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On the Addressing Problem of Loop Switching

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R. L. Graham and H. O. Pollak recently proposed an addressing scheme to J. R. Pierce's Loop Switching Network–a network system which would exploit the one-way nature of much data transmission. Our paper provides a generalization of the scheme and proposes a new approach to certain unanswered questions.

I. INTRODUCTION

Recently, J. R. Pierce[1] proposed a data communication system concept whereby subscribers are placed on interconnected "loops". A message originating in one loop and destined for a recipient in another loop must find its way through the network of interconnections. Such a system is potentially efficient for data communication because it exploits the one-way nature of many such transmissions by eliminating the need to set up an entire line between two parties in the conventional manner.

The realization of this potential efficiency depends in part on the existence of an efficient routing scheme. Such a scheme is discussed by Graham and Pollak in Ref. 2. Briefly, they assigned to each loop a ternary address; a message destined for a certain loop is tagged with the address of its destination, and upon entering a loop, a computation

involving that loop address and the message destination address yields the distance of the message from its destination. The message then moves in such a way as to decrease this distance. In particular, it is assumed that there are n loops and each address is a sequence of N (fixed) 0's, 1's and d's; let H be the $n \times N$ matrix the ith row H_i of which is the address of the ith loop. The addresses are chosen in such a way that if D_{ij} is the distance between the ith and jth loop (by D_{ii} is meant the fewest number of loops which it is necessary to cross to to get from the ith to the jth), then

$$D_{ij} = \sum_{k=1}^{N} (H_{ik} \oplus H_{jk}), \quad \text{where } a \oplus b = \begin{cases} 1 \text{ if the set } \{a, b\} = \{0, 1\} \\ 0 \text{ otherwise} \end{cases}$$

(this is an "extended" Hamming distance between H_i and H_j).

Alternatively, the problem may be considered in graph-theoretic terms: associate with the loop network a graph the vertices of which correspond to the loops and the edges of which correspond to direct connections between two loops. Then D_{ij} is the minimal number of edges in a path between the ith and jth vertices.

Central to the efficiency of the above scheme was the possibility of keeping N small; if G is a connected graph on n vertices, then Graham and Pollak showed that an address matrix H with corresponding N–call it N_H–could always be found to satisfy $N_H \leqq (n-1) \times \text{diam } G$ (diam $G = \max_{i,j} D_{ij}$). Unfortunately, $(n-1) \times$ diam G can be unpleasantly large. However, they conjectured that, in fact, for any connected graph G, one could always find an H matrix such that $N_H \leqq n - 1$. This was proved in the case of complete graphs, trees and cycles. Furthermore, defining $N(G) = \min_H N_H$, they showed that, $N(G) = n - 1$ for G a cycle on an odd number of vertices, a complete graph or a tree; and $N(G) = n/2$ for G a cycle on an even number of vertices. In any case, it was shown that $N(G) \geqq \max \{n^+, n^-\}$ where $n^+(n^-)$ is the number of positive (negative) eigenvalues of the distance matrix D with elements D_{ij} defined above.

For a graph G (which we will always assume to be connected) referred to implicitly or explicitly, let us reserve the letter D to denote the distance matrix of G, and n to denote the number of vertices of G. Then the conjecture of Ref. 2 becomes

$$N(G) \leqq n - 1. \tag{1}$$

Incidentally, this conjecture is equivalent to the following conjecture: that any graph G on n nodes can be embedded in a "factor" F of an $(n-1)$-cube in such a way that the distance matrix for G is a submatrix

of that for F; by a "factor" of a cube C is meant a graph F defined as follows: Let $\mathfrak{F} = \{f_1, \cdots, f_k\}$ be a set of pairwise disjoint faces (of various dimensions) of C; let V be the set of vertices of C appearing in no f_i. Then the set of vertices of F is $V \cup \mathfrak{F}$ (i.e., the vertices V of C along with the faces of $\mathfrak{F}$). There is an edge between two vertices x, $y \in V \cup \mathfrak{F}$ if:

(i) $x, y \in V$ and x is connected to y by an edge in C;
(ii) $x, y \in \mathfrak{F}$ and some vertex of (the face) x is connected by an edge to some vertex of (the face) y;
(iii) $x \in V$, $y \in \mathfrak{F}$ and some vertex of (the face) y is connected by an edge to x.

More descriptively, F is formed from C by shrinking each f_i to a vertex.

In this paper, we discuss the conjecture (1) by formulating the addressing problem in matrix-algebra terms and then present an algebraic generalization of this that immediately yields an addressing scheme more efficient than that of Ref. 2.

Specifically, we show that a graph G has an $n \times N$ address matrix H if and only if there exist $n \times N$ binary (0, 1)-valued matrices A and B such that

$$D = AB^t + BA^t \tag{2}$$

(where t means "transpose"). Using this, we expand the class of graphs known to satisfy (1). Then we rewrite (2) as

$$D = [A \vdots B]\begin{bmatrix} B^t \\ \cdots \\ A^t \end{bmatrix} \tag{3}$$

(the product of the two pairs of concatenated matrices). Note that the problem of finding $n \times N$ matrices A and B satisfying (2) [and (3)] can be generalized to the problem of finding $n \times 2N$ binary (0, 1)-valued matrices P and Q satisfying

$$D = PQ^t. \tag{4}$$

Clearly, any decomposition (2) yields the corresponding decomposition (4). On the other hand, any decomposition of the form (4) with P and Q (binary-valued) $n \times L$ matrices gives rise to a binary addressing of length L which, as in Ref. 2, will enable a message to compute its distance from its destination. In particular, the ith vertex is tagged with Q_i, the ith row of Q, while a message destined for the ith vertex is tagged with P_i, the ith row of P. Then the message, at vertex j, makes the

computation $P_i Q_j^t = D_{ij}$ and determines its distance from its destination. This computation involves summing L products, each of the form $a \cdot b$ where $a, b \varepsilon \{0, 1\}$, as compared to summing N sums of the form $a \oplus b$ defined above where $a, b, \varepsilon \{0, 1, d\}$. Note that two sets of addresses are required: the set $\{Q_i\}$ is stored in the network, and members of the set $\{P_i\}$ are tagged onto messages. But users of the network only need a directory of the P_i's.

Hence, assuming that the addresses of Ref. 2 are to be transmitted in binary, the decomposition (4) gives rise to no larger, and in certain cases, smaller addresses than those derived in Ref. 2. If $L(G)$ is the minimum value of L over all such decompositions (4), then corresponding to the conjecture (1) [and a necessary consequence, if (1) is true] we conjecture that

$$L(G) \leqq 2(n - 1). \tag{5}$$

In any case, we show that

$$L(G) \leqq \sum_i \max_i D_{ij} \leqq n(\text{diam } G.) \tag{6}$$

This improves on the upper bound derived in Ref. 2, which after translation of addresses into the form of (3), gives $L \leqq 2(n - 1)$ diam G.

