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A Cost Optimization Model for Seismic Design of Structures

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Considering the earthquake susceptibility of structures located in seismic regions, the question arises as to what level of protective measures should be provided in order to achieve a certain degree of reliability against possible damage. To address this question, engineering risk and optimal design of structures located in a seismic area are studied. The basic concept is to obtain a tradeoff between the cost of providing a protective measure and the expected cost of earthquake damage.

A simple mathematical approach is presented to determine the optimal earthquake intensity which the structure is designed to withstand. The objective is to minimize the total construction cost of the structure plus the expected cost of earthquake damage throughout the entire service life of the structure. For the case of deterministic structural resistance, and for structural response processes having Poisson (independent) crossings, an objective function is derived in terms of the building and earthquake variables. The optimal design intensity can then be determined by minimizing the objective function with respect to the intensity variable. The resulting equations are relatively simple and can be easily handled for numerical studies and sensitivity analysis. Generalizations of the results for nondeterministic structural resistance and for structural response processes different from those having Poisson crossings are also indicated.

As an illustration of the proposed approach, a hypothetical building with realistic seismicity and structural parameters is analyzed for its optimal design earthquake intensity. The construction, damage and total costs are obtained in terms of the intensity variable. The implications and sensitivity of the results are also discussed.

I. INTRODUCTION

Structures constructed in seismic regions are required to function properly in a forcing environment characterized by random earthquake occurrences and intensities. The seismic environment including the expected earthquake-magnitude levels and the corresponding frequency of occurrence for different seismic-risk zones was described previously.[1] The study was based on a statistical analysis of nationwide seismic data and may be used as a guide for the development of seismic design requirements on a global basis. Under localized situations, however, the seismic requirement for structures that are expected to adequately withstand the earthquake environments should be based on cost-reliability studies. During an earthquake of given intensity, there exists a probability that the response of the structure is greater than its resistance capability and, therefore, a probability of damage to the structure. The cost associated with this probable damage may be referred to as the "earthquake risk cost". Increasing the design intensity of the structure reduces the probability of damage, but at an increased cost of construction. Therefore, an optimal design earthquake intensity can be determined by achieving an appropriate balance between the construction cost and the earthquake risk cost.

This paper presents a new analytical approach to the determination of the economically optimal earthquake intensity or other design variables for structures. The construction and earthquake risk costs are expressed in terms of design intensity and other parameters reflecting the earthquake and structural characteristics. Minimization of the total expected cost of the structure yields the optimum structural design intensity in terms of such parameters as estimated cost of earthquake damage, unit construction cost, expected earthquake duration, and statistics obtained from seismological data for the particular site.

II. ANALYSIS OF DESIGN INTENSITY MODEL

2.1 *Objective Function*

Consider a certain seismic region in which a structure is located. Let $K_c(i_o)$ and $K_d(i_o)$ represent the construction and the earthquake risk or damage costs of the structure respectively, both being functions of the design intensity i_o measured from I to XII on the Modified Mercalli scale. The function $K_c(i_o)$ may be regarded as a monotonically increasing function of i_o, while the function $K_d(i_o)$, as would be expected, is a monotonically decreasing function of i_o.

The optimum design intensity i_o^*, may be obtained as a trade-off between these two functions by minimizing the total cost

$$K(i_o) = K_c(i_o) + K_d(i_o) \tag{1}$$

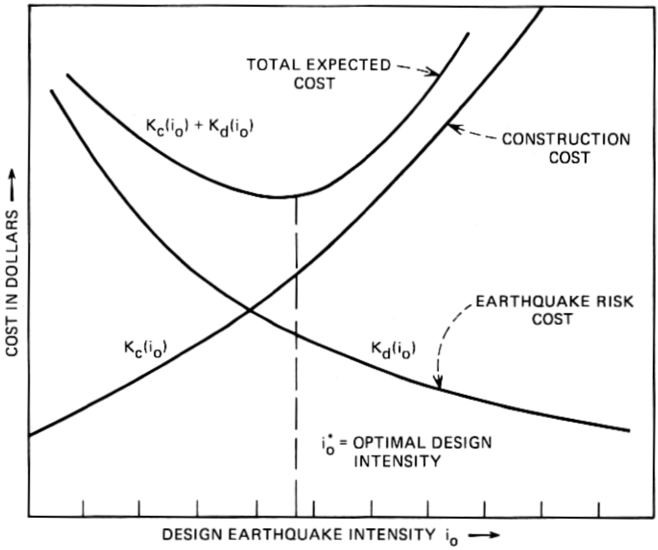

Fig. 1—Sketch showing cost of structure against design intensity of earthquake.

as shown in Fig. 1. The function $K(i_o)$ in eq. (1) is the objective function and i_o is the decision variable.

