{i=0}^{i=7} a_i Y^i. \tag{17a}$$

Let the 256th (i.e., nth) position contain the bubble position a_n at the same instant of time. Four cycles of rotation of the field causing a clockwise circulation of binary positions yields

$$u_q = X^{96}\left\{\sum_{i=0}^{i=3} a_i Y^{252+i} + \sum_{i=4}^{i=7} a_i Y^{i-4}\right\}. \tag{18a}$$

In the first four terms in the polynomial (38), $(i - 4) \bmod 256$ would correspond to 252, 253, 254 and 255 for i ranging from 0 to 3.

Now if the field is rotated for 5 clock cycles in the opposite direction, resulting in an anticlockwise shift of bubble positions, then the resulting polynomial is

$$u = X^{101}\left\{\sum_{i=0}^{i=3} a_i Y^{(257+i) \bmod 257} + a_n Y^4 + \sum_{i=4}^{i=7} a_i Y^{i+1}\right\},$$

or

$$u = X^{101}\left\{\sum_{i=0}^{i=3} a_i Y^i + a_n Y^4 + \sum_{i=4}^{i=7} a_i Y^{i+1}\right\}. \tag{20a}$$

In effect it is seen that the bubble position after the nth location has been inserted at the 4th location in the bubble stream.

4.4 *Functions of One Stream Resulting in Two or More Streams*

Let u_p be the initial polynomial about to undergo the function resulting in two polynomials u_q and u_r. In general, functions of this type may be represented as

$$u_q + u_r = F_m(u_p); \quad \text{or,} \quad F_m(u_p) \rightarrow u_q + u_r,$$

where F* represents duplication or addressing sections of the initial polynomial u_p into one or the other branch of a circuit. A finite number of clock cycles (m) are allowed during which the operation takes place.

4.4.1 *Duplication and Replication of Bubble Streams*

Duplicators may require a certain finite number of clock cycles to

* These functions are used extensively in the companion paper[8] dealing with the applications of the algebra.

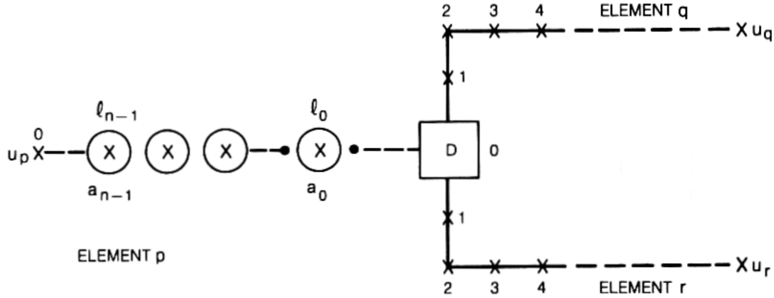

Fig. 7—The bubble stream u_p entering the duplicator D.

operate. In the T-bar propagation, one clock cycle is necessary to duplicate. The effect may easily be included in the algebra by considering the duplicator as an extra period of element p with u_p (see eq. 5), and the zero location of elements q and r in which u_q and u_r will be positioned after m clock cycles. (See Fig. 7).

Consider a string of binary data represented as u_p in eq. (5) to be duplicated during the m clock cycles yielding u_q and u_r represented as

$$u_q = X^m Y^{m-d} \cdot u_p = X^{i+m} \sum_{i=0}^{i=n-1} a Y_q^{l+m-z-d} \tag{21}$$

$$u_r = X^m Y^{m-d} \cdot u_p = X^{i+m} \sum_{i=0}^{i=n-1} a Y_r^{l+m-z-d}, \tag{22}$$

where a and l are a_i and l_i respectively, the duplicator is located after z periods in k, and d is the number of clock cycles to accomplish the duplication. It is to be noted that when the bubbles occupy adjoining locations, d cannot exceed 1 for satisfactory duplication. Under such conditions*

$$u_q = X^{i+m} \sum_{i=0}^{i=n-1} a Y_q^{\langle l \rangle -i+m-z-1} \tag{23}$$

and a similar expression for u_r, where $\langle l \rangle$ denotes l_0.