Using the decomposition (4) which yields (6), we discuss a secondary coding of the addresses which reduces the number of bits actually transmitted; by using this decomposition rather than one with possibly smaller L, we achieve an addressing scheme the addresses of which are very easy to compute and which is hence amenable to continual updating of the loop network.

II. EQUIVALENT FORMULATIONS OF THE GRAHAM–POLLAK SCHEME

Here we present three theorems: the first characterizes in various terms the property that a graph G admits an $n \times N$ address matrix H, the second and third present some sufficient (but not necessary) conditions for (1) to hold.

We will call a matrix *binary-valued* if its entires are either 0 or 1. Clearly, G admits an $n \times N$ H if and only if $N \geqq N(G)$.

Theorem 1: Let D be the distance matrix of a graph G on n vertices. Then the following conditions are equivalent:

(*i*) $N \geqq N(G)$;

(*ii*) *There exist binary-valued* $n \times N$ *matrices* A *and* B *such that* $D = AB^t + BA^t$;

(*iii*) *There exist* $n \times N$ *matrices* C *and* E, C *binary-valued and* E *with entries* $0, \pm 1$ *such that* $C_{ij} = |E_{ij}|$ *and* $2D = CC^t - EE^t$.

(*iv*) *There exist* $n \times n$ *matrices* F_i, $i = 1, \cdots, N$ *with each* F_i *binary valued, symmetric, rank* $\leqq 2$, *and* $D = \sum_{i=1}^{N} F_i$.

Proof: (*i*) $\Rightarrow$ (*ii*): Since $N \geqq N(G)$ there exists for G an $n \times N$ address matrix H. By construction we know that D_{ij} is the extended Hamming distance between H_i and H_j, which is in fact the number of positions where H_i and H_j differ without either being a d. Define for each H_i a binary-valued N-vector (i.e., $1 \times N$ matrix) A_i by $A_{ik} = 1$ if $H_{ik} = 1$ and $A_{ik} = 0$ otherwise, and define a binary-valued N-vector B_i by $B_{ik} = 1$ if $H_{ik} = 0$ and $B_{ik} = 0$ otherwise. It is then clear that D_{ij} is the sum of inner products $\langle A_i, B_j \rangle + \langle B_i, A_j \rangle$, i.e., $D_{ij} = A_i(B_j)^t + B_i(A_j)^t$. We thus obtain $D = AB^t + BA^t$, where $A[B]$ is the $n \times N$ matrix whose ith row is $A_i[B_i]$.

(*ii*) $\Rightarrow$ (*i*): Suppose we have $n \times N$ binary-valued matrices A and B such that $D = AB^t + BA^t$. For all i, since $D_{ii} = 0$, $A_i(B_i)^t = 0 = B_i(A_i)^t$ and A_i and B_i can hence have no 1's in common positions. It should be clear from the converse case above that by defining H as

$$H_{ij} = \left\{ \begin{array}{lll} 1 & \text{if} & A_{ij} = 1 \\ 0 & \text{if} & B_{ij} = 1 \\ d & \text{if} & A_{ij} = B_{ij} = 0 \end{array} \right\},$$

we obtain the desired address matrix.

(*ii*) $\Rightarrow$ (*iii*): Given (*ii*), define matrices C and E as $C = A + B$ and $E = A - B$. Since A and B have no common ones, C is binary-valued with zeros only in those positions where A and B have common zeros and one elsewhere. Moreover, E has values $0, \pm 1$ and has zeros only in those positions where A and B have common zeros and is ± 1 elsewhere. The definition of C and E immediately leads to verification of the relation

$$2D = CC^t - EE^t.$$

(*iii*) $\Rightarrow$ (*ii*): Given (*iii*), define $A = \frac{1}{2}(C + E)$ and $B = \frac{1}{2}(C - E)$. Clearly, A and B are binary-valued (with no ones in common positions) and (*ii*) follows directly from (*iii*).

(*i*) $\Rightarrow$ (*iv*): Let H^i, $i = 1, \cdots N$ be the ith column of matrix H and let F_i be the $n \times n$ matrix of Hamming distances between the rows (actually sequences of length one) of H^i. Then clearly

$$D = \sum_{i=1}^{N} F_i$$

and we must show that each F_i has the required structure. Suppose that H^i has n_1 0's, n_2 1's, and n_3 d's. Reorder the elements of H^i so that all of the 0's appear first, followed by all of the 1's. Then, the rows and columns of F_i can be reordered (symmetrically) to yield

$$\begin{array}{c} \\ n_1 \\ \\ \\ n_2 \\ \\ \\ n_3 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \left[\begin{array}{c|c|c} 0\cdots0 & 1\cdots1 & 0\cdots0 \\ \vdots\ \ \vdots & \vdots\ \ \vdots & \vdots\ \ \vdots \\ 0\cdots0 & 1\cdots1 & \cdots\cdots \\ \hline 1\cdots1 & 0\cdots0 & \cdots\cdots \\ \vdots\ \ \vdots & \vdots\ \ \vdots & \vdots\ \ \vdots \\ 1\cdots1 & 0\cdots0 & \cdots\cdots \\ \hline 0\ \cdots & \cdots\cdots & 0\cdots0 \\ \vdots\ \ \vdots & \vdots\ \ \vdots & \vdots\ \ \vdots \\ 0\ \cdots & \cdots\cdots & 0\cdots0 \end{array}\right] \end{array} = F_1 \tag{7}$$

Clearly, F_i is binary-valued, symmetric, and rank two, (or rank zero if n_1 or $n_2 = 0$) and these properties are invariant with respect to symmetric permutation of rows and columns.

(iv) $\Rightarrow$ *(i):* Let $D = \sum_{i=1}^{N} F_i$ with each F_i satisfying the conditions of the theorem. We show that by appropriate permutation of rows and columns, F_i can be partitioned as in (7). Note first that since diagonal elements of D are zero, the same must be true of each F_i. Hence rank $(F) \leqq 2$ implies either rank $(F_i) = 2$ or $F_i \equiv 0$ since F_i is symmetric.

