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# Conditions of High Gain in Mixers and Their Relation to the Jump Phenomenon

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*A study of the stability of periodically driven nonlinear networks (mixers), motivated by recent work on low-noise down-conversion with Schottky barrier diodes, is presented. Necessary and sufficient conditions for the unconditional stability of a mixer are derived and discussed. It is shown that potential instability is always associated with the jump phenomenon in the sense that a mixer will (under suitable circumstances) exhibit the phenomenon if, and only if, the above stability conditions are violated. Application of these conditions to frequency multipliers is also discussed.*

## I. INTRODUCTION

The Schottky barrier diode down-converter is a frequency converter that is capable of noise figures below 1 dB in the microwave range with operation at room temperature.[1] However, this converter is potentially unstable, i.e., is capable of arbitrarily high conversion gain. Evaluation of the noise performance at high gain requires a knowledge of the mechanism of instability, and of the conditions necessary and sufficient for instability. Torrey and Whitmer[2] derived a simple stability condition assuming weak reciprocity and also studied a particular case in detail, but their results are not applicable to the down-converter of Ref. 1. Here we derive general stability conditions, in closed form, and show that instability is intimately related to the jump phenomenon, a type of instability peculiar to periodically driven nonlinear networks. These stability conditions are applicable to any periodically driven nonlinear network (henceforth simply called a mixer) provided it is driven by a source (pump) that generates power at a single frequency $\omega_1$. Because they are very general, these conditions can be used for a variety of purposes; for instance, suitable design criteria for harmonic generators can be determined in order to obviate the jump phenomenon and related instabilities in these devices. A brief discussion of this application is given in Section 3.2 following a detailed discussion of the stability of the Schottky barrier diode down-converter in Section 3.1.

Mixer stability is reexamined using the method of Torrey and Whitmer[2] in their phenomenological theory of frequency conversion. This method determines the small-signal terminal behavior of a mixer at the input ($\omega_1 \pm p$), image ($\omega_1 \mp p$), and output ($p$) frequencies without any knowledge or assumptions regarding the internal structure of the mixer. The method requires that the output frequency $p$ be very small, in which case the behavior can be derived directly from the terminal behavior of the mixer at dc and at the pump frequency $\omega_1$. No other assumptions are made.

The small-signal terminal behavior of a mixer can be represented by a nonreciprocal three-terminal-pair network, but no simple stability criterion in closed form is known for such a network; Ku[3] has resorted to graphical and numerical methods. However, because $p$ is assumed small, the three-port assumes special properties that permit study of its stability analytically.

## II. THEORY

### 2.1 *Description*

A mixer that is potentially unstable can exhibit the jump phenomenon and, vice versa, a mixer exhibiting this phenomenon is potentially unstable. To acquaint the reader with our definitions and notation we begin in Section 2.2 with some preliminary considerations, including derivation of a result of Torrey and Whitmer (Ref. 2)‡ [eq. (11)]. In Section 2.3 conditions necessary and sufficient for avoiding the jump phenomenon are derived. In Sections 2.3.3 and 2.3.4, these conditions are shown to be necessary and sufficient for unconditional stability of the mixer; their sufficiency is shown by proving that if they are fulfilled the mixer has passive behavior at $\omega_o \pm p$, provided it is terminated in a passive impedance at $p$. In Section 2.4, it is pointed out that because of such behavior at $\omega_o \pm p$, a certain type of interconnection of stable nonlinear networks is unconditionally stable.

In Section 2.5, we introduce the concept of a stable nonlinear impedance, and discuss its significance.

### 2.2 *Preliminary Considerations*

Suppose the mixer is represented (Fig. 1) by a two-terminal-pair network M with two ideal filters $F_1$ and $F_0$ permitting currents to flow

‡ We are indebted to H. E. Rowe for suggesting including derivation of eqs. (11).

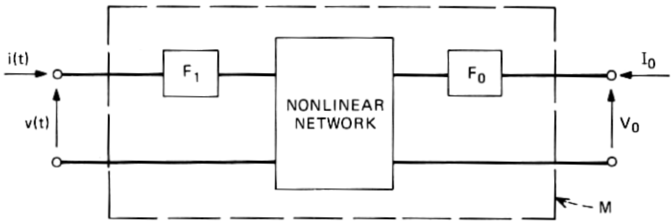


Fig. 1—Periodically driven nonlinear network M.

only in narrow bands centered about $\omega_1$ and dc. This network is assumed to be nonlinear and to contain no sources of energy.

Let the dc and sinusoidal terminal currents be $I_o$ and $i(t)$, the latter with complex amplitude $I$,

$$i(t) = 2\,Re\,(Ie^{i\omega_1 t}); \tag{1}$$

a periodic steady-state is assumed. Let $v(t)$ and $V_o$ be the terminal voltages arising at $\omega_1$ and dc, and let $V$ denote the complex amplitude of $v(t)$,

$$v(t) = 2\,Re\,(Ve^{i\omega_1 t}). \tag{2}$$

Both the dc voltage and the impedance presented by the network at $\omega_1$ are functions of $I_o$ and $|I|$. We write

$$\begin{aligned} V_o &= \mathcal{V}_o(I_o\,,\,|I|) \\ V &= \mathfrak{z}(I_o\,,\,|I|)\cdot I. \end{aligned} \tag{3}$$

It is convenient to choose the time origin so that $i(t)$ is a cosine wave,‡

$$i(t) = 2I\cos\omega_1 t, \qquad I = |I|, \quad \angle I = 0. \tag{4}$$

If we superimpose small perturbations $\delta I_o$ and

$$\delta i(t) = 2\,Re\,(\delta I e^{i\omega_1 t}), \tag{5}$$

on $I_o$ and $i(t)$, and note that

$$|I + \delta I| = I + Re\,\delta I \tag{6}$$

because $I$ is real, then eqs. (3) lead to the variational relationships

$$\begin{aligned} \delta V_o &= \frac{\partial \mathcal{V}_o}{\partial I_o}\,\delta I_o + \frac{\partial \mathcal{V}_o}{\partial |I|}\,Re\,\delta I \\ \delta V &= I\,\frac{\partial \mathfrak{z}}{\partial I_o}\,\delta I_o + I\,\frac{\partial \mathfrak{z}}{\partial |I|}\,Re\,\delta I + \mathfrak{z}\,\delta I, \quad \text{for} \quad I = |I| \end{aligned} \tag{7}$$

‡ This assumption is not used in the following section.

where $\delta V_o$ and $\delta V$ are perturbations on $V_o$ and $V$.

When the network of Fig. 1 is used as a mixer, a small signal is applied at the input frequency ($\omega_1 + p$, or $\omega_1 - p$), and terminations are provided at the output frequency ($p$) and at the image frequency ($\omega_1 - p$, or $\omega_1 + p$). The input signal causes small perturbations at frequencies $\omega_1 \pm p$ and $p$ to appear at the terminals of M. We wish to derive from eqs. (7) the relations among the various frequency components of these perturbations. Equations (7) hold without change even if $\delta I$ and $\delta I_o$ vary with time, provided the variations are *very slow.* Let

$$\begin{aligned} \delta I &= \delta I(t) = I_\alpha e^{ipt} + I_\gamma e^{-ipt} \\ \delta I_o &= \delta I_o(t) = I_\beta e^{ipt} + I_\beta^* e^{-ipt}. \end{aligned} \tag{8}$$

Then the terminal currents of M become

$$\begin{aligned} i(t) + \delta i(t) &= 2Re(Ie^{i\omega_1 t} + I_\alpha e^{i(\omega_1+p)t} + I_\gamma e^{i(\omega_1-p)t}) \\ I_o + \delta I_o(t) &= I_o + 2Re(I_\beta e^{ipt}). \end{aligned} \tag{9}$$

Substituting eq. (8) in eqs. (7), after replacing $I$ with $|I|$,

$$\begin{aligned} \delta V &= \delta V(t) = V_\alpha e^{ipt} + V_\gamma e^{-ipt} \\ \delta V_o &= \delta V_o(t) = V_\beta e^{ipt} + V_\beta^* e^{-ipt}, \end{aligned} \tag{10}$$

where

$$\begin{bmatrix} V_\alpha \\ V_\beta \\ V_\gamma^* \end{bmatrix} = \begin{bmatrix} \mathfrak{z} + \frac{1}{2}|I|\frac{\partial \mathfrak{z}}{\partial |I|} & |I|\frac{\partial \mathfrak{z}}{\partial I_o} & \frac{1}{2}|I|\frac{\partial \mathfrak{z}}{\partial |I|} \\ \frac{1}{2}\frac{\partial \mathcal{V}_o}{\partial |I|} & \frac{\partial \mathcal{V}_o}{\partial I_o} & \frac{1}{2}\frac{\partial \mathcal{V}_o}{\partial |I|} \\ \frac{1}{2}|I|\frac{\partial \mathfrak{z}^*}{\partial |I|} & |I|\frac{\partial \mathfrak{z}^*}{\partial I_o} & \mathfrak{z}^* + \frac{1}{2}|I|\frac{\partial \mathfrak{z}^*}{\partial |I|} \end{bmatrix} \begin{bmatrix} I_\alpha \\ I_\beta \\ I_\gamma^* \end{bmatrix}$$

$$(\text{for } I = |I|). \tag{11}$$

Thus we have the Torrey and Whitmer result.[2] Note that according to eqs. (2) and (10) the terminal voltages produced by the currents of eqs. (9) can be written

$$\begin{aligned} v(t) + \delta v(t) &= 2Re(Ve^{i\omega_1 t} + V_\alpha e^{i(\omega_1+p)t} + V_\gamma e^{i(\omega_1-p)t}) \\ V_o + \delta V_o(t) &= V_o + 2Re(V_\beta e^{ipt}). \end{aligned} \tag{12}$$

$V_\alpha$, $V_\gamma$ and $V_\beta$ are the complex amplitudes of the terminal voltages at $\omega_1 + p$, $\omega_1 - p$ and $p$, respectively.