* An alternate way to visualize this transformation is to consider that the duplicator is located at the intersection of Y_p^{z+1}, Y_q^0 and Y_r^0 which leads to $Y_p^\alpha = Y_q^{\alpha-(z+1)} = Y_r^{\alpha-(z+1)}$, when $\alpha > z + 1$, thus yielding

$$u_q' = X^{i+m} \sum_{i=0}^{i=n-1} a Y_{p^{\langle l \rangle -i+m}},$$

where $\langle l \rangle$ is the location l_0 of the first bubble position a_0 prior to duplication. This leads to u_q in eq. (23).

Replication leads to a series of resulting polynomials u_q, u_r, $\cdots$, etc., but the algebraic treatment is exactly the same.

4.4.2 *Addressing Sections of a Stream into Different Branches of a Circuit*

Consider a polynomial u_p in element p which will approach a gate. The gate operating for g clock cycles will address the first n' bubble positions in q and the rest into r. This function can be visualized as the effect of two independent translatory functions: (see Section 4.3.1) (*i*) the first n' positions are translated in the forward direction and change elements from p to q, and (*ii*) the remaining positions are translated in the forward direction and change element from p to r. Algebraically this can be expressed as follows:

$$u_p = X^i \sum_{i=0}^{i=n-1} aY_p^l = X^i \sum_{i=0}^{i=n'-1} aY_p^l + X^i \sum_{i=n'}^{i=n-1} aY_p^l \tag{24}$$

with

$$u_q = X^{i+m} \sum_{i=0}^{i=n'-1} aY_q^{l+m-z} \tag{25}$$

and

$$u_r = X^{i+m} \sum_{i=n'}^{i=n-1} aY_r^{l+m-z}, \tag{26}$$

where a and l are a_i and l_i respectively (see Section 5.3.1). The gate should be located at Y_p^z, Y_q^0 and Y_r^0. It can be seen that the first bubble position does not reach the gate till $(z - (l_0 + 1))$ clock cycle, and to divert the first n' positions the gate should be operating to divert into element q for exactly $(l_0 - l_{n'} + 1)$ clock cycle, leading to the design detail that the gate should divert into q for g clock cycles where $g = (z - l_{n'} + 1)$. If the bubble positions are in the adjoining location, then the gate has to operate diverting into q for n' clock cycles starting after $(z - (l_0 + 1))$ clock cycles. Further, it has to act for the next $(n - n')$ clock cycles to divert the bubble position into r.

When a gate addresses various sections of a data stream into more than two elements, the algebraic representation is similar. Such a condition exists if m is not chosen large enough in the previous case, and then there will be two bubble strings u_q and u_r together with a section of u_p, which has not been processed by the gate.

Example 9: Consider a data stream* in element 0 of a circuit. Eight data

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* Such a data stream is encountered in general rate change circuits represented in Fig. 1 and 1a of Ref. 9.

bits a_0 through a_7, 12 clock cycles after a prechosen origin of time, occupy location Y_0^{12} through Y_0^5. The polynomial representing the stream is

$$u_p = X^{12} \sum_{i=0}^{i=7} a Y_0^{12-i}. \tag{24a}$$

If the data stream passes through a gate at Y^{13}_0 which diverts the first 4 bits into element Y_1 and the last four into Y_2, then after 14 clock cycles the resulting polynomial u_q in 1 and u_r in 2 may be represented as

$$u_q = X^{26} \sum_{i=0}^{i=3} a_i Y_1^{13-i}, \tag{25a}$$

and

$$u_r = X^{26} \sum_{i=0}^{i=3} a_{i+4} Y_2^{9-i}. \tag{26a}$$

4.5 *Functions of Two or More Streams Resulting in One Stream*

Consider two streams u_p and u_q (Fig. 8) interacting to yield one stream u_r at a gate ⓖ, where ⓖ may denote the function of logical gating, combining, etc. This function is different from the previous functions, since the individual binary values a_i in the polynomials are likely to be changed by this function. Let

$$u_p = X^i \sum_{i=0}^{i=n-1} a_{ip} Y_p^{l(ip)}, \tag{27}$$

and

$$u_q = X^i \sum_{i=0}^{i=n'-1} a_{iq} Y_q^{l(iq)}, \tag{28}$$

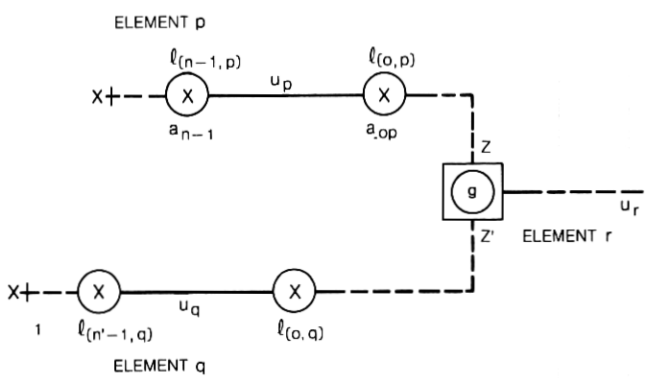

Fig. 8—Gating or merging of u_p and u_q to yield u_r.