Now, let F be any binary-valued, symmetric, rank two, $n \times n$ matrix with zeros on the main diagonal. Let $e_1, \cdots e_n$ represent the diagonal elements of F. (The value of each e_i is 0). Two elements e_i and e_j will be defined as equivalent, $e_i \sim e_i$ if $F_{ij} = 0$. We now show that under the assumptions concerning F, $\sim$ is a true equivalence relation with respect to the e_i's that do not index a row or column of zeros in F[†]. Clearly, $e_i \sim e_i$, $i = 1, \cdots n$, and $e_i \sim e_j$ implies $e_j \sim e_i$. In order to

† The equivalence class method of proof was suggested to the authors by H. S. Witsenhausen.

prove transitivity, let us (temporarily) remove from F all rows and columns consisting entirely of zeros. Suppose now that $e_i \sim e_j$, $e_j \sim e_k$, but that $F_{ik} = 1$. Then F has the form

	i	j	k		m	
i	0	0	1		y	
j	0	0	0		1	
k	1	0	0			
m		1	x		0	

Since rows and columns of zeros have been removed, $F_{jm} = F_{mj} = 1$ for some m. The unspecified values of F_{mk} and F_{im} are denoted by x and y respectively. Clearly the 3×3 submatrix of F

$$\begin{bmatrix} 0 & 1 & y \\ 0 & 0 & 1 \\ 1 & x & 0 \end{bmatrix}$$

consisting of columns j, k, m and rows i, j, m, is nonsingular regardless of how x and y are specified, thus contradicting the rank two condition. Hence, $\sim$ is transitive, and F can be permuted in such a way that all diagonal blocks consist entirely of zeros, and all off-diagonal blocks consist entirely of ones. This partitioned matrix can then be bordered on the right and bottom with the previously removed rows and columns of zeros. The resulting matrix then resembles (7) except that we must discount the possibility of having more than two equivalence classes. However, it is obvious from the partitioned structure of F that the

number of equivalence classes equals rank (F) which is two by assumption. To each F_i we can thus associate a column vector H^i consisting of 0's, 1's and d's. We place d's in positions of H^i corresponding to the elements e_k that index rows and columns of 0's in F_i. We place 0's in positions of H^i corresponding to the elements e_k that determine one of the equivalence classes, and we place 1's in the remaining positions of H^i. The matrix of Hamming distances associated with H^i is thus equal to F_i. If rank $(F_i) = 0$ for some i, then $F_i = 0$ and the corresponding column H^i can be chosen to consist entirely of d's. This completes the proof.

We now present a method for joining address matrices of graphs to form an address matrix of their union. Suppose a graph G contains (connected) subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ are exactly a vertex of G (i.e., G is found by connecting the two graphs G_1 and G_2 at some vertex). Then we will say that G is *separable into* G_1 *and* G_2.

Theorem 2: Suppose the graph G is separable into the subgraphs G_1 and G_2. Then $N(G) \leqq N(G_1) + N(G_2)$.

Proof: Let n_i be the number of vertices of G_i and let H_i be an $n_i \times N(G_i)$ address matrix for G_i for $i = 1, 2$. We may assume that $G_1 \cap G_2$ is the n_1th vertex of G_1 and the 1st vertex of G_2. Define H as

$$H = \begin{array}{c} \\ n_1 \\ \\ n_2 \end{array} \overbrace{\phantom{\begin{bmatrix} H_1 & J_2 \\ J_1 & H_2 \end{bmatrix}}}^{N(G_1)} \overbrace{\phantom{\begin{bmatrix} H_1 & J_2 \\ J_1 & H_2 \end{bmatrix}}}^{N(G_2)} \begin{bmatrix} H_1 & J_2 \\ J_1 & H_2 \end{bmatrix}$$

where $J_1(J_2)$ is the $(n_2 - 1) \times N(G_1)((n_1 - 1) \times N(G_2))$ matrix each row of which is the n_1th (1st) row of $H_1(H_2)$. Clearly H is an address matrix for G with addresses of length $N(G_1) + N(G_2)$.

Corollary: If the graph G is separable into subgraphs each of which satisfies the conjecture (1), then G also satisfies (1): i.e., $N(G) \leqq n - 1$.

Proof: With notation as above, if $N(G_1) \leqq n_1 - 1$ and $N(G_2) \leqq n_2 - 1$ then $N(G) \leqq n_1 + (n_2 - 1) - 1 = n - 1$.

Clearly, this corollary can be extended to any number of subgraphs.

We will now present a class of graphs for which $N(G) \leqq n - 1$. Call a linear tree (i.e., $D_{ij} = |i - j|$) contained in a graph G a *spine* if each vertex of G is either a vertex of the linear tree or else distance one away

from the linear tree. Clearly, any diameter two graph (i.e., the maximum distance between each pair of vertices is two) admits a spine; any maximal linear tree will do. It can also be shown that any diameter three graph admits a spine, while, on the other hand, there exists a 14-point nonseparable diameter four graph admitting no spine.

If L is a spine for a graph G, label the vertices of L consecutively: $v_{10}, v_{20}, v_{30} \cdots$ and label each vertex v of G not on L v_{ij}, $j = 1, 2, \cdots$ where $i = \min \{k \mid d(v, v_{k0}) = 1\}$.

Theorem 3: *A sufficient condition for* $N(G) \leqq n - 1$ *is that* G *admit a spine* L *satisfying*

EITHER (1) $d(v_{ie}, v_{ik}) \geqq d(v_{j0}, v_{ik}) - \gamma_k$ for $i = 1, 2, \cdots ; j > i$; $k, e = 0, 1, \cdots$ *(when defined) where* $\gamma_k = 0$ *if* $k = 0$ *and* 1 *if* $k \neq 0$.

OR (2) *There are no six vertices* $x_1, y_1, y_2, x_2, x_3, x_4$, *distinct except possibly* $y_1 = y_2$, *and increasing from left to right in the lexicographic order of the vertices' double-indexing, with each* x_i *a vertex of* L *and such that:*

$$\begin{aligned} i)\ & d(x_2, x_1) = d(y_2, x_1) + 1 + \gamma(y_2), \\ ii)\ & d(x_4, x_2) = d(x_3, x_2) + 1, \\ iii)\ & d(x_4, x_1) = d(x_3, x_1) + 1, \\ iv)\ & d(x_3, y_1) = d(x_4, y_1) + 1 + \gamma(y_1), \end{aligned}$$

where $\gamma(y_i) = 0$ *if* y_i *is a vertex of* L *and* 1 *otherwise.*

[*Note:* (1) $\Rightarrow$ (2), *and there exist even weaker (but more complicated) sufficient conditions than* (2); *however, no graph is known to the authors which admits a spine and fails to satisfy* (1) *for some spine*].

Corollary 1: *For any diameter* 3 *graph* G *admitting a spine of* 3 *or fewer vertices,* $N(G) \leqq n - 1$.