Equation (11) describes the mixer performance subject to three assumptions

(*i*) Quasi-static ($p$ small)
(*ii*) Small-signal ($| \delta I | \ll | I |$, $\delta I_o \ll | I_o |$)
(*iii*) Zero phase for $I$ [eq. (4)] (not restrictive).

The matrix elements in (11), and hence the performance of the mixer, depend exclusively upon the same coefficients $\mathfrak{z}$, $| I | (\partial \mathfrak{z} / \partial | I |)$, etc. that characterize the small-signal terminal behavior at $\omega_1$ and dc [see eqs. (7)].

*Stability:* The network M is unconditionally stable, for a given steady-state condition, if the powers

$$Re(V_\alpha I_\alpha^*), \qquad Re(V_\beta I_\beta^*), \qquad Re(V_\gamma I_\gamma^*) \tag{13}$$

absorbed at $\omega_1 + p$, $p$ and $\omega_1 - p$ cannot *simultaneously* become negative. On the other hand, if

$$Re(V_\alpha I_\alpha^*) < 0, \qquad Re(V_\beta I_\beta^*) < 0, \qquad Re(V_\gamma I_\gamma^*) < 0 \tag{14}$$

*simultaneously*, then M is potentially unstable. In this case, spurious oscillations at $\omega_1 \pm p$ and $p$ can be produced (without sources at these frequencies) by terminating M with appropriately chosen impedances at $\omega_1 \pm p$ and $p$. A potentially unstable mixer can have (in principle) unlimited conversion gain.

### 2.2.1 *The Jump Phenomenon*‡ *and Stability*

Now suppose a one-terminal-pair network $N$ is constructed by connecting M to a linear circuit consisting of a fixed resistance $R_o$ in series with a constant voltage $E_o$, as shown in Fig. 2a. For given values of $R_o$ and $E_o$, the impedance

$$Z = R + jX = \frac{V}{I} \tag{15}$$

presented at $\omega_1$ is now a function only of the magnitude of $I$. Let us connect in series to this network a linear passive impedance $Z_1$ as shown in Fig. 2b, and let $E$ denote the complex amplitude of the voltage $e(t)$ arising at the terminals. The behavior at $\omega_1$ is described by the equation $E = I(Z + Z_1)$. Thus, if $R_1$ and $X_1$ denote the real and

‡ This phenomenon is discussed in various texts on nonlinear differential equations (e.g. Ref. 4) for systems governed by Duffing's equation.

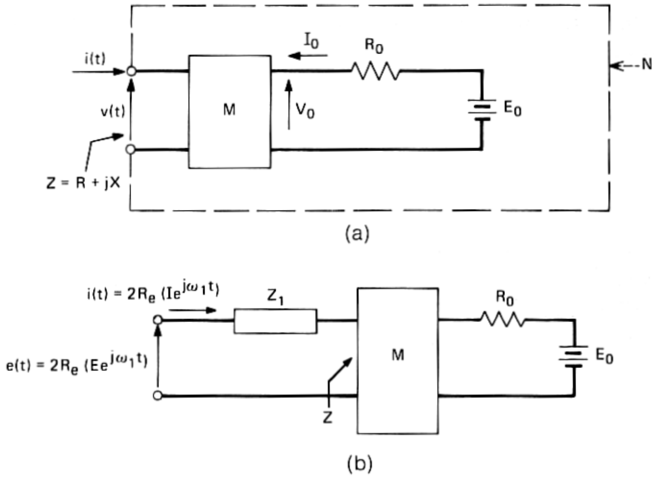


Fig. 2—Networks consisting of (a) M connected to a dc supply, and (b) M connected to a dc supply and a passive impedance $Z_1$.

imaginary parts of $Z_1$, we can write the following relation for the magnitude of $E$

$$| E | = | I | \sqrt{(R_1 + R)^2 + (X_1 + X)^2}, \tag{16}$$

where it is important to keep in mind that $R$ and $X$ are functions of $| I |$. The form of these functions depends, as seen from Fig. 2a, upon the values of $R_o$ and $E_o$ and the behavior of M.

The jump phenomenon occurs when $| E |$ is not a strictly monotonic function of $| I |$ (i.e., when $| E |$ has a negative slope for some $| I |$). For instance, suppose that $| E |$ has the behavior of Fig. 3, and let an ideal voltage source with zero internal impedance and variable $| E |$ be connected to $N$ as indicated in Fig. 3. Then, if $| E |$ is gradually increased, starting from $| E | = 0$, $| I |$ will increase smoothly until it

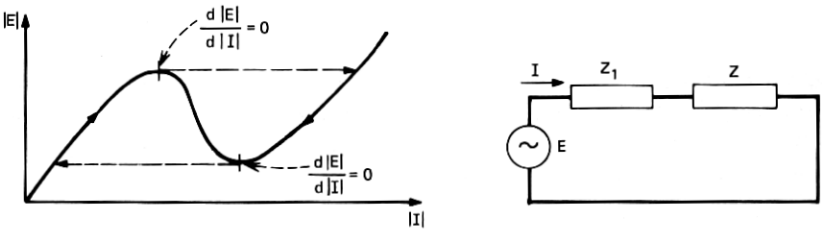


Fig. 3—Jump phenomenon.

reaches a critical value for which

$$\frac{d \mid E \mid}{d \mid I \mid} = 0.$$

At this point $\mid I \mid$ will suddenly jump to another value‡, as indicated in Fig. 3. If $\mid E \mid$ is then decreased, $\mid I \mid$ will decrease smoothly until another jump occurs, as shown in Fig. 3, for $d \mid E \mid / d \mid I \mid = 0$.

The following stability criterion, the validity of which will be proven in the following two sections, plays a central role in this paper.

*Stability Criterion:* Suppose one wants to determine whether or not M is unconditionally stable for a given steady-state condition. Assume in Fig. 2b that $R_o$ and $Z_1$ are arbitrary, but that $E_o$ and $\mid I \mid$ have been chosen to produce the given steady-state condition in M. It will be shown that M is unconditionally stable if, and only if, the following property is obeyed:

$$\frac{d \mid E \mid}{d \mid I \mid} > 0 \quad \text{for all} \quad R_o \geqq 0, \qquad R_1 \geqq 0, \quad \text{and} \quad X_1 . \tag{17}$$

This result implies that, if for a given steady state M is unconditionally stable, then discontinuous jumps from the steady state in question cannot occur, no matter what the values of $R_o$, $R_1$ and $X_1$ may be. If, on the other hand, M is potentially unstable, then the jump phenomenon can be produced by certain choices of $R_o$, $R_1$, and $X_1$.

### 2.3 *Stability Criteria*

In the first part of this section we will show that requirement (17) demands that the behavior of $R + jX$ as a function of $\mid I \mid$ satisfy the inequality

$$4R \frac{d(\mid I \mid R)}{d \mid I \mid} > \left( \mid I \mid \frac{dX}{d \mid I \mid} \right)^2 . \tag{18}$$

Since the derivatives of $R$ and $X$ in this inequality depend not only upon the properties of M but also upon the value of $R_o$, this inequality must be fulfilled for all $R_o \geqq 0$. In Section 2.3.2, we determine the relationship between $R_o$ and the derivatives of $R$, $X$, and show that if inequality (18) is fulfilled in the two particular cases

$$R_o = \infty \tag{19}$$

‡ Assuming, of course, that for the steady-state condition corresponding to this new value of $|I|$ the circuit is stable, and that a transient leading to this new condition (from the unstable condition) exists.

and

$$R_o = 0, \tag{20}$$

then it is in general also fulfilled for all positive $R_o$ ("in general" means: except when $\partial \mathcal{V}_o/\partial I_o \leqq 0$, as we shall see). Furthermore, it will be shown (see Section 2.3.2) that in these two cases inequality (18) becomes, respectively,

$$4\mathcal{R} \frac{\partial(|I|\mathcal{R})}{\partial |I|} > |I|^2 \left(\frac{\partial \mathcal{X}}{\partial |I|}\right)^2 \tag{21}$$

and

$$4R \frac{\partial \mathcal{V}_o}{\partial I_o} \left[\frac{\partial \mathcal{V}_o}{\partial I_o} \frac{\partial(|I|\mathcal{R})}{\partial |I|} - \frac{\partial \mathcal{V}_o}{\partial |I|} \frac{\partial(|I|\mathcal{R})}{\partial I_o}\right]$$
$$> \left(|I| \frac{\partial \mathcal{X}}{\partial |I|} \frac{\partial \mathcal{V}_o}{\partial I_o} - |I| \frac{\partial \mathcal{X}}{\partial I_o} \frac{\partial \mathcal{V}_o}{\partial |I|}\right)^2 \tag{22}$$

where the functions $\mathcal{R} = \mathcal{R}(I_o, |I|)$ and $\mathcal{X} = \mathcal{X}(I_o, |I|)$ are the real and imaginary parts of $\mathfrak{z} = \mathfrak{z}(I_o, |I|)$. At the end of Section 2.3.2 we will find that for requirement (17) to be fulfilled it is necessary and sufficient that the above two inequalities be fulfilled, and that

$$R > 0, \qquad \frac{\partial \mathcal{V}_o}{\partial I_o} > 0. \tag{23}$$

In Sections 2.3.3 and 2.3.4, these inequalities are shown to be necessary and sufficient for unconditional stability of the mixer.