The construction cost, $K_c(i_o)$, can be written as

$$K_c(i_o) = A_f(f + C) \tag{2}$$

where A_f is the floor area of building, f is the building cost per unit floor area, and C is the cost of earthquake protection per unit floor area. The earthquake protection cost obviously increases with the protection level, such as i_o in this case. Therefore, $C = C(i_o)$ is a monotonically increasing function of i_o. It should be noted that the determination of the function $C(i_o)$ depends on the type of structure, method of design, and is greatly influenced by the designer's personal judgment and experience. A reasonable first approximation can be made assuming $C(i_o)$ is linear with a coefficient c. Under this assumption eq. (2) can be written as

$$K_c(i_o) = A_f(f + ci_o) \tag{3}$$

where $A_f f$ equals a fixed initial cost of building, and c is the earthquake resistance cost of the building per unit floor area per unit intensity.

To determine the function $K_d(i_o)$, analytical procedures can be effectively employed utilizing some basic knowledge about the structure and the random forcing environment. Let $N(t)$ be a random variable representing the total number of earthquakes to occur in time t. Furthermore, for $k = 1, 2, \cdots, N(t)$, let p_k be the probability that the structure fails given, the kth earthquake occurs, and let e_k, an identically distributed random variable, be the associated loss. Therefore, the total cost of earthquake damage, Z, is

$$Z = \sum_{k=1}^{N(t)} e_k p_k \,. \tag{4}$$

Taking expectation (denoted by overbar) of both sides of eq. (4), one obtains

$$\bar{Z} = \overline{N(t)}\bar{e}\bar{p} \tag{5}$$

in which $\overline{N(t)}$ equals the expected number of earthquakes (of all intensities) in time t, $\bar{e}$ equals the expected value of the random earthquake loss, and $\bar{p}$ equals the mean failure probability of structure given an earthquake occurs. The present worth of the expected value of Z is the earthquake risk cost $K_d(i_o)$:

$$K_d(i_o) = \bar{Z}g(t) = \bar{N}(t)\bar{e}\bar{p}g(t) \tag{6}$$

where $g(t)$ is a discount factor.

The quantities on the right-hand side of eq. (6) will be discussed next in terms of the related design parameters which fall into two categories: the earthquake parameters and the building parameters. The earthquake parameters include the regional seismicity constants, the earthquake magnitude (m), intensity (i), duration (l_o), amplitude (a) of the ground motions, and the statistics of these quantities. The building parameters include the mass (ρ), stiffness (k), natural frequency (ω_o), damping (ξ_o), height (h), the resistance or strength (x) of the structure, etc.

2.2 *The Average Number of Earthquakes*, $\bar{N}(t)$

The quantity $\bar{N}(t)$ depends on and can be estimated from the regional seismicity. Earthquakes can be considered to be a series of events randomly distributed on a real line (representing time), and the sequence of original times $\{t_n\}$ forms a point process.[2] It is further assumed that the joint statistics of the respective number of shocks in any set of intervals are invariant under a translation of these intervals; this implies that $\{t_n\}$ is a stationary point process. The stationary point process generalizes certain aspects of renewal processes; in particular, the interval lengths $\tau_k = t_k - t_{k-1}$ between successive events need not be independently or identically distributed.

The simplest stationary point process is the Poisson process. Intuitively, the process $\{t_n\}$ can be approximated as a Poisson process if it represents rare events. More rigorously, it requires that τ_k be independently and identically distributed and follow a negative exponential function. The main deficiency of the simple Poisson model is its inability to describe the aftershocks which are often triggered by a large main shock. However, for most practical engineering purposes, the simple Poisson model for earthquakes appears to be adequate. In practice, an engineer is concerned with the earthquake risk of structures located in some specific geographic areas. The risk depends heavily on the statistics of large earthquakes in these areas, and the omission of small earthquakes or aftershock processes is relatively unimportant in terms of earthquake risk.

If $\{N(t);\ t \geqq 0\}$ is assumed to be Poisson with a constant rate, α, then

$$\text{prob}\ [N(t) = n] = \frac{(\alpha t)^n}{n!} e^{-\alpha t} \tag{7}$$

and

$$\bar{N}(t) = \alpha t. \tag{8}$$

The parameter α per unit time t can be determined from regional seismicity data.[1]

2.3 *The Mean Failure Probability of Structure,* $\bar{p}$

The quantity $\bar{p}$ depends on both earthquake and building parameters. The earthquake intensity "i" to which the structure is designed will be the only decision variable considered in this formulation and all other parameters are assumed known. Let $Y(t) = \{\max |y(t)|;\ t\ \varepsilon\ [0, t_o]\}$, where, t_o equals the duration of the structural vibration which is assumed approximately equal to the duration of the earthquake ground motion, and $y(t)$ equals the response parameter (displacement, velocity, acceleration, stress, etc.) of the structure. For an earthquake with intensity i, failure could occur when the resulting structural response $y(t)$ equals or exceeds the actual resistance, x, of the structure. The corresponding failure probability, $p(i)$, can be expressed by

$$p(i) = \text{prob}\ [Y \geqq x \mid \text{earthquake with intensity } i \text{ has occurred}]. \tag{9}$$