Corollary 2: *For any diameter* 2 *graph* G *admitting a spine* L *such that, either no four vertices of* L *form a* 4*-cycle in the original graph, or else* L *has at most* 4 *points, we have* $N(G) \leqq n - 1$; *this includes all complete bipartite graphs.*

Proof: Both statements follow directly from condition (2) of the theorem.

Proof of Theorem: Suppose we have a graph which admits a spine; double-indexing the vertices as above, the associated distance matrix D will have the form

$D =$

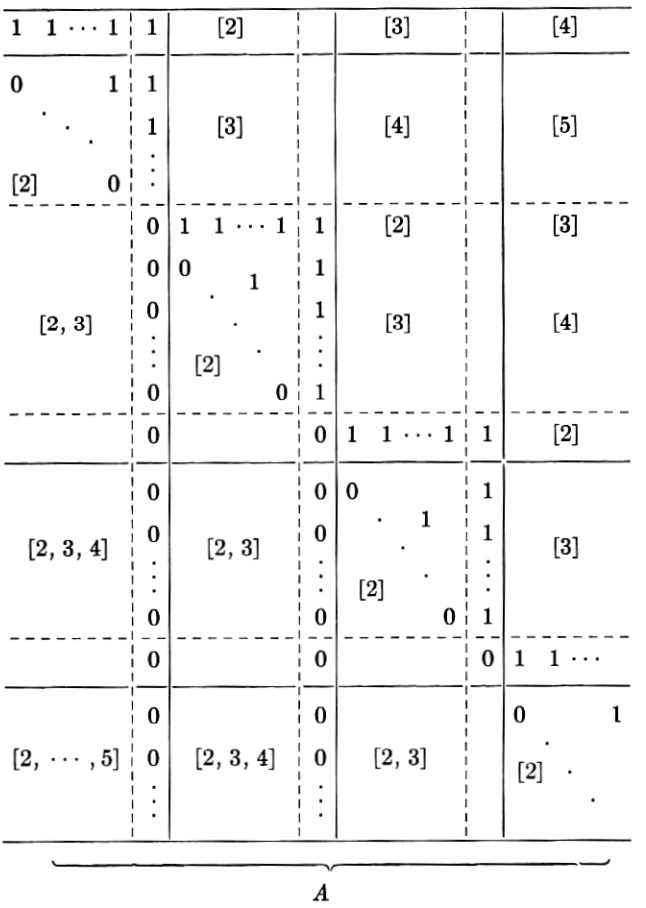
	v_{10}	v_{11}	v_{12}	$\cdots$	v_{20}	$v_{21}v_{22}$	$\cdots$	v_{30}	$v_{31}v_{32}$	$\cdots$	v_{40}	$v_{41}v_{42}$
v_{10}	0	1	1 $\cdots$	1	1	$\leqq 2$			$\leqq 3$			$\leqq 4$
v_{11}	1	0		$\leqq 2$								
v_{12}	1					$\leqq 3$			$\leqq 4$			$\leqq 5$
$\vdots$	$\vdots$											
	1	$\leqq 2$		0								
v_{20}	1				0	1	1 $\cdots$ 1	1	$\leqq 2$			$\leqq 3$
v_{21}					1	0						
v_{22}	$\leqq 2$	$\leqq 3$			1		$\leqq 2$		$\leqq 3$			$\leqq 4$
$\vdots$												
					1	$\leqq 2$	0					
v_{30}					1			0	1 1 $\cdots$ 1		1	$\leqq 2$
v_{31}								1	0	$\leqq 2$		
v_{32}	$\leqq 3$	$\leqq 4$		$\leqq 2$	$\leqq 3$			1				$\leqq 3$
$\vdots$								$\vdots$	$\leqq 2$			
								1		0		
v_{40}								1			0	1 1 $\cdots$
v_{41}											1	0 $\leqq 2$
v_{42}	$\leqq 4$	$\leqq 5$		$\leqq 3$	$\leqq 4$			$\leqq 2$	$\leqq 3$		1	
								$\vdots$			$\vdots$	$\leqq 2$

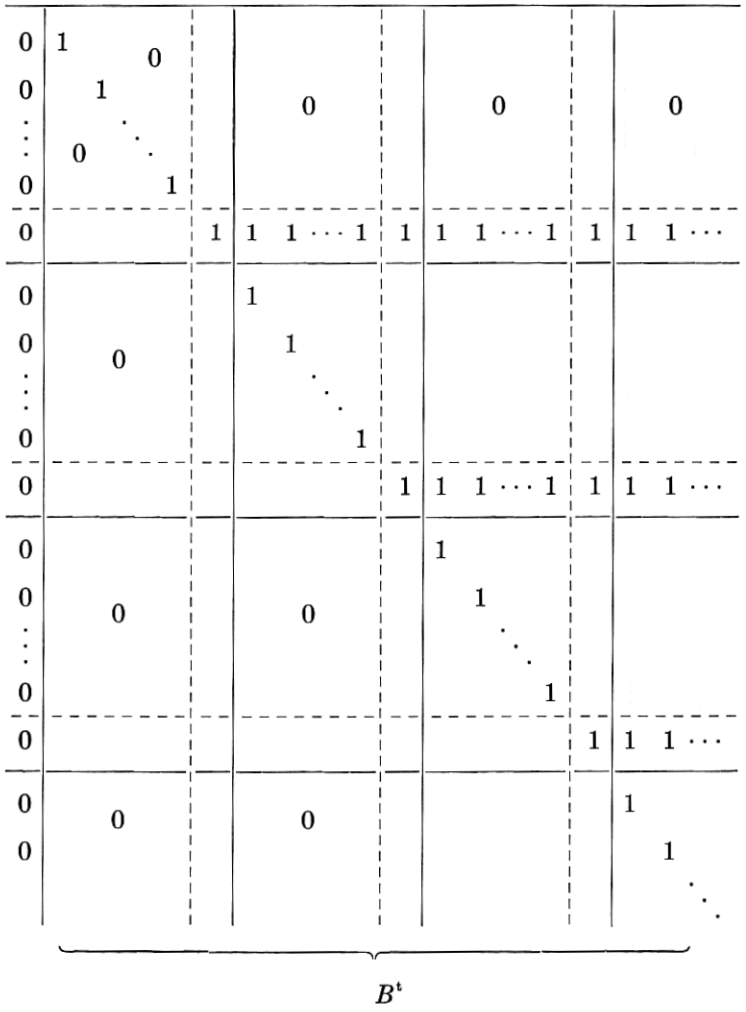
Conditions (1) and (2) both place certain restrictions on the extent of non-monotonicity of any column of D below the main diagonal; (1) says that the columns under each v_{i_0} shall be monotonically increasing and that the other columns shall contain decreases of at most one unit per square (roughly speaking). Condition (2) allows a greater degree of non-monotonicity but excludes an unmanageable combination of such. It is not difficult to see that (1) $\Rightarrow$ (2).