#### 2.3.1 *Significance of Inequality (18)*

For a given steady-state of M, and given values of $R_o$ and $E_o$ in Fig. 2, we wish to show that the requirement

$$\frac{d|E|}{d|I|} > 0 \quad \text{for all} \quad R_1 \geqq 0, X_1 \tag{24}$$

is fulfilled if, and only if, $R > 0$ and inequality (18) is fulfilled.

First, note that if $R \leqq 0$ then requirement (24) is certainly violated because one can verify, using eq. (16), that $d|E|/d|I| = 0$ for

$$R_1 = -R, \qquad X_1 = -X - |I| \frac{dX}{d|I|}.$$

Thus for fulfillment of requirement (24) it is necessary that $R > 0$.

Next, we show that if $R > 0$, then for fulfillment of (24) it is necessary and sufficient that inequality (18) be satisfied. We begin by noting that requirement (24) is equivalent to

$$\frac{|E|}{|I|}\frac{d|E|}{d|I|} > 0 \quad \text{for all} \quad R_1 \geqq 0,\ X_1\ . \tag{25}$$

(Note that $|E| \neq 0$ because $R > 0$.) Let us therefore examine the dependence of the quantity

$$\frac{|E|}{|I|}\frac{d|E|}{d|I|} = \frac{1}{2|I|}\frac{d(|E|^2)}{d|I|} \tag{26}$$

upon $X_1$ and $R_1$. If one calculates this quantity, using eq. (16), it is found that its minimum value as a function of $X_1$ occurs for

$$X_1 = -\frac{1}{2}\left[X + \frac{d(|I|X)}{d|I|}\right], \tag{27}$$

while its minimum value as a function of $R_1$‡ occurs either for

$$R_1 = -\frac{1}{2}\left[R + \frac{d(|I|R)}{d|I|}\right] \tag{28}$$

or for $R_1 = 0$, according to whether the value given by eq. (28) for $R_1$ is positive or negative. In the former case one finds, using eqs. (16), (26), (27) and (28),

$$\left(\frac{|E|}{|I|}\frac{d|E|}{d|I|}\right)_{\min} = -\frac{1}{4}\left[|I|^2\left(\frac{dR}{d|I|}\right)^2 + |I|^2\left(\frac{dX}{d|I|}\right)^2\right].$$

Thus, requirement (24) is surely violated if the value given by eq. (28) for $R_1$ is positive. To satisfy (24) it is therefore necessary that (28) be negative. That is, it is necessary that

$$R + \frac{d(|I|R)}{d|I|} > 0, \tag{29}$$

in which case one has

$$\left(\frac{|E|}{|I|}\frac{d|E|}{d|I|}\right)_{\min} = R\frac{d(|I|R)}{d|I|} - \frac{1}{4}|I|^2\left(\frac{dX}{d|I|}\right)^2.$$

This expression is positive if (and only if) inequality (18) is satisfied. Thus, requirement (24) is fulfilled provided inequalities (29), (18) and $R > 0$ are satisfied; since the first of these is implied by the latter two,

‡ Note that we assume $R_1 \geqq 0$.

it is necessary and sufficient that only inequality (18) be satisfied, and $R > 0$.

2.3.2 *Derivation of Inequalities (21), (22) and (23)*

We can write

$$\frac{dX}{d\,|I|} = \frac{\partial \mathfrak{X}}{\partial I_o}\frac{dI_o}{d\,|I|} + \frac{\partial \mathfrak{X}}{\partial\,|I|} \quad (\mathfrak{X} \equiv \mathfrak{X}(I_o\,,\,|I|)). \tag{30}$$

An analogous expression can be written for $d\,(|I|\,R)/d\,|I|$ [substitute $X \to |I|\,R$ throughout eq. (30)]. Thus, we can write for inequality (18)

$$4R\left[\frac{\partial(|I|\,\mathfrak{R})}{\partial\,|I|} + \frac{\partial(|I|\,\mathfrak{R})}{\partial I_o}\frac{dI_o}{d\,|I|}\right] - |I|^2\left[\frac{\partial \mathfrak{X}}{\partial\,|I|} + \frac{\partial \mathfrak{X}}{\partial I_o}\frac{dI_o}{d\,|I|}\right]^2 > 0. \tag{31}$$

From Fig. 2a, $\mathfrak{V}_o(I_o, |I|) + R_oI_o = E_o$. Differentiating this relation we obtain

$$\frac{\partial \mathfrak{V}_o}{\partial I_o}\,dI_o + \frac{\partial \mathfrak{V}_o}{\partial\,|I|}\,d\,|I| + R_o\,dI_o = 0.$$

Thus,

$$\frac{dI_o}{d\,|I|} = -\frac{\partial \mathfrak{V}_o}{\partial\,|I|}\left[\frac{\partial \mathfrak{V}_o}{\partial I_o} + R_o\right]^{-1}. \tag{32}$$

Using this relation we obtain from inequality (18) in the two cases $R_o = \infty$ and $R_o = 0$, inequalities (21) and (22) respectively, as stated at the beginning of this section.

The conditions necessary and sufficient for fulfillment of inequality (31), for all $R_o \geqq 0$, are obtained by noting that this requirement demands that

$$\frac{\partial \mathfrak{V}_o}{\partial I_o} > 0, \tag{33}$$

because the magnitude of $dI_o/d\,|I|$ becomes infinite‡ (and consequently inequality (31) is violated) for $R_o \cong -(\partial\mathfrak{V}_o/\partial I_o)$. Therefore let $\partial\mathfrak{V}_o/\partial I_o > 0$. Then $dI_o/d\,|I|$ is (for $R_o \geqq 0$) a continuous function of $R_o$, and it varies from the value

‡ We assume $(\partial \mathfrak{Z}/\partial I_o)\cdot(\partial \mathfrak{V}_o/\partial|I|) \neq 0$. One can show that the necessary and sufficient conditions for the stability of $M$ are also given by inequalities (21) through (23) in the special case $(\partial \mathfrak{Z}/\partial I_o)\cdot(\partial \mathfrak{V}_o/\partial|I|) = 0$.

$$\beta = -\frac{\partial \mathfrak{V}_o}{\partial \mid I \mid} \Big/ \frac{\partial \mathfrak{V}_o}{\partial I_o}$$

to 0, as $R_o$ varies from 0 to $\infty$. If $y$ denotes the left-hand side of inequality (31), and $x$ denotes $dI_o/d \mid I \mid$, we have from inequality (31)

$$\frac{d^2 y}{dx^2} = -2 \left| I \frac{\partial \mathfrak{X}}{\partial I_o} \right|^2 < 0.$$

It follows that $y$ cannot have interior minima in the interval $\beta \leqq x \leqq 0$; therefore the lowest value attained by the left-hand side of inequality (31), as $dI_o/d \mid I \mid$ varies from $\beta$ to 0, must occur at one of those end points. Since those two points correspond to two cases $R_o = 0$ and $R_o = \infty$, one concludes that if inequality (31) is fulfilled in these two cases then it is also fulfilled for all $R_o > 0$. The conditions necessary and sufficient that inequality (31) be fulfilled for all $R_o > 0$ are, therefore, inequalities (21), (22) and (33).

### 2.3.3 *Necessity of Inequalities (21) through (23) for the Unconditional stability of* M

We have derived inequalities (21) through (23) from the behavior of the network of Fig. 2a at $\omega_1$, by requiring $Z$ to satisfy inequality (18) for all $R_o \geqq 0$. Alternatively, these inequalities could have been derived from the dc behavior of the network of Fig. 4, by requiring that the derivative of $V_o$ (with respect to $I_o$) be positive for all passive $Z_1$.‡ In fact it is shown in the Appendix that this requirement and requirement (17) are equivalent; this implies that if inequalities (21) through (23) are violated, then by properly choosing $Z_1$, one can make the network of Fig. 4 exhibit a negative differential resistance at dc as illustrated in Fig. 5. Since such a network is potentially unstable, we conclude that inequalities (21) through (23) are necessary conditions

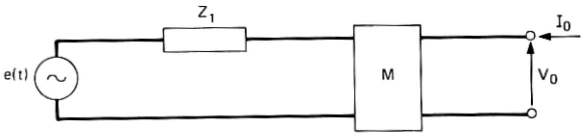


Fig. 4—Network consisting of M driven by a pump with internal impedance $Z_1$.

‡ This requirement is discussed in Ref. 2. We have chosen to derive our stability conditions from requirement (17), rather than the requirement $dV_o/dI_o > 0$, because one of the purposes of our derivation is to point out the relation between inequalities (21) and (22) and inequality (18). This relation is essential for the proof in the following section. The significance and practical importance of inequality (18) is pointed out in Section 2.5.

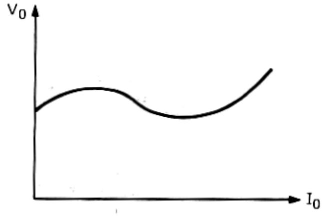


Fig. 5—Example of a dc characteristic with a negative slope.

for the unconditional stability of M. In the following section it is shown that they are also sufficient conditions.