The quantity $p(i)$ is a function of the random variable i representing the earthquake intensity whose probability density function $f_I(i)$ can be found in terms of the regional seismicity and earthquake source geometry. Since i, in the Modified Mercalli intensity scale, takes only discrete integer values from one to twelve, the mean failure probability of structure, $\bar{p}$ is given by

$$\bar{p} = \sum_{i=i_o}^{12} p(i)f_I(i), \tag{10}$$

in which i_o equals the design earthquake intensity. From eqs. (9) and (10) it is clear that earthquake parameters enter the formulation of the problem through $p(i)$ and $f_I(i)$, while building parameters enter the formulation through $p(i)$ only.

The density function $f_I(i)$ can be derived from an expression obtained by Cornell[3] for the distribution function of earthquake intensity i:

$$F_I(i) = 1 - \frac{1}{l}\,\Gamma J \exp\left(-\frac{\beta}{c_2}\,i\right). \tag{11}$$

The governing assumptions for eq. (11) are:

(*i*) The earthquake magnitude m is a random variable with independent and negative-exponential distribution function

$$F_M(m) = 1 - e^{-\beta m}, \quad m \geqq m_o \tag{12}$$

where m_o is some magnitude small enough, say 4, that events of lesser magnitude may be ignored by engineers, and β is a constant the inverse and inverse square of which represent the mean and variance of earthquake magnitude $m > 0$ respectively;

(*ii*) The intensity attenuation law is given by

$$i = c_1 + c_2 m - c_3 \ln r \tag{13}$$

in which c_1, c_2 and c_3 are regional seismicity constants,† and r, the focal distance in miles, is the random variable representing the distance from the structural site to the location of an earthquake source on the fault line.

(*iii*) The earthquake has a line source (fault line) of length l (in miles) with uniform distribution.

The parameters Γ and J are given by:

$$\Gamma = \exp\left[\beta\left(\frac{c_1}{c_2} + m_o\right)\right] \tag{14}$$

$$J = 2\int_d^{r_o} \frac{dr}{r^\gamma\sqrt{r^2 - d^2}}, \qquad \gamma = \beta\,\frac{c_3}{c_2} - 1 \tag{15}$$

† For example, c_i, $i = 1, 2, 3$ are semi-empirical constants on the order of 8, 1.5, and 2.5, respectively, for firm ground in Southern California.[4]

in which $d = \min r$ and $r_o = \max r$.

It follows from eq. (11) that

$$f_I(i) = \frac{dF_I(i)}{di} = \frac{\beta \Gamma J}{lc_2} \exp\left(-\frac{\beta}{c_2} i\right). \tag{16}$$

2.4 *Determination of the Failure Probability of Structure, $p(i)$*

To find the structure's failure probability $p(i)$, as defined by eq. (9), it is necessary to specify the failure mechanisms of the structure. It is also necessary to establish stochastic models for the response parameters $y(t)$ and the corresponding resistance x of the structure. For a linear and deterministic structure which is assumed to experience no plastic deformations and the properties of which are governed by given constants, the response model can be obtained given a stochastic model for the input earthquake ground motion. For the case of random strength, x, it becomes necessary to determine the distribution function $F_X(x)$ based on statistical and laboratory tests on individual building components as related to the overall structural resistance, e.g., Kennedy.[5] Similar tests were used by Freudenthal and Wang to establish a representative distribution of the ultimate strength of aircraft structures.[6]

In this study, consideration will be limited to the first excursion failure only, and the input and response of the structure are both treated as random processes. The probability of the duration of the response amplitude excursion and other failure mechanisms such as wearing and fatigue are not considered. In the first excursion failure, a structure is said to have failed if the response parameter, $y(t)$, exceeds a prescribed resistance or strength level, x, during the vibration cycles caused by the earthquake. Let the duration of the structural vibration be approximated by t_o, then, for any $t \in [0, t_o]$

$$\begin{aligned} p(i) &= \text{prob}\,[\,|\,y(t)\,| \geqq x; 0 \leqq t \leqq t_o] \\ &= 1 - W_i(t_o) \end{aligned} \tag{17}$$

in which $W_i(t_o) = \text{prob}\,[\,|\,y(t)\,| < x; 0 \leqq t \leqq t_o]$ = the reliability of the system. Two different situations in the structure's resistance characteristics will be considered below.