Now assume that (2) is satisfied. We will construct $n \times (n-1)$ matrices A and B satisfying $D = AB^t + BA^t$. Roughly speaking, one

can think of A as being superimposed on the $n \times (n-1)$ matrix D' remaining after removing from D its first column; we will use the notation $[k_1, k_2, \cdots]$ to indicate that the designated spot in A shall be a 1 if the associated spot in D' is either k_1 or k_2 or $\cdots$; all non-1 entries in A are 0. The matrix B^t will basically be an $n-1 \times n-1$ identity matrix with a concatenated first column of zeros; 1's will then be added above the existing diagonal of 1's for the purpose of "repeating" the associated column of A.

First, consider the case in which (1) holds. In this case (it is easily checked) AB^t can have the form




B^t

$AB^t =$

0	1 1 ⋯ 1	1	1→1, 2→2		1→1, 2→2, 3→3	
0 0 ⋮ 0	0 ⋱ 1 2→1 ⋱ 0	1 1 ⋮ 1	1, 2→1 3→2		1, 2→1 3→2 4→3	
0		0	1 1 ⋯ 1	1		
0 0 ⋮ 0	2, 3→1	1 1 ⋮ 1	0 ⋱ 1 2→1 ⋱ 0	1 1 ⋮ 1	1, 2→1 3→2	
0		1		0	1 1 ⋯ 1	1
0 0 ⋮ 0	2, 3, 4→1		2, 3→1	1 1 ⋮ 1	0 ⋱ 1 2→1 ⋱ 0	1 1 ⋮ 1
0				1		0 1 1 ⋯
0 0						1 0 1 ⋱ ⋮

where $k_1 k_2 \cdots \to l$ means that whenever k_1 or k_2 or $\cdots$ appears in D, then l appears in AB^t. It is then clear that $AB^t + (AB^t)^t = D$.

The general case of (2) now follows in an analogous fashion; the matrix B^t will be as above, except that certain of the off-diagonal 1's will be replaced by 0's. The matrix A will be as above, except that the bracket notation will have to be written for each column above the diagonal, each row below the diagonal and the entries in the brackets

will have to be altered to meet the needs of the situation. It can be shown that given the conditions of (2), this can actually be done.

III. EXTENSION TO THE PQ^t SCHEME

As explained in the introduction, an alternative way of looking at the addressing problem is to determine the smallest L for which there are $n \times L$ binary-valued matrices P and Q such that $D = PQ^t$. This formulation also allows for a possibility where D is nonsymmetric. A nonsymmetric D can arise if the given graph is directed, or can arise if the elements of D are chosen so as to obtain a desirable routing scheme not strictly dependent on the actual distances between vertices. In these terms, the analog of the Graham and Pollak conjecture (1) is that $L(G) \leqq 2(n-1)$ (where $L(G)$ is the smallest L). We know this is true when $N(G) \leqq n-1$ since in any case (from (3) and (4))

$$L(G) \leqq 2N(G). \tag{8}$$

In Theorem 4 we will show that $L(G) \leqq \sum_i \max_j D_{ij} \leqq n$ diam G.

On the other hand, we have the lower bounds

$$L(G) \geqq \text{rank } D \tag{9}$$

and

$$L(G) \geqq 2 \text{ diam } G. \tag{10}$$

The bound (9) follows since $L(G) \geqq \text{rank } P \geqq \text{rank } D$. To see (10), find i and j such that $D_{ij} = \text{diam } G$. Then $P_i(Q_j)^t = \text{diam } G = P_i(Q_i)^t$. Hence P_i and Q_i have diam G 1's in common and the same is true for P_j and Q_j . But $D_{ii} = 0$, so that in each spot that P_i has a 1, Q_i has a zero. Thus, Q_i must contain at least diam G 1's and diam G 0's.

Theorem 4: $L(G) \leqq \sum_i \max_j D_{ij} \leqq n$ diam G

Proof: Let $s_i = \max_j D_{ij}$ and $r_k = \sum_{i=1}^{k} s_i$ with $r_o = 0$. It is easy to see that the following construction for P and Q gives an address of length r_n . For every k, the kth row of P, P_k is zero everywhere except in positions $r_{k-1} + 1$ to r_k , where it is one. For every j, the jth row of Q, Q_j is divided into cells, the kth cell defined as positions $r_{k-1} + 1$ to r_k . In the kth cell is placed exactly D_{kj} 1's with the rest being equal zero. Obviously $P_k Q_j^t$ will then be the number of 1's in the kth cell of Q_j ., i.e., D_{kj} . Notice that this proof is not dependent on D being the distance matrix of a graph; the theorem is true for any nonnegative integer-valued matrix D.

A comparison of (3) and (4) suggests that the inequality (8) can be

strict. This is illustrated in several of the following special cases where we show, in particular, that if G is a cycle with n odd or a complete graph, then $L(G) = N(G) + 1$, and for certain trees, $L(G) \leqq \frac{3}{2}N(G)$.

(*i*) *Complete Graphs*

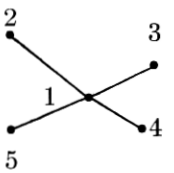
For a complete graph, all off-diagonal elements of D are 1's. Let $P = I$, the n-dimensional identity matrix, and let $Q = D$. Then obviously $D = ID = PQ^t$, and in light of (9), since rank $D = n$, this factorization is minimal.

For a binary-valued matrix X, define $\bar{X}$ to be X with all the 1's changed to 0's and all the 0's changed to 1's. Let $X^* = \bar{X}^t$. Then the decomposition for the complete graph above has the special form $D = PP^*$. This property has important implications which will be discussed later.

(*ii*) *Trees*

In Ref. 2 (Theorem 3), it was shown that for any tree G, $N(G) = n - 1$, and in fact, the address matrix H could be chosen to be binary-valued–without d's. Hence, by (3), for any tree G, $L(G) \leqq 2(n - 1)$. For G a linear tree, i.e., $D_{ij} = |i - j|$, in fact $L(G) = 2(n - 1)$ by (10), and hence for an arbitrary graph G there is no general upper bound for $L(G)$ lower than $2(n - 1)$. However, we shall show that for some trees $L(G) < 2(n - 1)$. Also, since the address matrix for a tree can be chosen to have no d's, it is clear from (3) and Theorem 1 that the distance matrix for a tree can be written as $D = PP^*$.