2.3.4 *Sufficiency of Inequalities (21) through (23) for the Unconditional stability of* M

In Fig. 2a, assume that the internal impedance of the linear circuit connected to M is, instead of a frequency-independent resistance $R_o$, a passive impedance $Z_o(\omega)$ with the arbitrary value $Z_\beta$ at $\omega = p$. Let a small perturbation $\delta i(t)$ containing the frequencies $\omega_1 \pm p$ be superimposed on the terminal current $i(t)$ of this network, as shown in Fig. 6. According to the definition of Section 2.2, M is unconditionally stable if it is impossible that

$$\text{Re}\,[I_\alpha V_\alpha^*] < 0, \qquad \text{Re}\,[I_\gamma V_\gamma^*] < 0, \tag{34}$$

simultaneously. Recall that $I_\alpha$, $I_\gamma$, $V_\alpha$ and $V_\gamma$ are the Fourier coefficients of $\delta i(t)$ and $\delta v(t)$ of $\omega_1 \pm p$ [see eqs. (9) and (12)]. In this section we show that inequalities (21) through (23) guarantee

$$\text{Re}\,(I_\alpha V_\alpha^* + I_\gamma V_\gamma^*) > 0. \tag{35}$$

An obvious consequence of this result is that Re $[I_\alpha V_\alpha^*]$ and Re $[I_\gamma V_\gamma^*]$ cannot simultaneously become negative, that is, M is unconditionally stable if inequalities (21) through (23) are fulfilled.

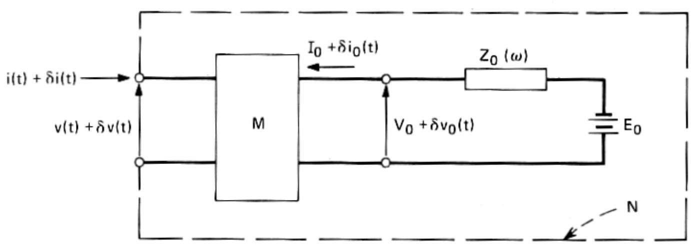


Fig. 6—Network N with perturbations at $\omega_1 \pm p$ and $p$.

*Theorem:*[‡] *If inequalities* (21) *through* (23) *are fulfilled and Re* $(I_\beta V_\beta^*) < 0$ *then necessarily Re* $(I_\alpha V_\alpha^* + I_\gamma V_\gamma^*) > 0$.

*Proof:*[§] The relations imposed by M among $I_\alpha$, $I_\beta$, $I_\gamma$, $V_\alpha$, $V_\beta$ and $V_\gamma$ are given by eq. (11). By using the constraint $V_\beta = -Z_\beta I_\beta$, which is imposed by the linear circuit at $\omega = p$, one may easily eliminate from eq. (11) the variables $V_\beta$ and $I_\beta$, so as to obtain the following relations among $I_\alpha$, $I_\gamma$, $V_\alpha$ and $V_\gamma$,

$$\begin{bmatrix} V_\alpha \\ V_\gamma^* \end{bmatrix} = [Z_{\alpha,\gamma}] \begin{bmatrix} I_\alpha \\ I_\gamma^* \end{bmatrix}, \tag{36}$$

where

$$[Z_{\alpha,\gamma}] = \begin{bmatrix} \mathfrak{z} + \frac{1}{2}\left[|I| \frac{\partial \mathfrak{z}}{\partial |I|} + |I| \frac{\partial \mathfrak{z}}{\partial I_o}\xi\right], & \frac{1}{2}\left[|I| \frac{\partial \mathfrak{z}}{\partial |I|} + |I| \frac{\partial \mathfrak{z}}{\partial I_o}\xi\right] \\ \frac{1}{2}\left[|I| \frac{\partial \mathfrak{z}^*}{\partial |I|} + |I| \frac{\partial \mathfrak{z}^*}{\partial I_o}\xi\right], & \mathfrak{z}^* + \frac{1}{2}\left[|I| \frac{\partial \mathfrak{z}^*}{\partial |I|} + |I| \frac{\partial \mathfrak{z}^*}{\partial I_o}\xi\right] \end{bmatrix} \tag{37}$$

$$\xi = -\frac{\partial \mathfrak{V}_o}{\partial |I|}\left[\frac{\partial \mathfrak{V}_o}{\partial I_o} + Z_\beta\right]^{-1}. \tag{38}$$

Condition (35) demands that $[Z_{\alpha,\gamma}] + [Z_{\alpha,\gamma}]^\dagger$ (the superscript $(\ )^\dagger$ denotes the Hermitian conjugate) be positive definite. If we introduce the new quantities

$$Z_u = R_u + jX_u = \frac{\partial(|I|\,\mathfrak{R})}{\partial |I|} + \frac{\partial(|I|\,\mathfrak{R})}{\partial I_o}\xi \tag{39}$$

$$Z_v = R_v + jX_v = |I| \frac{\partial \mathfrak{X}}{\partial |I|} + |I| \frac{\partial \mathfrak{X}}{\partial I_o}\xi \tag{40}$$

then from eq. (37) we obtain

$$[Z_{\alpha,\gamma}] + [Z_{\alpha,\gamma}]^\dagger = \begin{bmatrix} R_u + \mathfrak{R} - X_v & R_u - \mathfrak{R} + jR_v \\ R_u - \mathfrak{R} - jR_v & R_u + \mathfrak{R} + X_v \end{bmatrix}. \tag{41}$$

[‡] Note that the condition $Re(I_\alpha V_\alpha^* + I_\gamma V_\gamma^*) > 0$ if $Re(I_\beta V_\beta^*) < 0$ is more restrictive than the condition imposed by the requirement of stability (see Section 2.2.1).

[§] Throughout the proof it is assumed implicitly $Re(I_\beta V_\beta^*) < 0$, since the impedance $Z_\beta = Z_o(p)$ in Fig. 6 is assumed passive.

One can verify that this matrix is positive definite if, and only if, $R > 0$ and

$$4R_u\mathcal{R} - |Z_v|^2 > 0. \tag{42}$$

Thus, in order to prove the above, theorem one must show that inequality (42) is satisfied if inequalities (21) through (23) are satisfied.

Comparison of (38) and (32) shows that for $Z_\beta = R_o$ the quantity $\xi$ appearing in eqs. (39) and (40) equals $dI_o/d\,|I|$. Furthermore, if $\xi = dI_o/d\,|I|$, then inequality (42) reduces to inequality (31), as one can verify using eqs. (39) and (40). Thus, for $Z_\beta = R_o$, inequalities (42) and (31) are equivalent. It follows that if inequalities (21) through (23) are fulfilled then inequality (36) is certainly satisfied for $X_\beta = 0$. We will now show that if inequalities (21) through (23) are fulfilled, inequality (42) is satisfied even if $X_\beta \neq 0$.

It is convenient to introduce the quantity

$$Q = (4R_u\mathcal{R} - |Z_v|^2)\left[\left(\frac{\partial \mathcal{V}_o}{\partial I_o} + R_\beta\right)^2 + X_\beta^2\right],$$

a product of two factors. This second factor is always positive and the first is the expression appearing in inequality (42); it follows that inequality (42) is equivalent to the condition $Q > 0$. Now let us consider the behavior of $Q$ as a function of $X_\beta$. Using eqs. (38) through (40), it can be verified that

$$\frac{\partial Q}{\partial X_\beta} = 2X_\beta\left[4R\,\frac{\partial(|I|\,\mathcal{R})}{\partial\,|I|} - \left(|I|\,\frac{\partial \mathcal{X}}{\partial\,|I|}\right)^2\right].$$

From this relation we see that if inequality (21) is fulfilled, then the minimum value of $Q$ (as a function of $X_\beta$) occurs for $X_\beta = 0$. That is, $Q$ is positive for all $X_\beta$ provided it is positive for $X_\beta = 0$. Since we already know that inequalities (21) through (23) insure $Q > 0$ for $X_\beta = 0$, we conclude that they also insure $Q > 0$ for all $X_\beta$. Thus, if inequalities (21) through (23) are fulfilled, then $\mathcal{R} > 0$, $Q > 0$ and condition (35) is fulfilled.

### 2.4 *Lossless Interconnection of n Nonlinear Networks*

In this section certain properties of a lossless interconnection of stable nonlinear networks are discussed; these illustrate the significance of the theorem of the preceding section.

Consider $n$ networks $N_1, N_2 \cdots, N_n$ of the same type as the network of Fig. 6. Let them be connected as shown in Fig. 7, through a $(n + 1)$-terminal-pair linear time-invariant lossless network L, resulting in a

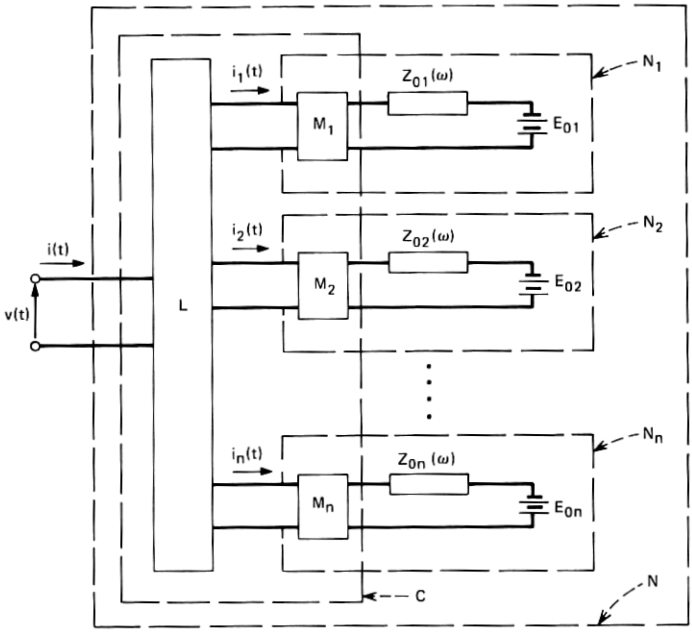


Fig. 7—Lossless interconnection of $n$ networks $N_1, \cdots, N_n$.

one-terminal-pair network N. Let N be driven by the sinusoidal current $i(t)$ of eq. (1) and assume that $i(t)$ produces in N a periodic steady-state with frequency $\omega_1$. Assume that $M_1, M_2, \cdots, M_n$ are unconditionally stable for this steady-state. Then the small-signal terminal behavior of N satisfies condition (35) (i.e., N is passive at $\omega_1 \pm p$), no matter what the values of the passive impedances $Z_{01}(\omega)$, $Z_{02}(\omega)$, $\cdots$, $Z_{on}(\omega)$ for $\omega = p$ may be. This is a direct consequence of the theorem of the preceding section, which shows that if we superimpose on $i(t)$ the perturbation $\delta i(t)$ of eqs. (9), then the total power absorbed at $\omega_1 \pm p$ by $N_r$ $(r = 1, 2, \cdots, n)$ is necessarily positive. Thus, the power absorbed by N (the sum of the powers absorbed by $N_1, \cdots, N_n$, because L is lossless) is positive.