2.4.1 *Deterministic Resistance Variable, x*

A random process model for the response of structures subjected to a stationary earthquake excitation can be established as follows. A simple structure can generally be treated as a lightly damped linear oscillator and its response, $y(t)$, is related by a second-order differential equation to the excitation, e.g., $a_i(t)$, the ground acceleration of an earthquake with intensity i. A multistory structure can be treated similarly in generalized coordinates considering normal mode vibrations. The structural response $y(t)$ in our case is a Gaussian process which approaches stationarity after a few cycles of initial transient motions. Let $p_1(t)dt$ denote the probability that $y(t)$ exceeds the threshold $y = x$ during the interval $[t, t + dt]$ for the first time since the initial time $t = 0$. The probability density function $p_1(t)$, referred to as the first-crossing density, is related to the reliability function by $-(dW/dt) = p_1(t)$. While establishing the precise behavior of $p_1(t)$ for small t poses some difficulty, for most practical purposes some approximations can be made for large mean failure time t. The simplest approximation to the first-crossing density is to assume that the up-crossings of the threshold occur rarely in the stationary response, so that they can be considered as statistically independent events. If so, the instants at which $| y(t) |$ cross the level x from below would constitute a Poisson process with a constant rate $2\nu_x$, where ν_x is the level crossing rate of $y(t)$ at the level $y = x$. In this situation it can be easily shown[7] that

$$\begin{aligned} p(i) &= 1 - \exp(-2\nu_x t_o) \\ &= 1 - \exp[-2\nu_o t_o \exp(-x^2/2\sigma_y^2)] \end{aligned} \tag{18}$$

in which $\nu_o = \sigma_{\dot{y}}/(\pi\sigma_y)$ = zero crossing rate of $y(t)$; $\nu_x = \nu_y|_{y=x} = \nu_o \exp(-x^2/2\sigma_y^2)$; where σ_y = standard deviation of $y(t)$, $\sigma_{\dot{y}}$ = the standard deviation of $\dot{y}(t) = dy(t)/dt$. All these quantities are dependent on earthquake parameters (therefore, on intensity i) and building parameters. These dependences will be derived later in this study. More specifically, it will be shown that σ_y and $\sigma_{\dot{y}}$ are directly proportional to i and that ν_o is a constant.

Expressions for $p(i)$ under other assumptions on the response process are presented in Appendix A.

2.4.2 *Random Resistance Variable, x*

The resistance variable x is a random variable with probability density function $f_X(x)$. In this situation, the failure probability is given by

$$p(i) = \int_0^\infty \text{prob}\,[\max_t | y(t) | \geq x; 0 \leq t \leq t_o] f_X(x)\, dx. \tag{19}$$

Let $Y(t_o) = (\max_t | y(t) |, 0 \leq t \leq t_o)$; and $N_y(t)$ = the number of peaks of $|y(t)|$ in the time t; then $\{N_y(t); 0 \leqq t \leqq t_o\}$ is a random process. Assuming it is a stationary Poisson process of intensity λ_y, it follows from eq. (18) that $\lambda_y = 2\nu_x$. For situations as described in Appendix A, $\lambda_y = 2v_o \ln[1 - \exp(-x^2/2\sigma_g^2)]$ from assumption (*i*); and $\lambda_y = 2v_x\sqrt{2\xi_o}x/\sigma_y$ from assumption (*ii*).

The first excursion probability can be expressed in terms of λ_y as:

$$\text{prob}\,[Y(t_o) \leqq x] = \exp\,[-\lambda_y t_o(1 - F_x(x))] \tag{20}$$

in which $F_X(x) = \int_{-\infty}^{x} f_X(x')\,dx'$. From eqs. (19) and (20) the expression for $p(i)$ becomes

$$p(i) = \int_0^\infty \{1 - \exp\,[-\lambda_y t_o(1 - F_X(x))]\} f_X(x)\,dx. \tag{21}$$

A closed form solution of eq. (21) is possible when $f_X(t)$ has a simple expression such as a uniform, Gaussian, or Rayleigh density function. In general, eq. (21) can be conveniently solved by numerical integration.

2.5 *Ground Motion Statistics*

The statistics characterizing the random ground motion shall now be brought into the formulation. Let $a_i(t)$ be the ground accelerations of earthquakes of intensity i and assume $\{a_i(t); 0 \leqq t \leqq t_o\}$ be a stationary process with a power spectral density function $G_{ai}(\omega)$, where ω is the frequency variable. Such a stochastic ground motion model has been proposed and used extensively, e.g., by Liu[8] and by Jennings et al.[9] Further, assume that the process $\{a_i(t)\}$ is a filtered white noise with a constant power spectrum density G per unit intensity, and that the ground filter is a linear, single-mode oscillator with constant frequency and damping characteristic values ω_g and ξ_g, respectively. The following is a derivation of eq. (22) showing the direct relationship between $G_{ai}(\omega)$ and intensity i.

$$G_{ai}(\omega) = \frac{i^2 G}{(\omega^2 - \omega_g^2)^2 + 4\xi_g^2\omega^2\omega_g^2}. \tag{22}$$

The earthquake intensity value is a measure of the damage potential which is represented by the corresponding response spectrum $S_{\dot{y}} = \max_t |\dot{y}(t)|$ for a structure with natural frequency and damping parameters ω_o and ξ_o. Therefore, $S_{\dot{y}} = S_{\dot{y}}(i, \omega_o, \xi_o, t_o)$ is clearly an increasing function of i. The precise functional relationship between $S_{\dot{y}}$ and i is not yet known, but can be obtained from data fittings of calculated response spectral values of past earthquakes with known i.

