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The General Second-Order Twin-T and Its Application to Frequency-Emphasizing Networks

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The general conditions for reducing the third-order transfer function of a twin-T by one are derived using Euclid's algorithm. The conditions presently used impose narrower constraints than necessary on the twin-T, thus leaving fewer free parameters to optimize the circuit. With the new method the zeros of the twin-T transfer function can be placed in both the left- and the right-half s-plane.

The advantages of the twin-T with additional free parameters in second-order RC-active filters are appreciable. For example, in the medium-selectivity frequency-emphasizing network (MSFEN), the gain needed to realize a given pole Q may be up to 70 times smaller than that required with previous methods, while the stability of the pole is improved typically by a factor of 2. Thus, an MSFEN with the general second-order twin-T is capable of realizing a wider range of pole Q's than was possible previously, while the sensitivity of the pole Q is reduced.

I. INTRODUCTION

The twin-T as represented in Fig. 1 consists of the three resistors R_1, R_2, R_3 and the three capacitors C_1, C_2, C_3. A straightforward analysis provides its Y-matrix as

$$Y = \begin{bmatrix} \dfrac{as^3 + s^2(b+f) + s(c+g) + 1}{R_s(R_pC_3s+1)(R_3C_ss+1)} & -\dfrac{as^3 + bs^2 + cs + 1}{R_s(R_pC_3s+1)(R_3C_ss+1)} \\ -\dfrac{as^3 + bs^2 + cs + 1}{R_s(R_pC_3s+1)(R_3C_ss+1)} & \dfrac{as^3 + s^2(b+d) + s(c+e) + 1}{R_s(R_pC_3s+1)(R_3C_ss+1)} \end{bmatrix}, \tag{1a}$$

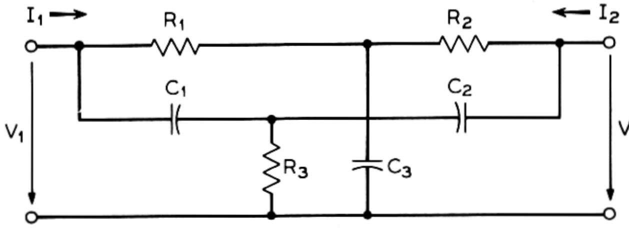

Fig. 1—The twin-T.

where

$$\left.\begin{aligned} a &= R_1R_2R_3C_1C_2C_3 & e &= R_1C_3 + R_sC_2 \\ b &= R_sR_3C_1C_2 & f &= R_2C_3(R_1C_1 + R_3C_s) \\ c &= R_3C_s & g &= R_sC_1 + R_2C_3 \\ d &= R_1C_3(R_2C_2 + R_3C_s) & R_s &= R_1 + R_2\,; \quad C_s = C_1 + C_2 \\ & & R_p &= \frac{R_1R_2}{R_s} \end{aligned}\right\}. \tag{1b}$$

The transfer function of the unloaded twin-T is

$$T(s) = \frac{V_2}{V_1}\bigg|_{I_2=0} = -\frac{Y_{21}}{Y_{22}} = \frac{as^3 + bs^2 + cs + 1}{as^3 + (b+d)s^2 + (c+e)s + 1}. \tag{2}$$

The elements of Y and the transfer function $T(s)$ are of the third degree in s. The properties of the twin-T are very useful in second-order RC-active filter sections.[1,2] One of these properties is to allow right-half-plane zeros of $T(s)$ in the unshaded region of Fig. 2 which is bounded by the line with an angle of 60 degrees.[3] An important application in RC-active filters is based on the fact that one needs less gain of the amplifier to realize a high-pole Q if $T(s)$ has right-half-plane zeros.[4] However, as is well known, for the RC-active filter applications, the degree in s of either $T(s)$ or of the elements of Y has to be reduced to second order by creating a common divider in the numerator and the denominator of $T(s)$ or the elements of Y. To achieve this several special solutions are known,[5] some of which will be listed later. All of them, however, either impose more constraints than necessary on the values of the components of the twin-T, or they destroy the possibility of right-half-plane zeros. Some solutions are approximations which hold only in the neighborhood of the imaginary axis. This paper will derive the general condition for the reduction by one of the degree in s. This will lead to only one constraint, leaving additional free components of the twin-T. This fact has a variety of applications in network theory. As an example it will be used in Section V to optimize and to extend the capabilities of the second-order FEN. Another application is the precision tuning of second-order RC-active filter sections.[6] The reduction of the degree of $T(s)$ by one will be dealt with in the following section.