Note that this result implies that when $M_1, \cdots, M_n$ are unconditionally stable, then the $(n + 1)$-terminal-pair network C consisting of $M_1, \cdots, M_n$ interconnected through L (Fig. 7) is also unconditionally stable. Thus, a lossless interconnection (of the type represented by the network C) of unconditionally stable networks $M_1, \cdots, M_n$ is unconditionally stable.

Another consequence of the above result is that the impedance $Z$

presented by N at $\omega_1$ must satisfy inequality (18) no matter what the values of the positive dc resistances $R_{01} = Z_{01}(0)$, $R_{02} = Z_{02}(0)$, $\cdots$, $R_{0n} = Z_{0n}(0)$ may be. In fact, in the following section we will see that if inequality (18) were violated for some choice of $R_{01}$, $R_{02}$, $\cdots$, $R_{0n}$ then the small-signal terminal behavior of N at $\omega_1 \pm p$ would be potentially unstable, which cannot be, because we have already seen that C is unconditionally stable.

### 2.5 *Concept of a Stable Nonlinear Impedance*

*Definition:* The nonlinear impedance $Z$ presented at $\omega_1$ by a one-terminal-pair nonlinear network N which can exchange power only in the vicinity of $\omega_1$ and *does not* contain time-varying sources of energy (such as the network N shown in Fig. 2a or the network N of Fig. 7) is said to be stable if (and only if) it satisfies inequality (18) and $R > 0$.

An important property of a stable, nonlinear impedance has already been pointed out in Section 2.3, where it was shown that such an impedance cannot give rise to the jump phenomenon. We now want to point out another property of this impedance in connection with the small-signal terminal-behavior of N at $\omega_1 \pm p$.

If the frequency $p$ is so small that the value of $Z_o(\omega)$ (Fig. 6) for $\omega = p$ can be assumed equal to its value for $\omega = 0$, $Z_\beta = R_o$, then the small-signal terminal behavior of the network N of Fig. 6 at $\omega_1 \pm p$ is uniquely specified by $Z$ and the derivative of $Z$ with respect to $|I|$. In fact, if in eq. (38) we set $Z_\beta = R_o$, then $\xi$ can be identified as the derivative of $I_o$ with respect to $|I|$ [see eq. (32)], and therefore according to eq. (37) the small-signal terminal behavior at $\omega_1 \pm p$ can be expressed in the form

$$\begin{bmatrix} V_\alpha \\ V_\gamma^* \end{bmatrix} = \begin{bmatrix} Z + \frac{1}{2}|I|\dfrac{dZ}{d|I|} & \frac{1}{2}|I|\dfrac{dZ}{d|I|} \\ \frac{1}{2}|I|\dfrac{dZ^*}{d|I|} & Z^* + \frac{1}{2}|I|\dfrac{dZ^*}{d|I|} \end{bmatrix} \begin{bmatrix} I_\alpha \\ I_\gamma^* \end{bmatrix}, \tag{43}$$

provided $I = |I|$. This equation‡ is also applicable to the network of Fig. 7, in which case $p$ must be sufficiently small such that $Z_{or}(p) \cong R_{or}$ $(r = 1, \cdots, n)$. Now, one can easily verify using this equation that the conditions necessary and sufficient for passivity are identical to the

‡ Note that this equation can be obtained by the same method used in Section 2.2 to derive eq. (11). In fact, the matrix of eq. (43) can be formed directly from the matrix of eq. (11) by deleting from this matrix the second row and second column and then replacing $\partial Z/\partial|I|$ with $dZ/d|I|$ throughout the resulting $2 \times 2$ matrix.

conditions necessary and sufficient for unconditional stability, and are given by inequality (18) and $R > 0$. Thus, we can say that a stable nonlinear impedance insures passive behavior against small perturbations at frequencies very close to $\omega_1$ (this property is in accord with the theorem of the preceding section).

An interesting consequence of these results is now discussed in connection with the circuit of Fig. 7, which has been redrawn schematically in Fig. 8. Consider the small-signal terminal behavior of this network about some given steady-state condition and assume that the impedances $Z_1, \cdots, Z_n$ presented at $\omega_1$ by the nonlinear networks $N_1, \cdots, N_n$ are stable. Let $p$ be sufficiently small so that eq. (43) is applicable to each nonlinear network $N_r$ (i.e., let $Z_{or}(p) \cong R_{or}$). Then each nonlinear network is passive at $\omega_1 \pm p$ and therefore N is also passive at $\omega_1 \pm p$; this implies that the impedance resulting from a lossless interconnection (of the type shown in Fig. 8) of stable nonlinear impedances is a stable impedance.‡ In particular, if two stable nonlinear impedances are connected in series, or in parallel, the resulting nonlinear impedance is stable.

This last result has an important application in connection with harmonic generators. Often such nonlinear networks are driven by pumps that are *not* linear and that can be represented by an equivalent circuit consisting of a nonlinear impedance $Z_1$ in series with an ideal voltage $e(t)$. The result in question shows that, even in the case of a harmonic generator driven by such a pump, the jump phenomenon can be prevented by designing the pump and harmonic generator so that both of their impedances ($Z_1$ and $Z$) satisfy inequality (18). A particular

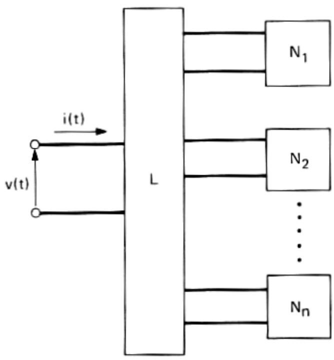


Fig. 8—Network N.

‡ Such an interconnection will therefore satisfy requirement (24).

case discussed in the following section will show that a harmonic generator can actually be designed to satisfy inequality (18) for *all* magnitudes of its input current I.

## III. APPLICATIONS

Two applications are now discussed, but first we summarize some results of a previous study[1], concerning stability and noise in a Schottky barrier down-converter. That study motivated the present theory; conversely the present theory was needed in that study. In Section 3.2 results of a study of the jump phenomenon (following certain experimental work on solid-state power sources[5]) are given; other related effects (e.g. starting problems) are also discussed.

### 3.1 *Schottky Barrier Down-Converter*[1]

Figure 9 shows schematically a network consisting of a Schottky barrier diode connected to two filters $F_1$ and $F_0$, which permit currents to flow only in narrowbands centered about $\omega = \omega_1$ and $\omega = 0$ respectively, and which have zero impedance at those frequencies. The diode is represented (to good approximation) by the equivalent circuit of Fig. 9b, consisting of a small resistance $R_s$ and two nonlinear elements, the barrier capacitance $C_b(v_b)$ and the barrier resistance $R_b(v_b)$. $C_b(v_b)$ and the current $i_R$ through $R_b(v_b)$ are assumed to obey the familiar relationships

$$C_b(v_b) = \frac{C_{\min}\sqrt{\phi - V_B}}{\sqrt{\phi - v_b}} \tag{44}$$

$$i_R = i_s\left[\exp\left(\frac{qv_b}{kT}\right) - 1\right], \tag{45}$$

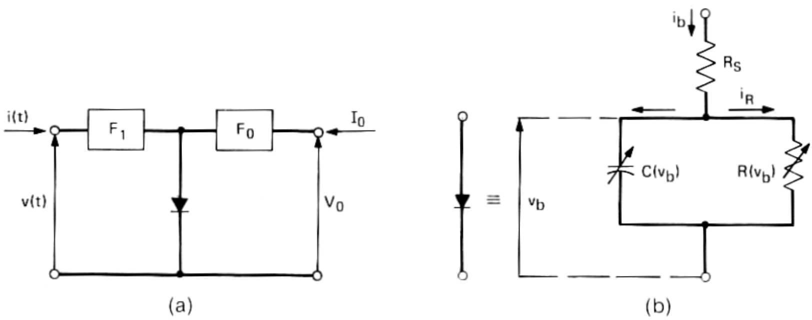


Fig. 9—Down-converter consisting of a Schottky barrier diode and two filters $F_0$ and $F_1$.

where $V_B$ is the breakdown voltage of the diode and $\phi$ the contact potential.