A simple linear approximation may be made for our analysis by assuming

$$S_{\dot{y}}(i, \omega_o, \xi_o, t_o) = \text{k}i \tag{23}$$

where k is a constant of proportionality. From the ground acceleration model defined above, it can be shown[10] that

$$\bar{S}_{\dot{y}} = \omega_o K \bar{\sigma}_y \tag{24}$$

where

$$K = (2 \ln \nu_o t_o)^{\frac{1}{2}} + 0.577(2 \ln \nu_o t_o)^{-\frac{1}{2}}. \tag{25}$$

Since K is independent of i, as will be shown later in this section, it is obvious that

$$G_{ai}(\omega) = i^2 G_o(\omega) \tag{26}$$

satisfies eqs. (23) and (24); and according to the ground motion model as defined earlier, $G_o(\omega)$ is given[10] by

$$G_o(\omega) = \frac{G}{(\omega^2 - \omega_g^2)^2 + 4\xi_g^2 \omega^2 \omega_g^2}. \tag{27}$$

Finally, eq. (22) follows directly from eqs. (26) and (27).

It may be noted from eqs. (23), (24), and the relation $\sigma_y^2 = \int_{-\infty}^{\infty} G_{ai}(\omega) \, | H(\omega) |^2 \, d\omega$, that the power spectrum density $G_{ai}(\omega)$ of the earthquake process is proportional to i^2, which agrees with eq. (26). Housner and Jennings[11] have used the relationship $G_{ai}(\omega) = \text{const.}\, S_{\dot{y}}^2$, which also leads to our assertion of eq. (26).

A difficulty exists in determining the value of the constant spectral density G corresponding to a unit Modified Mercalli intensity level. Because the intensity cannot be precisely related to the earthquake waveform parameters such as the amplitude of acceleration, velocity, displacement, response spectrum intensity, etc., some normalization procedures based on these parameters must be used to determine G. For example, a constant power spectral density level for the input white noise to the ground filter is determined by matching the corresponding expected velocity spectra of the filter's response to Housner's average velocity spectra.[12]

Using the well-known relation $\sigma^2 = \int_{-\infty}^{\infty} G(\omega) \, d\omega$, (i.e., the variance of a random process is equal to the integral of its power spectral density over the entire real line representing frequency), it follows from eq. (24) that the variances of $a_i(t)$ and $\dot{a}_i(t)$ are respectively $\sigma_a^2 = i^2 G\pi/(2\xi_g\omega_g^3)$ and $\sigma_{\dot{a}}^2 = \omega_g^2 \sigma_a^2$. Also, from the relation $G_y(\omega) = | H(\omega) |^2 \, G_{ai}(\omega)$ in which $H(\omega) = (\omega^2 - \omega_o^2 - 2j\xi_o\omega_o\omega)^{-1}$ = the transfer function of the simple structure for displacement output $y(t)$ and input $a_i(t)$, where j represents the complex unit, it can be shown that[1]

$$\sigma_y = \left(\frac{\pi G B}{A_0 A_1 B - A_0^2 A_3^2}\right)^{\frac{1}{2}} i = \theta_y i \tag{28}$$

$$\sigma_{\dot{y}} = \left(\frac{\pi G A_3}{A_1 B - A_0 A_3^2}\right)^{\frac{1}{2}} i = \theta_{\dot{y}} i \tag{29}$$

in which $A_0 = \omega_g^2 \omega_o^2$, $A_1 = 2\omega_o\omega_g(\xi_o\omega_g + \xi_g\omega_o)$, $A_2 = \omega_o^2 + \omega_g^2 + 4\xi_o\xi_g\omega_o\omega_g$, $A_3 = 2(\xi_o\omega_o + \xi_g\omega_g)$ and $B = A_2 A_3 - A_1$.

From eqs. (28) and (29), the zero crossing rate of $y(t)$ is

$$\nu_o = \sigma_{\dot{y}}/\pi\sigma_y = \frac{1}{\pi}\left(\frac{A_0 A_3}{B}\right)^{\frac{1}{2}}. \tag{30}$$

To show that K is independent of i in eq. (25), it is sufficient to show that ν_o is likewise independent of i. This is obvious from eq. (30).

This completes the discussion on the determination of the failure probability of structure $p(i)$. It is shown that eqs. (18), (21), and eqs. (37) through (39) in Appendix A define $p(i)$ for various failure mechanisms. Furthermore, substituting eqs. (27) through (30) in the appropriate terms in $p(i)$, indicates that $p(i)$ is a function of intensity i. Finally, substituting eq. (16) and various expressions for $p(i)$ into eq. (10) determines $\bar{p}$, which is a function of i_o.