Now let G be a star on five vertices.



Then

$$D = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix},$$

and

$$Q^t = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then $D = PQ^t$ (note that we no longer have $Q^t = P^*$) and $L(G) \leqq 6 < 8 = 2N(G)$. Conversely, we show $L(G) \geqq 6$. To see this, suppose there exists P and Q of order 5×5 such that $D = PQ^t$. Then for D_1, the submatrix of D defined below we have

$$D_1 = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix} = RS^t = SR^t,$$

where R and S are the obvious submatrices of P and Q, respectively. Observe first that each row of R and S must contain at least two 1's. Also, at least one of the rows of R or S must contain at most two 1's, since if this were not the case, the diagonal elements of D_1 would not vanish. Since the roles of R and S are interchangeable, we assume without loss of generality that the first two elements of the first row R_1 of R are 1's and the rest are zeros. Now since

$$R_1 S_j^t = 2, \qquad j = 2, 3, 4,$$

we have

$$S_{j1} = S_{j2} = 1, \qquad j = 2, 3, 4;$$

thus

$$R_{j1} = R_{j2} = 0 \qquad j = 2, 3, 4$$

because

$$R_j S_j^t = 0 \quad \text{for all } j.$$

Now, at least two of the numbers R_{2j}, $j = 3, 4, 5$ must be 1, and hence, at most one of the numbers S_{2i}, $j = 3, 4, 5$ can be 1, which implies $R_j S_2^t = \leqq 1$, which contradicts the condition that $R_3 S_2^t = 2$. Thus $L(G) > 5$.

If G is a star with $n = 4k + 1$, then $L(G) \leqq 6k < 8k = 2N(G)$. This follows from the above example and a result analogous to Theorem 2: viz., if G is separable into G_1 and G_2 then $L(G) \leqq L(G_1) + L(G_2)$.

(iii) Cycles–n odd

Suppose G is a cycle with distance matrix D and n odd. Let $m = (n - 1)/2$ and let L_m be the distance matrix for the linear tree on m vertices. Let A be the $m \times m$ triangular matrix with 1's on and below the diagonal, and let U be the $m \times m$ matrix all the entries of which are 1:

$$A = \begin{bmatrix} 1 & & & \\ 1 & 1 & & 0 \\ 1 & 1 & 1 & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then

$$D = \begin{array}{|c|c|c|} \hline L_m & \begin{matrix} m \\ m-1 \\ \vdots \\ 2 \\ 1 \end{matrix} & mU - L_m + A^* \\ \hline m \quad m-1 \cdots 2 \quad 1 & 0 & 1 \quad 2 \cdots m-1 \quad m \\ \hline mU - L_m + \bar{A} & \begin{matrix} 1 \\ 2 \\ \vdots \\ m-1 \\ m \end{matrix} & L_m \\ \hline \end{array}.$$

Define

$$P = \begin{array}{|c|c|c|} \hline A & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & A^t \\ \hline 1 \cdots 1 & 1 & 0 \cdots 0 \\ \hline \bar{A} & \begin{matrix}1\\ \vdots \\ 1\end{matrix} & A \\ \hline \end{array} .$$

Then, by direct computation, we get

$$PP^* = \begin{array}{|c|c|c|} \hline AA^* + A^t\bar{A} & \sum^r A^t & AA^t + A^tA^* \\ \hline \sum^c (A^* + I) & 0 & \sum^c A^t \\ \hline \bar{A}A^* + A\bar{A} + U & \sum^r A & \bar{A}A^t + AA^* \\ \hline \end{array}$$

where $\sum^r (\sum^c)$ means: sum the rows (columns) of, and I is the $m \times m$ identity matrix. It can be shown that $PP^* = D$. Note that P is $n \times n$ and hence $L(G) \leqq n$; but D is nonsingular (see Ref. 2, Theorem 4) and so by (9), $L(G) = n$.

(iv) Cycles–n even

In Ref. 2 (Theorem 4), it was shown that for G, a cycle with n even, $N(G) = n/2$, and hence by (8), $L(G) \leqq n$; but diam $G = n/2$ and by (10) $L(G) \geqq n$, so $L(G) = n$. Furthermore, the minimal address matrix H for G can be chosen without d's and hence we can write $D = PP^*$ with P $n \times n$.

(v) Cube

Let G be the vertex and edge structure of the m-dimensional cube; then $n = 2^m$. It can be shown that $N(G) = m$, and since diam $G = m$,

$L(G) = 2m$. Also, it is apparent that we can write $D = PP^*$ with P $n \times 2m$.

(vi) Diameter 2 Graphs

We assume that the graph has at least four vertices, since a graph on three or fewer vertices must be a complete graph or a tree, and we know how to address such graphs. There are now two cases to be considered.
Case 1. Here, we assume that the graph is a star. We know $N(G) = n - 1$, so $L(G) \leqq 2(n - 1)$.
Case 2. Here we assume that the graph is not a star. Then, since the graph is connected, there must be four vertices, say v_1, v_2, v_{n-1}, and v_n, such that v_1 is adjacent to v_2 and v_{n-1} is adjacent to v_n. The distance matrix has the structure

$$D = \left[\begin{array}{cc|c|cc} 0 & 1 & & & \\ 1 & 0 & & & \\ \hline & & & & \\ \hline & & & 0 & 1 \\ & & & 1 & 0 \end{array}\right],$$

where the unspecified elements are 0's, 1's, and 2's. Let D_1 represent the distance matrix of a complete graph on n vertices and let $D = D_1 + D_2$. The remainder matrix D_2 has the form

$$D_2 = \left[\begin{array}{cc|c|cc} 0 & 0 & & & \\ 0 & 0 & & & \\ \hline & & & & \\ \hline & & & 0 & 0 \\ & & & 0 & 0 \end{array}\right],$$

and the unspecified elements are 0's and 1's. The idea now is to take 1's from D_1 and put them into D_2 in such a way as to reduce the rank of D_2. This can be done by taking ones from the first two and last two rows of D_1 and placing them into the first two and last two rows of D_2. The result is $D = D_1' + D_2'$ where

$$D_2' = \left[\begin{array}{cc|c|cc} 0 & 0 & 1 \cdots 1 & 1 & 1 \\ 0 & 0 & 1 \cdots 1 & 1 & 1 \\ \hline \multicolumn{2}{c|}{X} & Y & \multicolumn{2}{c}{Z} \\ \hline 1 & 1 & 1 \cdots 1 & 0 & 0 \\ 1 & 1 & 1 \cdots 1 & 0 & 0 \end{array}\right] \begin{matrix} \\ \\ \\ \\ \end{matrix} \overbrace{}^{n} \; \Big\} n,$$