This network is of the same type as the network M considered in the preceding sections. Therefore, its stability can be studied using inequalities (21) and (22) (for this network, inequalities (23) are always fulfilled). Its impedance $Z$ at $\omega_1$ can be written

$$Z = R_s + R_b + jX_b , \tag{46}$$

where $Z_b = R_b + jX_b$ is the impedance presented by the barrier of the diode. Because $F_1$ and $F_o$ allow current to flow only in the vicinity of $\omega = \omega_1$ and $\omega = 0$, the current through the diode cannot have components at the harmonics $2\omega_1$, $3\omega_1$, $4\omega_1$, etc. (and at their side frequencies $2\omega_1 \pm p$, $3\omega_1 \pm p$, $4\omega_1 \pm p$, etc.). This condition is an important requirement for low noise down-conversion. Another important requirement is that the diode should be fully pumped. That is, the current $I$ should have the largest magnitude allowed (for a given $I_o$) by the breakdown voltage $V_B$ of the diode; we assume that this is so. Then, if $V_B$ is sufficiently large, this circuit has the following properties.[1]

First, for unconditional stability, it is sufficient (and of course necessary) that inequality (21) be fulfilled [for this circuit, inequality (22) is always fulfilled if inequality (21) is fulfilled]. Second,

$$| I | \frac{\partial \mathfrak{X}}{\partial | I |} \cong \frac{1}{2\omega_1 C_{\min}} , \tag{47}$$

where, according to eq. (44), $C_{\min}$ is the value of $C_b(v_b)$ for $v_b = V_B$. Third, the power absorbed by the barrier resistance is very small‡, so that inequality (21) is violated provided $R_s$ is sufficiently small. To find out how small $R_s$ should be for the circuit to be potentially unstable, one can neglect $R_b$ with respect to $R_s$ in eq. (46). Then, inequality (21) requires

$$R_s > \frac{1}{2} \left| I \frac{\partial \mathfrak{X}}{\partial | I |} \right| . \tag{48}$$

From this inequality and eq. (47), we find that high gain is possible provided

$$\frac{4\omega_1}{\omega_c} < 1, \tag{49}$$

where $\omega_c$ is the cutoff frequency of the diode, $\omega_c = (R_s C_{\min})^{-1}$. According to this inequality the highest pump frequency for which a given diode

‡ For this to be true, $\omega_1$ must be large, such that the diode behaves essentially like a variable capacitance for $v_b < 0$ (see Ref. 1).

can be made to exhibit arbitrarily high gain is approximately $\omega_c/4$. This simple relation is valid provided $V_B$ is very large. If $V_B$ is finite, high gain can be obtained only if $\eta\omega_1/\omega_c < 1$, where $\eta$ is a parameter which is typically less than 6.25 (but always greater than 4; see Ref. 1).

### 3.2 *Abrupt-Junction Varactor Doubler*[6]

Experimental varactor multipliers[7] exhibit instabilities of the type considered in this paper. In practice it is often found that empirical techniques are necessary to make a varactor multiplier self-starting. Furthermore, the range of frequencies, temperatures and powers over which a varactor multiplier shows stable and efficient operation are often seriously limited by the jump phenomenon. Little is known about the restrictions that should be imposed on multiplier design to prevent such undesirable effects.

In this section, the simplest varactor multiplier, the doubler with abrupt-junction varactor, is considered. Our main result is a stability diagram which gives, for any given varactor characteristics, the input frequencies $\omega_1$ and load resistances $R_L$ for which discontinuous jumps and starting problems may occur. It is shown that these nonlinear effects can always be prevented in an abrupt-junction doubler by properly choosing the output load. In particular, these effects will not occur if the output load is optimized for maximum efficiency. This result, obtained with the help of data from Penfield and Rafuse,[6] shows the importance of optimizing the efficiency of a multiplier in order to reduce starting problems, discontinuous jumps and spurious oscillations.

#### 3.2.1 *Assumptions*[6]

The model of Fig. 9b is assumed for the diode. Since we are interested in converting power from pump frequency $\omega_1$ to its second harmonic $2\omega_1$, the diode is assumed to be terminated by a load impedance $Z_L = R_L + jX_L$ at $2\omega_1$ and to be open-circuited at $3\omega_1$, $4\omega_1$, $5\omega_1$, etc. Furthermore, we assume that it is biased at $\omega = 0$ by a *fixed voltage* $V_o$.

The doubler can then be represented as in Fig. 10. The three networks $F_o$, $F_1$ and $F_2$ are ideal filters. The impedance of $F_r$ ($r = 0, 1, 2$) is assumed to be zero for $\omega \cong r\omega_1$ and infinite for $|\omega - r\omega_1| > \omega_1/2$ ($\omega > 0$). We are interested exclusively in the behavior of this circuit in the particular case $V_o =$ constant.‡

‡ The behavior for the two cases $V_o =$ constant and $I_o =$ constant is discussed in Ref. 1, for the limiting case $R_L = \infty$ (some of the results of Ref. 1 have been pointed out in Section 2.6). It is shown in Ref. 1 that the condition $V_o =$ constant yields greater stability than for $I_o =$ constant.

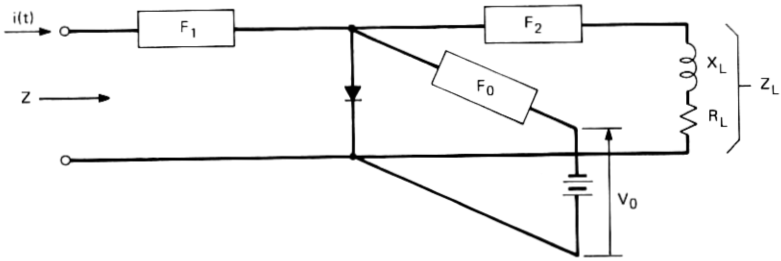


Fig. 10—Doubler.

Let $I$ be the complex amplitude of the diode current at $\omega_1$, and $v_b(t)$ the voltage across the barrier. The doubler will normally be operated about a particular steady state, corresponding to some value $I_c$ of $I$. Let $v_{bc}(t)$ denote the voltage across the barrier for this particular steady-state, and assume that this steady-state is characterized by the condition

$$X_L = \frac{S_M - S_m}{4\omega_1}, \tag{50}$$

where $S_M$ and $S_m$ denote, respectively, the maximum and minimum value of the elastance of the diode for $v_b = v_{bc}(t)$ (we make this assumption because maximum efficiency for a doubler occurs approximately when this condition is fulfilled[6]). Throughout this section we also assume that the operation of the diode is restricted to the range of voltages for which the barrier capacitance is predominant over the barrier resistance. We therefore neglect the barrier resistance (see Fig. 9b) and represent the diode simply by a resistance $R_s$ in series with a variable elastance $S(v_b) = C^{-1}(v_b)$.

According to Section 2.5, the stability of the doubler at frequencies close to $\omega_1$ depends on the sign of the quantity

$$\eta(|I|) = 4R\frac{d(|I|R)}{d|I|} - \left(|I|\frac{dX}{d|I|}\right)^2 \tag{51}$$

where $R$ and $X$ are the real and imaginary part of impedance $Z$ presented by the diode at $\omega_1$. If, for some value of $|I|$, $\eta(|I|) < 0$, then restrictions must be imposed on the diode terminations at frequencies close to $\omega_1$ in order to prevent spurious oscillations at these frequencies, for that value of $|I|$. Furthermore, restrictions must be imposed on the internal impedance $Z_1$ of the pump at $\omega_1$, in order to prevent the jump phenomenon for $0 \leqq |I| \leqq |I_c|$. If, on the other hand

$$\eta(|I|) > 0 \quad \text{for} \quad 0 < |I| < |I_c|, \tag{52}$$

then the jump phenomenon and the above spurious oscillations will not be possible for all values of $|I|$ in the interval $0 \leqq |I| \leqq |I_c|$.

3.2.2 *Results*

The functional relationship between $Z$ and $|I|$ has been obtained in a straightforward manner using the procedure described in Ref. 6 (pp. 299–335). For given diode characteristics, the form of this relationship depends upon the value of $R_L$. The effect of this parameter on the stability of $Z$ for $S_m/S_M = 0$ is shown in Fig. 11. The stable region of this diagram gives the values of $R_s$ and $R_L$ for which condition (52) is fulfilled. The boundaries of this region are characterized by the property that the minimum value of $\eta(|I|)$ over the interval $0 \leqq |I| \leqq |I_c|$ is zero. It is interesting to note that there are values of $R_L$ for which condition (52) is fulfilled even if $R_s = 0$. The dashed curve of Fig. 11 is the curve given by Penfield and Rafuse[6] for the load resistances required for maximum efficiency at $|I| = |I_c|$. Note that this curve is inside the stable region, as pointed out earlier in this section.

The unstable regions consist of the points for which $\eta(|I|) < 0$ for some values of $|I|$, $0 \leqq |I| \leqq |I_c|$. These regions can be divided into subregions having different properties, as indicated in Fig. 12. In subregions ② and ③ $\eta(|I_c|) > 0$. In ① and ④ $\eta(|I|) < 0$ even if

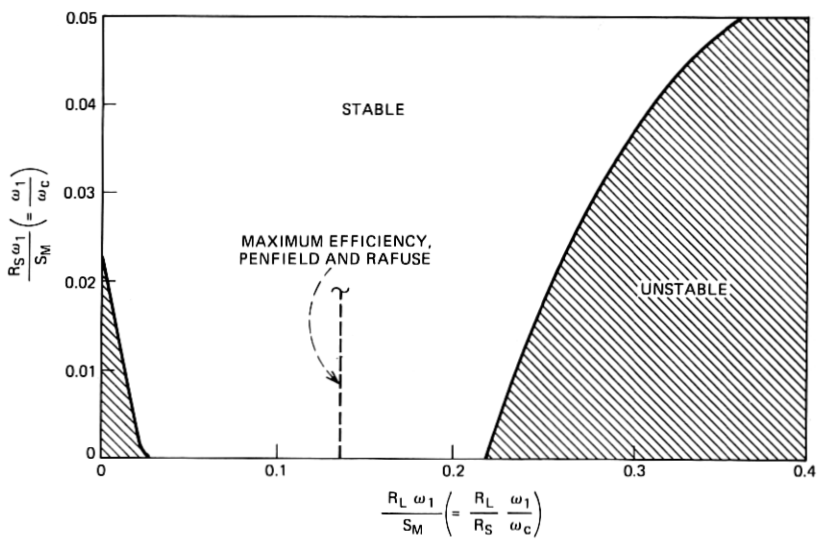


Fig. 11—Stability diagram of the abrupt-junction varactor doubler for $S_m/S_M \cong 0$.