where $l\langle ip\rangle$ and $l\langle iq\rangle$ represent l_{ip}, and l_{iq} respectively. The number of locations including the first and the last binary position of u_p is $(l_{0p} - l_{(n-1,p)} + 1)$; (i.e., n if adjoining locations). It can be seen (Fig. 8) that the length of the resulting bubble stream, u_r, is the largest number n'' of the following four numbers*

$$(i) \qquad l_{0p} - l_{(n-1,p)} + 1,$$

$$(ii) \qquad l_{0q} - l_{(n'-1,q)} + 1,$$

$$(iii) \qquad (l_{0p} - l_{(n-1,p)}) + 1 + z - l_{0p} - (z' - l_{0q}) = (z - l_{(n-1,p)}) - (z' - l_{0q}) + 1,$$

or

$$(iv) \qquad (l_{0q} - l_{(n'-1,q)}) + 1 + z' - l_{0q} - (z - l_{0p}) = (z' - l_{(n'-1,q)}) - (z - l_{0p}) + 1,$$

where z and z' denote the number or periods along elements p, and q at which the gate ⓖ is located (see Fig. 8). The polynomial u_r can be written as

$$u_r = X^{i+m} \sum_{i=0}^{i=n''-1} a_{ir} Y_r^{l\langle ir\rangle}, \tag{29}$$

where $l\langle ir\rangle$ represents l_{ir} respectively, and

$$a_{ir} = a_{ip} \circledS a_{iq} \tag{30}$$

$$l_{ir} = l_{ip} + m - z, \tag{31a}$$

or

$$l_{ir} = l_{iq} + m - z'. \tag{31b}$$

The subscripts for a and l should be chosen with adequate care to consider only the interacting binary positions in streams u_p and u_q that pass through the gate ⓖ simultaneously. It is important to note that each term in u_r results from a term in u_p, or in u_q, or from terms in both. When there is no term in one polynomial (u_q or u_p) corresponding to a particular term in the other (u_p or u_q), then the appropriate equation (31a or b) for the exponent of Y should be chosen. When there is a term in one polynomial (u_q or u_p), corresponding to a given term in (u_p or u_q),

* (*i*) or (*ii*) indicates the length u_p or u_q with the longer stream u_p or u_q completely overlapping u_q or u_p respectively. (*iii*) or (*iv*) indicate partial overlap, with u_q or u_p being nearer the gate.

then the location calculated from l_{ip} or l_{iq} will yield the same result for l_{ir}.

Example 10: Consider a bubble stream u_p with four binary bits a_{0p}, a_{1p}, a_{2p} and a_{3p}, after 41 clock cycles from an origin of time located at 3, 2, 1 and 0 of element 5 and represented as

$$u_p = X^{41} \sum_{i=0}^{i=3} a_{ip} Y_5^{3-i}. \tag{27a}$$

Consider a next stream u_q with seven bubble positions u_{0q} through u_{6q}, also 41 clock cycles from the prechosen origin of time occupying 10, 9, 8, $\cdots$, 4th locations of element 6, and represented as

$$u_q = X^{41} \sum_{i=0}^{i=6} a_{iq} Y_6^{10-i}. \tag{28a}$$

If u_p and u_q approach an exclusive-or gate ⓖ located at Y_5^6, Y_6^{11}, and Y_7^0, then the polynomial u_r after 12 cycles can be derived as follows: a_{0q}, a_{1q} pass through the gate during the first three cycles while u_p is still approaching the gate. The binary positions a_{2q} through a_{5q} interact with a_{0p} through a_{3p} in the gate ⓖ for the next four cycles. a_{6q} passes through the gate and the gating is now complete. During the last five cycles, the bubble stream in element Y_7 moves away from the gate thus leading to the resulting polynomial

$$u_r = X^{53} \sum_{i=0}^{i=6} a_{ir} Y_7^{11-i}, \tag{29a}$$

where a_{0r}, a_{1r} and a_{6r} are a_{0q}, a_{1q} and a_{6q} respectively, and

$$a_{ir} = a_{ip} \oplus a_{iq} \; (i = 2 \text{ through } 5). \tag{30a}$$

This example corresponds to the case (*i*) in Section 4.5.