2.6 *The Expected Random Earthquake Loss*, $\bar{e}$

The earthquake loss depends on earthquake and building parameters, and the extent to which human lives are in danger. The quantity $\bar{e}$ can be determined from statistical data of actual earthquake damage. Unfortunately, empirical values of $\bar{e}$ for different classes of constructions are not presently available. It is logical to assume that $\bar{e}$ is directly proportional to the damage potential of earthquakes, therefore, either of the following relationships, or their combinations, may be appropriate:

$$\bar{e} = C_1 \max_t \overline{a_i(t)} \tag{31a}$$

$$\bar{e} = C_2 \overline{t_o} \tag{31b}$$

$$\bar{e} = C_3\{\bar{S}_y, \text{ or } \bar{S}_{\dot{y}}, \text{ or } \bar{S}_{\ddot{y}}\}, \tag{31c}$$

in which C_1, C_2, C_3 are the constants of proportionality and $\bar{S}$ represents the expected response spectrum associated with the subscript response parameter. Equation (31c) in which $\bar{e}$ is expressed in terms of the expected velocity spectrum $\bar{S}_{\dot{y}}$ appears superior to others because the effects of the amplitude, duration as well as the frequency characteristics of the earthquake accelerogram, are all considered. Thus it will be used in the following analysis. Let $C_3 = \epsilon A_f f$, where ϵ is the percent loss in building cost per unit response velocity spectrum and ϵf is the expected earthquake loss per unit floor area per unit response spectrum. It should be noted that no upper bounds for ϵ can be established in situations where human lives are involved. In these situations, the value of ϵ would increase for large-occupancy structures such as hospitals, schools and office buildings, etc., and decrease for small-occupancy structures such as warehouses, machine rooms, unmanned equipment buildings, etc. The determination of ϵ, with due considerations to loss of human lives, needs actual earthquake life-loss statistics and a mathematical model which converts life-loss into dollars. These matters will need further studies and more data collection. Clearly, expressing $\bar{e}$ in terms of initial building investment is a convenient way of incorporating all possible losses in an earthquake environment. A sensitivity analysis for the parameter ϵ should provide some insight to the overall cost structure. From eqs. (24) and (31c)

$$\begin{aligned} \bar{e} &= C_3 \omega_o K \bar{\sigma}_y \\ &= \epsilon A_f f \omega_o K \theta_y \bar{i} \end{aligned} \tag{32}$$

where

$$\bar{i} = \sum_{i=1}^{12} i f_I(i).$$

It can be noted from the above that $\bar{e}$ is independent of the design intensity i_o and the expected service life of a building. This is because according to its definition, $\bar{e}$ is the expected loss associated with a "single" random event. The quantity $\bar{e}$ should not be confused with the total expected loss of building (see Appendix B for its derivation) throughout the entire service life, which should be expected to increase with service life but decrease with design intensity.

2.7 *Earthquake Risk Cost*

The expression for the earthquake loss function $K_d(i_o)$, for the case of deterministic structural resistance and independent crossings of response process can now be established. Substituting eqs. (8), (32), and (10) [$f_I(i)$, $p(i)$ and σ_y given by eqs. (16), (18) and (28)] into eq. (6) leads to:

$$\begin{aligned} K_d(i_o) = {} & \alpha t \epsilon A_f f \omega_o K \theta_y g(t) \left(\frac{\beta \Gamma J}{l c_2} \right)^2 \sum_{i=1}^{12} i e^{-\beta i / c_2} \sum_{i=i_o}^{12} e^{-\beta i / c_2} \\ & \cdot [1 - \exp(-2 t_o \nu_o e^{-x^2 / 2\theta_y^2 i^2})], \qquad i_o = 1, 2, \cdots, 12 \end{aligned} \tag{33}$$

in which K and ν_o are given by eq. (25) and eq. (30) respectively. To show eq. (33) is a monotonically decreasing function of i_o, let

$$\alpha t \epsilon A_f f \omega_o K \theta_u g(t) \left(\frac{\beta \Gamma J}{l c_2}\right)^2 = \Omega_1 ,$$

$$\sum_{i=1}^{12} i \exp\left(-\frac{\beta i}{c_2}\right) = \Omega_2 , \qquad \frac{\beta}{c_2} = \delta, \qquad 2t_o\nu_o = \mu, \qquad \frac{x^2}{2\theta_u^2} = \zeta,$$

and

$$\frac{K_d(i_o)}{\Omega_1\Omega_2} = Z_o(i_o).$$

Equation (33) is rewritten as

$$\left.\begin{aligned} Z_o(i_o) &= \sum_{k=i_o}^{12} z_k = T_{12} - T_{i_o} , \\ z_k &= e^{-\delta k}[1 - \exp(-\mu e^{-\zeta/k^2})], \\ &\qquad z_o = 0, \quad k, i_o = 1, 2, \cdots, 12 \end{aligned}\right\} \tag{34}$$