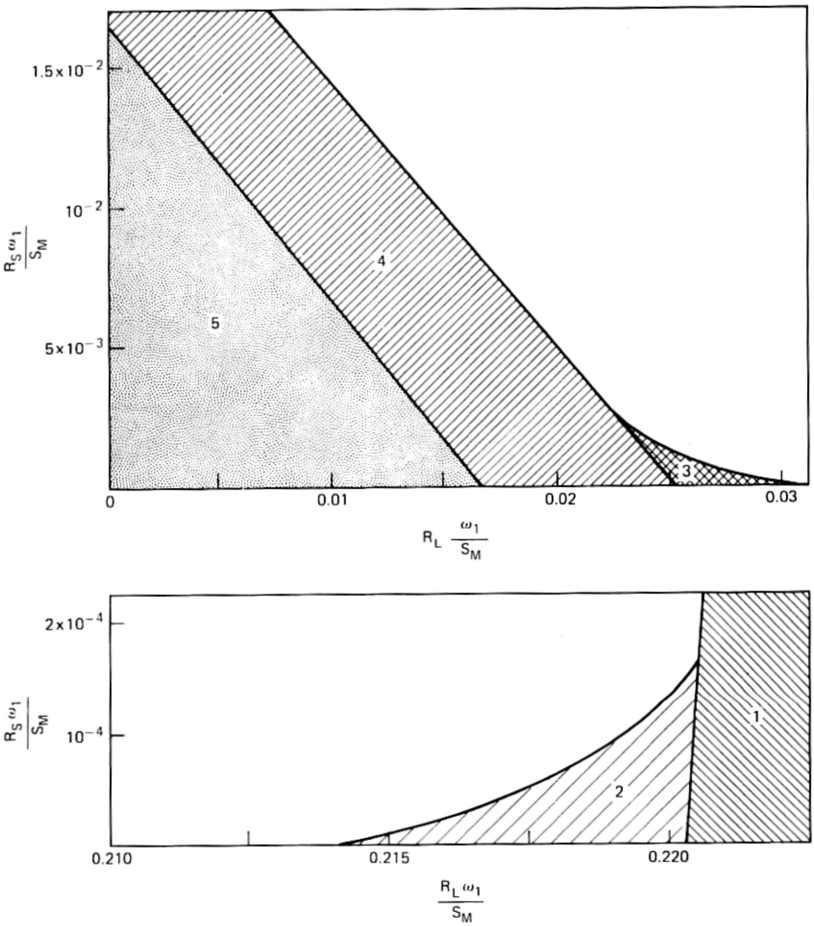


Fig. 12—Details of the diagram of Fig. 11.

$|I| = |I_c|$. For any point in one of these four subregions, the value of $I$ always determines uniquely the voltage $v_b(t)$ across the barrier of the diode. For any point in ⑤, on the other hand, there are values of $|I|$ for which $v_b(t)$ is not uniquely determined by $I$, as illustrated by the example in Fig. 13, where $V_m$ denotes the minimum value of $v_{bc}(t)$. (Thus, $S_M = C^{-1}(V_m)$). To prevent such undesirable behavior it is necessary (and sufficient) that

$$R_L + R_s > \xi \frac{S_M}{\omega_1}, \tag{53}$$

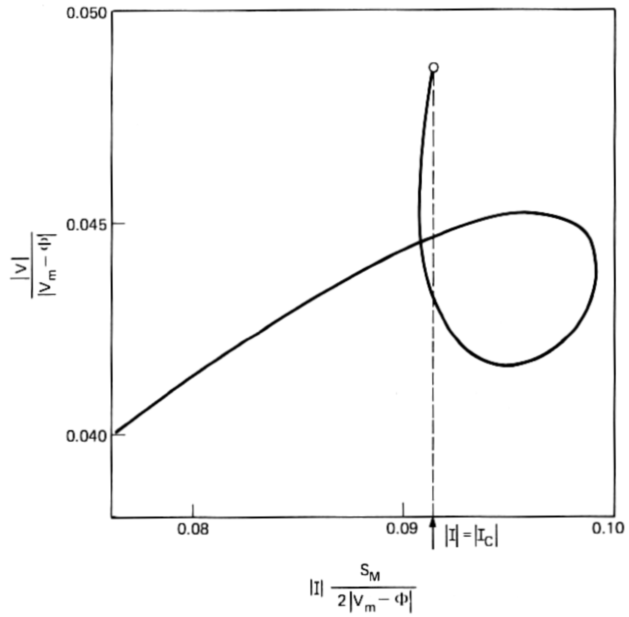


Fig. 13—Example of a $|V| - |I|$ characteristic in subregion ⑤ ($R_L = 1.14225$ $10^{-2}$ $S_M/\omega_1$, $R_s = 0$).

where $\xi \cong 0.0166$. Note that in the example of Fig. 13 the desired condition $v_b(t) = v_{bc}(t)$ cannot be obtained by simply increasing $|I|$ very slowly from zero to $|I_c|$.

For points in regions ①, ②, ③ and ④ in Fig. 12, constraints must be placed upon the pump impedances $Z_1$ in order to avoid discontinuous jumps and starting problems. For instance, consider the case $R_s = 0$ and $R_L = 0.3565\ S_M/\omega_1$. One can see from Figs. 11 and 12 that such a multiplier is potentially unstable, since it corresponds to a point located in subregion ①. The variation of the input voltage with current is shown in Fig. 14. It can be shown that if one connects in series to the input of this multiplier, an inductance (having reactance $jX_1$) chosen to tune $jX$ for $|I| = |I_c|$, the voltage $E$ across $Z + jX_1$ will exhibit the behavior given in Fig. 15 by the curve corresponding to $x_1 = 0.5$. Figure 15 also shows two examples of the behavior arising for $X_1 < S_M/2\omega_1$ (it can be shown that $S_M/2\omega_1$ is the value of $X_1$ needed to tune $X$ for $|I| = |I_c|$). All the characteristics of Fig. 15 exhibit a negative differential resistance over part of the range $0 \leqq |I| \leqq |I_c|$. Furthermore, in each case there is a range of voltages for which more than one value of $|I|$ is possible for a given value of $|E|$. In all cases the range

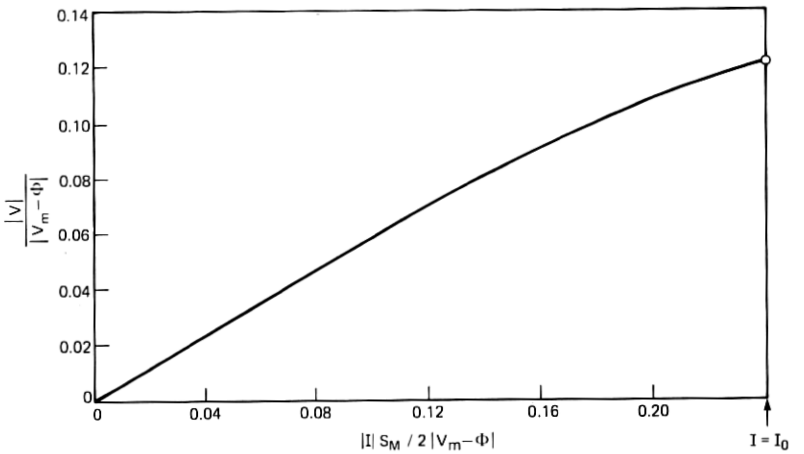


Fig. 14—Example of a $|V| - |I|$ characteristic in subregion ① ($R_L = 0.3565\ S_M/\omega_1$, $R_s = 0$).

in question contains the voltage for which $|I| = |I_c|$. The dotted curves of Fig. 15 show the effect of a small series resistance $R_s$; they have been calculated for $\omega_1/\omega_c = 5\ 10^{-3}$ ($R_s = 5\ 10^{-3}\ S_M/\omega_1$).

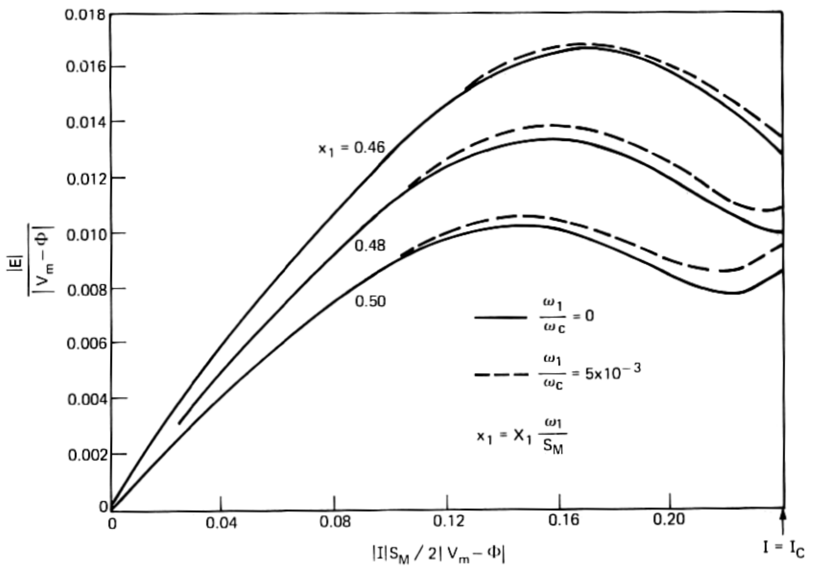


Fig. 15—$|E| - |I|$ characteristic corresponding to the example of Fig. 14 ($R_1 = 0$).