4.6 *Functions of Two or More Streams Resulting in Two or More Streams*

This function, though rarely encountered in normal bubble circuits, can still be conveniently represented as an integral procedure of many subfunctions in which two or more streams result in one stream. If g_1, g_2, g_3 $\cdots$ are individual functions yielding streams 1, 2, 3 $\cdots$, etc., then the algebraic representation of Section 4.5 may be extended to represent streams 1, 2, 3 $\cdots$, etc. One such example is presented in Section 2 of Ref. 8.

V. OVERALL BUBBLE CIRCUIT FUNCTIONS

We have a set of mathematical tools to predict the binary values and locations of individual bubble positions as the binary streams undergo different submodular functions within the circuit. The interval of time chosen for these submodular functions in Section IV is 'm' clock cycles. To study the overall circuit function, the function is divided into a series of submodular functions (F_1, F_2, $F_3 \cdots$); the circuit which performs the functions is divided into a series of elements

$$(1, 2, 3 \cdots, k, k + 1, \cdots),$$

and the time necessary to accomplish the function is divided into a series of (m_1, m_2, $m_3 \cdots$, etc.) clock cycles.

5.1 *Subdivision of Circuit into Elements*

After isolating the subfunctions within the overall circuit function, the elements that accomplish these subfunctions may be identified. A series of fine functional subdivisions may be necessary to identify the particular circuit elements.

Some of the examples of the elements are storage paths, transmission paths, loops, etc. Sometimes a particular element (k) of a circuit designed is very short, and it cannot accommodate the entire string of data. When it is still desired to study the contents of the n bit binary string of the polynomial u_p in that element, then it is possible to fictitiously extend the element to just accommodate the n data bits. With the observer located at a preselected period (say Y_k^b) in the element, the binary values of data which flow past this location would still be the values of a_0, a_1, $a_2 \cdots a_n$ in the calculated polynomial u_k. The instant of incidence of the leading bubble a_0 would be $(n - b)$ clock cycles prior to its value as predicted by the exponent of X(i.e., j_0) in the polynomial u_k.

5.2 *Subdivision of Time*

Subfunctions are accomplished by elements within specified intervals of time. The interval of time for a specific subfunction is almost entirely determined by the circuit parameters and the clock frequency. Any one particular interval of time may, however, be conveniently expressed as a certain number of clock cycles.

These different values of clock cycles m_1, m_2, m_3, $\cdots$, etc., are necessary to calculate the polynomials u_1, u_2, u_3, $\cdots$, etc., and they in turn uniquely define the values, intervals and positions of binary bits in the circuit. Generally, the subfunctions may proceed in series or in parallel, and different bubble streams may simultaneously undergo different functions in different elements of the circuit. However, the complete function of a circuit starts and finishes at an instant of time. Hence, whenever subfunctions are accomplished in parallel, it should be realized that the summation of m_i does not equal the total number of clock cycles necessary for the complete circuit function. Each subfunction proceeding simultaneously should be modelled independently by its corresponding algebraic operation.

VI. CONCLUSIONS

The operation of bubble circuits may be effectively analyzed by multidimensional polynomial algebra without actually constructing the circuits. The location of all the data positions in the circuit can be accurately predicted at any preselected instant of time during the operation of a circuit by this technique. When all the circuit parameters are not accurately known, the analysis helps in the calculation of some of the circuit parameters. Further, it helps to algebraically check the validity and effectiveness of a conceived circuit in the performance of specified functions.

The algebra may also be used for circuits that do not perform instantly but need a certain predetermined duration for movement, duplication, gating, etc. Several technologies (magnetic domain, charge coupled and charge transfer technologies) presently being developed in the Bell System fall into this category.

VII. ACKNOWLEDGMENT

The author thanks K. M. Poole for the numerous discussions while the ideas presented in this paper were being developed, and in particular for his suggestion to select one common origin of time, while the author was proposing a multi-origin system. Both systems yield the same results but in a slightly different format.

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