where $T_{i_o} = \sum_{k=1}^{i_o} z_k$ is a function of i_o and $T_{12} = \sum_{k=1}^{12} z_k$ is a constant. Notice in eq. (34) that for δ and $\zeta \geqq 0$, both $\exp(-\delta k)$ and $\exp(-\zeta/k^2)$ are bounded between zero and unity; furthermore, for $\mu \geqq 0$, the quantities $\exp[-\mu \exp(-\zeta/k^2)]$ and $1 - \exp[-\mu \exp(-\zeta/k^2)]$ are also bounded between zero and one. It is apparent that since $0 \leqq z_k \leqq 1$ and $T_i = \sum_{k=1}^{i} z_k \geqq 0$, therefore Z_o is a monotonically decreasing function of i_o, as is expected.

Substituting eqs. (3) and (34) into eq. (1), the final expression for the objective function is obtained:

$$\begin{aligned} K(i_o) &= K_c(i_o) + K_d(i_o) \\ &= A_f(f + ci_o) + \Omega_1\Omega_2 Z_o(i_o) \end{aligned} \tag{35}$$

in which the first term in the right-hand side increases with i_o and the second term decreases with i_o. The optimum intensity i_o^* is determined by setting equal to zero the first derivarive of eq. (35) with respect to design intensity i_o, i.e.,

$$\frac{\partial Z_o(i_o)}{\partial i_o} = -\frac{A_f c}{\Omega_1\Omega_2}. \tag{36}$$

Alternatively, i_o^* can be obtained from eq. (35) by direct, numerical evaluation of the function $K(i_o)$ for all i_o.

As an illustration of the presented approach, a hypothetical building design problem is numerically analyzed for its optimal design earthquake intensity. Figure 2 is a plot of i_o versus the normalized total cost $K(i_o)/A_f f$ computed using the following data: The cost data, ϵ = 1 percent, $g(t)$ = 5 percent; the building data, x = deflection = 0.9×10^{-2} ft; the earthquake data, l = 50 miles, α = 3 earthquakes per year, $\beta = 2$, $t_o = 25$ s, $t = 40$ yrs, $c_1 = 8.16$, $c_2 = 1.45$, $c_3 = 2.46$, $m_o = 5$, $\omega_g = 31.4$ rad/s, $\xi_g = 0.5$, $\omega_o = 3.14$ rad/s, $\xi_o = 0.5$, and $d = 40$ mi, and $G = 1.6\ g^2 s^{-3}$. Although these numerical values are used for illustration purposes, nevertheless, the earthquake parameters reflect realistic data based on seismicity in Southern California. Four different seismic protective ratios c/f = 0.01 to 0.04 are considered. The results indicate that for this specific design problem, the optimal design intensity for the building is VII.

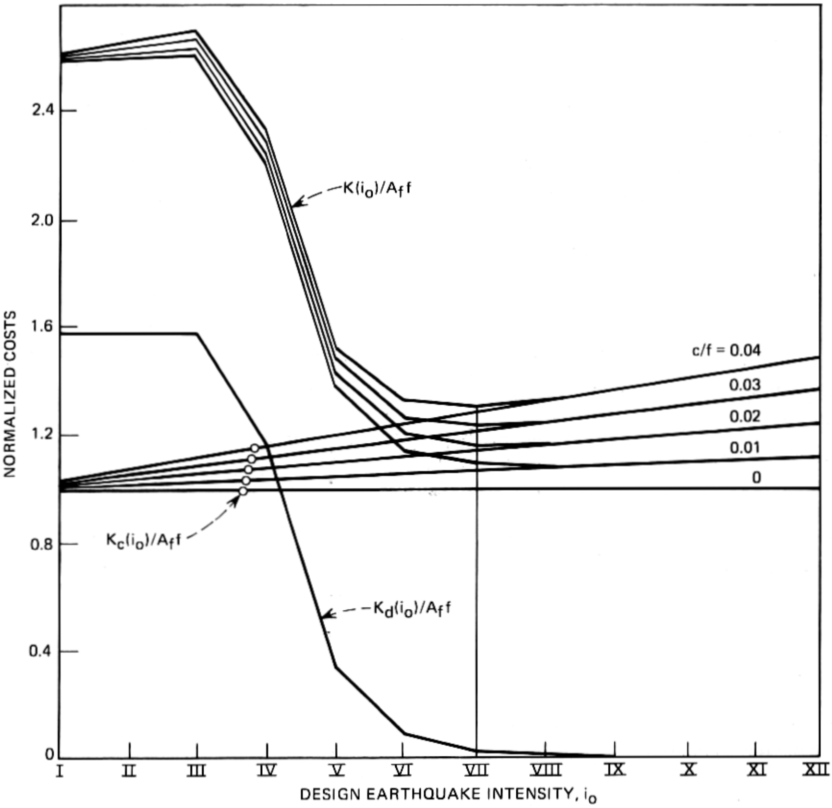

Fig. 2—Optimum design earthquake intensity analysis for a sample building.