and D_1' remains a 0, 1 valued matrix. Note that rank $(D_2') \leqq n - 2$. Let I_{n-2} and I_n be the identity matrices of order $n - 2$ and n respectively. Then $D = PQ^t$, where

$$P = \left[\begin{array}{c|c} & 1 \quad 0 \cdots 0 \\ I_n & I_{n-2} \\ & 0 \cdots 0 \quad 1 \end{array}\right] \begin{matrix} \overbrace{}^{n} & \overbrace{}^{n-2} \\ & \\ & \\ & \end{matrix} \Big\} n,$$

and

$$Q^t = \left[\begin{array}{c|c|c} \multicolumn{3}{c}{D_1'} \\ \hline 0 \quad 0 & 1 \cdots 1 & 1 \quad 1 \\ \hline \multicolumn{2}{c|}{X} & \multicolumn{1}{c}{Z} \\ \hline 1 \quad 1 & 1 \cdots 1 & 0 \quad 0 \end{array}\right] \begin{matrix} \overbrace{}^{n} \\ \\ \\ \\ \end{matrix} \begin{matrix} \Big\} n \\ \\ \\ \Big\} n - 2. \end{matrix}$$

Hence, $L(G) \leqq 2(n - 1)$. It is not yet known whether $N(G) \leqq n - 1$ for all diameter 2 graphs. Note that in the above construction $P^* \neq Q^t$ in general.

IV. SOME LINEAR ALGEBRA

In the previous section we realized the representation (4) for several classes of graphs. It is possible to evaluate lower bounds for $N(G)$ directly from these representations, (without having to calculate $n^{\pm}$). This and other results will follow directly from some simple linear algebra. In Lemmas 1 and 2 below, we use the following observation:

if $\mathfrak{C}^+$ and $\mathfrak{C}^-$ are two subspaces of an n-dimensional real linear space $\mathfrak{C}$, such that $\mathfrak{C}^+$ and $\mathfrak{C}^-$ have no non-zero vectors in common, then dim $(\mathfrak{C}^+)$ + dim $(\mathfrak{C}^-) \leqq n$.

Lemma 1: Let D be any real $n \times n$ matrix (not necessarily symmetric), and let $\mathfrak{C}^-$ be any subspace of $\mathfrak{C}$ such that for all $x \neq 0$, $x \in \mathfrak{C}^-$, we have $\langle x, Dx \rangle < 0$. If W and Z are $n \times n$ matrices such that $\langle x, Wx \rangle \geqq 0$ for all $x \in \mathfrak{C}$ and $D = W - Z$, then rank $Z \geqq$ dim $(\mathfrak{C}^-)$.

Proof: Let $\mathfrak{C}^+$ denote the null space of Z. Clearly $\langle x, Dx \rangle \geqq 0$ for $x \in \mathfrak{C}^+$, and dim $(\mathfrak{C}^+) = n$-rank (Z). Also, $\mathfrak{C}^+$ and $\mathfrak{C}^-$ have no non-zero vectors in common so that dim $(\mathfrak{C}^-)$ + dim $(\mathfrak{C}^+)$ = dim $(\mathfrak{C}^-)$ + n-rank $(Z) \leqq n$ or rank $(Z) \geqq$ dim $(\mathfrak{C}^-)$.

The following result is a dual version of the above:

Lemma 2: Let D be any real $n \times n$ matrix and let $\mathfrak{C}^+$ be any subspace of $\mathfrak{C}$ such that for all $x \neq 0$, $x \in \mathfrak{C}^+$, we have $\langle x, Dx \rangle > 0$. If $D = W - Z$ with $\langle x, Zx \rangle \geqq 0$ for all x, then rank $W \geqq$ dim $(\mathfrak{C}^+)$.

The following corollaries are immediate from the two lemmas.

Corollary: Let D be a real $n \times n$ symmetric matrix with n^+ and n^- positive and negative eigenvalues respectively, and let $D = W - Z$. If $\langle x, Zx \rangle \geqq 0$ [$\langle x, Wx \rangle \geqq 0$] for all x, then rank $(W) \geqq n^+$ [rank $(Z) \geqq n^-$].

Corollary: Let D be the distance matrix of a graph G. Then $N(G) \geqq$ max $\{n^+, n^-\}$.

Proof: From condition 3 of Theorem 1, we have $2D = CC^t - EE^t$ where C and E are $n \times N$ matrices. Then application of the previous corollary yields

$$N \geqq \text{rank } C \geqq \text{rank } CC^t \geqq n^+,$$

and

$$N \geqq \text{rank } E \geqq \text{rank } EE^t \geqq n^-.$$

Note that the proof of this result does not depend on the integer nature of the matrices involved.

We mentioned in the previous section that factorizations of the form $D = PP^*$ have special significance. We now demonstrate this.

Theorem 5: Let D be any symmetric non-zero matrix satisfying $D = PP^*$ for some binary-valued $n \times L$ matrix P. Then $n^+ = 1$ and $n^- =$ rank $(D) - 1$.

Proof: Let e_L and e_n be vectors of length L and n respectively, all of whose elements are 1. Then $e_L e_n^t$ is a rank one, $L \times n$ matrix of 1's

and $P^* = e_L e_n^t - P^t$. Thus $D = Pe_L e_n^t - PP^t$. But PP^t is non-negative definite and rank $(Pe_L e_n^t) = 1$. Hence, Lemma 2 implies $n^+ \leqq 1$. However, $D_{ii} \geqq 0$ for all i and j and $D \neq 0$ so that $e_n^t D e_n > 0$. Thus $n^+ = 1$, and since $n^+ + n^- = \text{rank } (D)$, we have $n^- = \text{rank } (D) - 1$.

Incidentally, the above shows that every row of P has the same number of 1's.

This Theorem is useful because it enables one to determine n^+ and n^- from rank D. Thus, if a distance matrix of a graph G satisfies $D = PP^*$, then $N(G) \geqq \max (1, \text{rank } D - 1) = \text{rank } D - 1$. This result has immediate application to complete graphs, trees, cycles, and cubes considered in the previous section, since in each of those cases, we exhibited a factorization of the form PP^*. We remarked previously that for any graph that admits an address matrix H having no d's as entries, $D = PP^*$ for some P. We now derive an upper bound on the number of rows of H that do not contain d's.

Theorem 6: Let $D = PQ^t$ be symmetric, and suppose that r rows of P are the complements of the corresponding r rows of Q. Then $r \leqq n - n^+ + 1$.