### 3.2.3 *Differences between this Analysis and that of Refs. 8–10*

For any point in the unstable region of Fig. 11 restrictions must be placed on the diode terminations at $\omega_1 \pm p$ (small $p$) in order to prevent the appearance (for some value of $|I|$ in the interval $0 \leqq |I| \leqq |I_c|$) of spurious oscillations at $\omega_1 \pm p$. The mechanism responsible for such spurious oscillations differs from that discussed in Refs. 8–10. The spurious oscillations considered in Refs. 8–10 can, in general, be prevented by imposing suitable restrictions on the diode terminations at $2\omega_1 \pm p$. In particular, they cannot occur if $p$ is low enough so that the terminations at $2\omega_1 \pm p$ are essentially equal to $Z_L$. On the other hand, we have just shown that spurious oscillations are possible even if this condition at $2\omega_1 \pm p$ is fulfilled. This discrepancy between the results of the two analyses arises because the analysis of Refs. 8–10 is not applicable to the circuit of Fig. 10, since in this circuit the diode is short-circuited for $\omega = p$, whereas in Refs. 8–10 the diode was assumed to be open-circuited at $\omega = p$.‡ Furthermore, in Refs. 8–10 the output load was assumed to be tuned, whereas in our analysis consideration has not been restricted to the particular pump level for which this condition is verified.

### 3.3 *Concluding Remarks*

The jump phenomenon is a form of instability. Thus it should not be surprising that a mixer capable of producing this nonlinear effect is potentially unstable, and vice versa. The derivation of inequalities (21) through (23), which are the stability conditions necessary and sufficient for the stability of a mixer, has been organized to demonstrate the important relationship between the jump phenomenon and mixer stability. A knowledge of this relation is requisite for an understanding of the mechanism of instability in a mixer; it is useful in experiments whenever one wants to determine whether or not a given mixer is potentially unstable. For that purpose the simplest procedure is to connect the mixer to a pump and a dc bias supply (as shown in Fig. 3) and then determine (in the two cases $R_o = 0$ and $R_o = \infty$) whether, by varying $|E|$ and $Z_1$, the circuit can be made to exhibit the jump phenomenon. This procedure is straightforward and has been used extensively in experimental work on down-converters.[1]

‡ More precisely, in Refs. 8–10 the impedance $Z$ terminating $C(v_b)$ at $\omega = p$ is assumed to be sufficiently large so as to insure negligible charge fluctuations at $\omega = p$ in $C(v_b)$. However, this condition cannot by realized in the limiting case $p \rightarrow 0$ because it can be shown that for $p \rightarrow 0$ this condition requires that

$$\lim_{p\rightarrow 0} pZ_\beta = \infty$$

This requirement is unrealizable.

In Sections 2.6 and 2.7, two applications of practical interest (down-converter and doubler) have been considered; they are quite different in many respects. A fully-pumped down-converter is, in general, potentially unstable if $R_s$ is sufficiently small, whereas a doubler can be unconditionally stable (even when $R_s = 0$), if it is properly designed. Furthermore, in the case of a down-converter, potential instability may be a desirable feature whereas it is highly undesirable in the case of a doubler. Another difference between the two cases is that in the down-converter of Fig. 9, the jump phenomenon always appears to be prevented by proper choice of $R_o$ and $Z_1$, while in the case of a doubler, the behavior of Fig. 13 may arise when the doubler is improperly designed, in which case the doubler is unusable for all practical purposes. However, in spite of these differences (which arise in part because the two circuits of Figs. 9 and 10 are intended for different purposes), the two cases are related, for in the limit $R_L \to \infty$ the circuit of Fig. 10 becomes that of Fig. 9.

We conclude by summarizing the derivation of inequalities (21) and (22). A nonlinear impedance $Z$ (with $R > 0$) obeying inequality (18) has the following property: if an arbitrary passive impedance $Z_1$ is connected in series with $Z$ (Fig. 3), and $E$ denotes the voltage $I(Z + Z_1)$ across $Z + Z_1$, then necessarily $d \mid E \mid / d \mid I \mid > 0$. In Section 2.5 such an impedance has been termed stable.

We derived inequalities (21) through (23) by connecting the network M to a dc voltage supply, and by requiring that the resulting impedance $Z$ be stable for all nonnegative internal resistances $R_o$ of the dc voltage supply. This procedure is analogous to that used in ordinary linear time-invariant network theory for deriving the stability conditions of a two-terminal-pair network. In fact, the stability conditions of such networks are usually derived by connecting one of its two terminal pairs to an arbitrary passive impedance, and then requiring that the resulting impedance at the other terminal pair be stable (i.e., that its real part be positive).

In Section 2.3 it was shown that if $\partial \mathcal{V}_o / \partial I_o > 0$, then $Z$ has the following property: if inequality (18) is fulfilled in the two particular cases, $R_o = 0$ and $R_o = \infty$, then it is also fulfilled for all positive $R_o$. Thus it was concluded that if $Z$ is to be stable for all nonnegative $R_o$, it is necessary and sufficient that $\partial \mathcal{V}_o / \partial I_o > 0$, $R > 0$, and that the two inequalities (21) and (22) [which are inequality (18) for $R_o = \infty$ and $R_o = 0$, respectively] be fulfilled. Then, in Section 2.3, we have proven a theorem showing (as a corollary) that these two inequalities, and the inequalities $\partial \mathcal{V}_o / \partial I_o > 0$ and $R > 0$, are necessary and sufficient conditions for the stability of M.

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## APPENDIX

Let us superimpose on $E_o$ and $|E|$ in Fig. 2 small perturbations $\delta E_o$ and $\delta|E|$, and let $\delta|I|$ and $\delta I_o$ denote the resulting perturbations of $|I|$ and $I_o$. We can write

$$\begin{bmatrix} \delta|E| \\ \\ \delta E_o \end{bmatrix} = \begin{bmatrix} \dfrac{\partial|E|}{\partial|I|} & \dfrac{\partial|E|}{\partial I_o} \\ \\ \dfrac{\partial E_o}{\partial|I|} & \dfrac{\partial E_o}{\partial I_o} \end{bmatrix} \begin{bmatrix} \delta|I| \\ \\ \delta I_o \end{bmatrix} \tag{54}$$

where

$$\begin{aligned} E &= I[Z_1 + \mathfrak{z}(I_o, |I|)] \\ E_o &= \mathcal{V}_o(I_o, |I|) + R_o I_o . \end{aligned} \tag{55}$$

Two particular cases $\delta E_o = 0$ and $\delta|E| = 0$ are of interest. In the former case, from eq. (54),

$$\left(\frac{\delta|E|}{\delta|I|}\right)_{\delta E_o = 0} = \frac{J}{\dfrac{\partial E_o}{\partial I_o}}, \tag{56}$$

where $J$ is the determinant (Jacobian) of eqs. (54),

$$J = \frac{\partial|E|}{\partial|I|}\frac{\partial E_o}{\partial I_o} - \frac{\partial|E|}{\partial I_o}\frac{\partial E_o}{\partial|I|}. \tag{57}$$

In the latter case, from eq. (54),

$$\left(\frac{\delta E_o}{\delta I_o}\right)_{\delta|E| = 0} \quad \frac{J}{\dfrac{\partial|E|}{\partial|I|}}. \tag{58}$$

Equation (56) gives the derivative of $|E|$ with respect to $|I|$ when $E_o$ is held constant. We have already considered this derivative in Section 2.3, where it was shown that inequalities (21) through (23) are necessary and sufficient for this derivative to be positive for all $R_o \geqq 0$, $R_1 \geqq 0$ and $X_1$. We now show that inequalities (21) through (23) can also be interpreted as the necessary and sufficient conditions for

$$\left(\frac{\delta E_o}{\delta I_o}\right)_{\delta|E| = 0} > 0. \tag{59}$$

We note from eqs. (56) and (58) that

$$\left(\frac{\delta E_o}{\delta I_o}\right)_{\delta|E|=0} = \frac{\left(\frac{\delta|E|}{\delta|I|}\right)_{\delta E_o=0} \frac{\partial|E|}{\partial|I|}}{\frac{\partial E_o}{\partial I_o}} = \frac{\frac{d|E|}{d|I|}\frac{\partial|E|}{\partial|I|}}{\frac{\partial E_o}{\partial I_o}}, \tag{60}$$

where $d|E|/d|I|$ is the derivative discussed in Sections 2.2 and 2.3. If inequalities (21) through (23) are fulfilled, then certainly

$$\frac{d|E|}{d|I|} > 0, \qquad \frac{\partial|E|}{\partial|I|} > 0, \qquad \frac{\partial E_o}{\partial I_o} > 0 \tag{61}$$

(note that requirement (17) implicitly demands $\partial|E|/\partial|I| > 0$, because for $R_o = \infty$, $d|E|/d|I|$ reduces to $\partial|E|/\partial|I|$). Thus, inequalities (21) through (23) are certainly sufficient conditions for requirement (59) to be fulfilled; they are also necessary because if requirement (59) is violated, then, according to eq. (59), at least one of inequalities (61) is violated for some $R_o \geqq 0$, $R_1 \geqq 0$ and $X_1$, and we already know from Section 2.3 that in such case, inequalities (21) through (23) are violated.

We have just shown that requirements (17) and (59) are equivalent. It can be shown, in an analogous way, that an equivalent requirement is

$$J > 0 \quad \text{for all} \quad R_o \geqq 0, \quad R_1 \geqq 0, \quad X_1 \tag{62}$$

[other equivalent requirements may be obtained by replacing > with ≠ in (17), (59) and (61)].


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