A preliminary sensitivity analysis on this numerical example indicates that the convexity of the total cost function becomes more apparent as the building resistance parameter x increases (in this case building deflection measured in feet). For the problem under consideration, the value of x above which the design intensity could be established is found to be 0.7×10^{-2} feet. For smaller values of x, the cost ratio $K(i_o)/A_f f$ becomes insensitive to design intensity i_o and ratio c/f as the total cost curve becomes flat for $i_o >$ V.

III. CONCLUSION

A simple mathematical approach is presented to determine the optimal design intensity of earthquakes for structures. The objective function to be optimized is taken as the total construction cost of the structure plus the expected cost of earthquake damage throughout the entire service life of the structure. For the case of deterministic structural resistance and probabilistic structural response with Poisson (independent) crossings, the objective function is derived in terms of the building and earthquake variables. The optimal design intensity can then be determined by minimizing the objective function with respect to the intensity variable. Other optimum design variables can also be obtained by simply regarding them as the decision variables in the objective function and by performing optimization analysis. The resulting equations are relatively simple and can be easily handled for numerical studies and sensitivity analysis. Generalizations of the results for nondeterministic structural resistance and for response processes different from those having Poisson crossings are also indicated.

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APPENDIX A

Expressions for $p(i)$ under Different Assumptions

(*i*) *Independent Peaks*—The dispersion in the number of peaks of the narrowband response $y(t)$ is neglected and the magnitudes of these peaks are assumed to be statistically independent variables, and each having the probability distribution $P[|y|_{\text{peak}} < x] = 1 - (\nu_x/\nu_o)$, then[7]

$$p(i) = 1 - \exp[-2\nu_o t_o \ln(1 - e^{-x^2/2\sigma_y^2})]. \tag{37}$$

(ii) Independent Envelope Crossings—The crossings of the envelope of $y(t)$ are assumed to be independent and in this situation[7]

$$p(i) = 1 - \exp[-2(2\xi_o)^{\frac{1}{2}}(x/\sigma_y)t_o \exp(-x^2/2\sigma_y^2)] \quad \text{for} \quad \xi_o \ll 1 \tag{38}$$

in which ξ_o equals the damping ratio of the structure.

(iii) Two-State Markov-Process Assumption[13]—The successive intervals that the envelope of $y(t)$ spends above and below the level x are assumed to be random variables with exponential distributions. In this case

$$p(i) = 1 - \exp[-(1 - \nu_x/\nu_o)\exp(-\delta_x t_o)] \tag{39}$$

where

$$\delta_x = n_x(1 - \nu_x/\nu_o)^{-1}$$

and $n_x = \int_0^\infty p(r, \dot{r})\dot{r}\, d\dot{r}\,|_{r=x}$ is the envelope crossing rate of $y(t)$.

APPENDIX B

Expressions for Failure Probability, p(t), and Total Expected Loss of Building, D

Let $h(t)$ be the expected loss in case of failure and $p(t)$ be the failure probability density function for the building, then

$$D = \int_0^{t_D} h(t)p(t)\, dt. \tag{40}$$

where t_D is the service life of the building. From the logic leading to eq. (32), and assuming a discount factor of cost $g(t) = 1 - (t/t_D)$, the function $h(t)$ can be written as

$$h(t) = K_1(i_o)(\eta + g(t)) \tag{41}$$

where η is percent of construction cost, representing the earthquake loss. The failure density $p(t) = dP(t)/dt$, where $P(t)$ is the failure probability of the building and is given by

$$P(t) = 1 - \sum_{i=i_o}^{12} \sum_{n=0}^{\infty} \text{prob}\,[N_i(t) = n](1 - p(i))^n \tag{42}$$

where $p(i)$ is given by eqs. (18) and (21), and $N_i(t)$ equals the total (random) number of earthquakes of intensity i in t years. According to eq. (7)

$$\text{Prob}\,[N_i(t) = n] = \frac{1}{n!}\exp\,(-\Delta F_i \alpha t)(\Delta F_i \alpha t)^n \tag{43}$$

where ΔF_i is the probability that given an earthquake occurs, this earthquake has an intensity equal to i. For a linear source of earthquakes it follows from eq. (11) that

$$\Delta F_i = \frac{1}{l}\,\Gamma J\,\exp\left(-\frac{\beta i}{c_2}\right)\left[1 - \exp\left(-\frac{\beta}{c_2}\right)\right]. \tag{44}$$

Equations (40) through (44) completely define the total loss expectation, D, and from above, it is obvious that D increases with t_D and decreases with i_o.

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