Proof: $D = PQ^t = P(Q^t + P^t - P^t) = P(Q^t + P^t) - PP^t$

From Lemma 2,

$$\text{rank } (Q^t + P^t) \geqq \text{rank } (P(Q^t + P^t)) \geqq n^+.$$

But $Q^t + P^t$ has n columns, r of which have ones in all positions. Hence rank $(Q^t + P^t) \leqq n - r + 1$, and the desired result follows.

V. AN ALTERNATE ADDRESSING SCHEME

In this section we discuss an addressing scheme that achieves minimum distance routing, and is very simple to construct for all graphs (in fact, D could be any non-negative integer-valued matrix; this could often arise whenever preference for routes is not dictated by just the path length). Let s be the diameter of G, a graph on n vertices. We consider a PQ^t addressing of G with P and Q matrices each of order $n \times ns$. The addressing that follows is a simplified version of the construction used in Theorem 4. The ith row of P, P_i, is zero everywhere except in positions $(s(i-1)+1)$ to si where it is one. The ith row of Q, Q_i, is constructed as follows: for every j, there are exactly D_{ji} 1's in positions $(s(j-1)+1)$ to sj with the rest of the entries of Q_i equal to zero.

$$P = \begin{array}{c} \begin{array}{cccccccc} & s & & 2s & & 3s & & ns \\ & \downarrow & & \downarrow & & & & \end{array} \\ \begin{bmatrix} 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 & \cdots & 0 \cdots 0 \\ 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 & & \\ & & & & \\ 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & \cdots & 1 \cdots 1 \end{bmatrix} \end{array}$$

$$Q = \begin{array}{c} \begin{array}{cccccccccc} & s & & D_{12} & & 2s & & D_{13} & 3s & & & D_{1n} & ns \\ & \downarrow & & & & \downarrow & & & \downarrow & & & & \downarrow \end{array} \\ \begin{bmatrix} 0 \cdots & 0 & 1 \cdots 1 \cdots & 0 & 1 \cdots 1 \cdots & 0 & \cdots & 1 \cdots 1 \cdots & 0 \\ 1 \cdots 1 & 0 & 0 \quad \cdots & 0 & 1 \cdots 1 \cdots & 0 & & 1 \cdots 1 \cdots & 0 \\ & & & & & & & & \\ \underbrace{1 \cdots 1}_{D_{n1}} & 0 & \underbrace{1 \cdots 1}_{D_{n2}} \cdots & 0 & \underbrace{1 \cdots 1}_{D_{n3}} \cdots & 0 & & 0 \quad \cdots & 0 \end{bmatrix} \end{array}$$

Obviously, the length of addresses in this scheme in ns. The P matrix defined above, rows of which are addresses to be prefixed onto messages, is the same for all graphs with diameter s. The Q matrix, rows of which are stored in the loops in the network, contains the information that identifies a particular graph. But the simplicity of the rows of P can be exploited to minimize the length of the message addresses in the following way: since the integer i ($\leqq n$) uniquely specifies row P_i, the message address need only consist of the binary representation of i, which, of course, requires at most $\log_2 n$ bits. The set of binary representations of the integers $1, \cdots n$ is the set of addresses to be prefixed to messages. In each loop, a device is placed which generates row P_i from the binary representation of i. The distance calculation, say in loop j is accomplished by forming the scalar product of the generated P_i sequence with the stored Q_j sequence. This calculation can be mechanized by an "and gate" followed by a counter (as can be any of the PQ^t decompositions discussed in Section III).

This scheme has the following advantages:

(*i*) The simplicity of constructing addresses by inspection of the D matrix is important in the case of large graphs with no special structure. Even if one were to find good minimum $N(G)$ and $L(G)$, we suspect

that the construction of a minimal length addressing will be very complicated.

(*ii*) Since present plans for length of message blocks envisage lengths of perhaps a few thousand bits, the length of the message address is an important parameter in any large loop system. The method of this section guarantees minimum length addresses ($\log_2 n$) provided that the graph is not constrained to have a particular structure.

One of the disadvantages of the present scheme is that it requires a large amount of storage in each loop. However, this can be remedied to a certain extent by coding the number of 1's in each cell of length s of the rows of Q in binary form with $\log_2 s$ bits. Thus the storage requirement can be reduced fron ns bits to $n \log_2 s$ bits. The engineering aspects of this scheme and some of its modifications will be the subject of a forthcoming memorandum.

VI. ANOTHER ALTERNATE ADDRESSING SCHEME

We present here a coding scheme of the type PQ^t, which can address all graphs, using ternary logic symbols $+1$, -1, 0, and requiring addresses of length $2(n-1)$ for a graph on n vertices. The scheme uses the fact that for any graph there exists a numbering of vertices such that every vertex i is adjacent to some vertex j where $j < i$. This, of course, implies for some pairs i and j that $|D_{im} - D_{jm}| \leqq 1$ for all m. We exhibit the construction inductively. Let $D(r)$ be the $r \times r$ submatrix of D consisting of the first r rows and columns of D.

Suppose $D(r) = P(r)Q^t(r)$ where $P(r)$, $Q(r)$ are $(\pm 1, 0)$-valued matrices of order $r \times 2(r-1)$. Then it is easily verified that

$$D(r+1) = \begin{bmatrix} P(r) & \underline{0} & A \\ P_s(r) & 1 & 0 \end{bmatrix} \begin{bmatrix} Q^t(r) & Q_s^t(r) \\ A^t & 0 \\ \underline{0}^t & 1 \end{bmatrix}$$

Here $P_s(r)$ is the sth row of $P(r)$ where the $(r+1)$st vertex is adjacent to the sth vertex, or $|D_{(r+1)m} - D_{sm}| \leqq 1$ for all m. The symbol $\underline{0}$ denotes the $r \times 1$ matrix of zeros, A_j, the jth component of the $r \times 1$ matrix A is $+1$, -1 or 0 according as

$$D_{(r+1)j} - D_{sj} = +1, -1, \text{ or } 0 \text{ respectively.}$$

Calling the obvious matrices $P(r+1)$ and $Q(r+1)$, we have $D(r+1) = P(r+1)Q(r+1)$ where $P(r+1)$, $Q(r+1)$ are $(r+1) \times 2r$ matrices. Since $D(2) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, the construction is complete. Because of the

structure induced on P and Q by this construction, one can code the rows P and Q making the length of the code smaller, at the expense of increasing the complexity of mechanization. The way this address is constructed simplifies changing the address when a new node is added to an existing coded graph. Notice also that the positivity of the elements of D is unnecessary for this construction.

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