II. THE TRANSFER FUNCTION T(s) OF SECOND DEGREE

The degree of $T(s)$ in equation (2) is reduced by one by creating a common divider in the numerator and the denominator of $T(s)$ or, in other words, by creating a coinciding zero and pole of $T(s)$ which can be cancelled. The condition under which a pole and a zero coincide may be found by using Euclid's algorithm.[7] This algorithm and its application to $T(s)$ are presented in the Appendix. The result is the following: $T(s)$ is of second degree in s if

$$d(d^2 + e^2 b) - e(e^2 a + cd^2) = 0. \tag{3}$$

The common divider of $T(s)$ is

$$D(s) = s + \frac{e}{d}. \tag{4}$$

Dividing the numerator and the denominator of $T(s)$ by $D(s)$ and considering equation (3) provides the transfer function of second degree in s

$$T(s) = \frac{s^2 + \left(\frac{b}{a} - \frac{e}{d}\right)s + \frac{d}{ea}}{s^2 + \left(\frac{b}{a} - \frac{e}{d} + \frac{d}{a}\right)s + \frac{d}{ea}}. \tag{5}$$

Now we have to check the condition (3) in further detail. Inserting

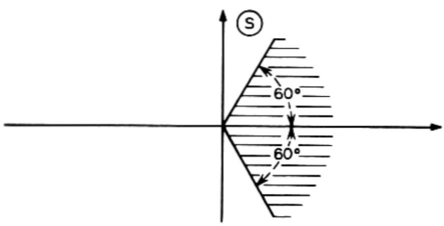

Fig. 2—The possible location of zeros of $T(s)$.

a, b, c, d, and e from (1b) in (3) and then rearranging equation (3) yields

$$(R_1R_2C_3 - R_3R_sC_s)\left(\left(\frac{d}{e}\right)^2 - R_1R_3C_1C_3\right) = 0. \tag{6}$$

Obviously equation (6) provides two separate conditions for the pole-zero cancellation; namely,

$$\left(\frac{d}{e}\right)^2 = R_1R_3C_1C_3 \tag{7}$$

or

$$R_1R_2C_3 = R_3R_sC_s\,. \tag{8}$$

The coefficients of $T(s)$ in equation (5) depend upon the choice of condition (7) or (8). The two solutions are:

Case 1. $T(s)$ as in equation (5), with $\left(\frac{d}{e}\right)^2 = R_1R_3C_1C_3$. (9a)

Case 2. $T(s)$ as in equation (5), with

$$R_1R_2C_3 = R_3C_sR_s\,. \tag{9b}$$

Case 2 should be worked out in further detail. Inserting (8) in (5) yields the result

$$T(s) = \frac{s^2 + \dfrac{d}{ae}}{s^2 + \dfrac{d}{a}s + \dfrac{d}{ae}} \tag{9c}$$

[where again constraint (8) holds].

In Case 1 the zeros of $T(s)$ may be in the left- or in the right-half plane since in the numerator the coefficient of s can be positive or negative. The zeros of $T(s)$ in Case 2 however always lie on the imaginary axis.

As can be seen, the cancellation of poles and zeros is guaranteed in all cases by only one condition; namely, either equation (7) or equation (8). Thus from the six parameters of the twin-T, five are left at our disposal.

III. THE Y-MATRIX OF SECOND DEGREE

The condition for the reduction of the degree in all four elements of Y in equation (1a) is obvious. One of the two first-order factors of the denominator has to be contained in all numerators. We divide the numerators by one of those first-order factors and set the rest of the division equal to zero. This provides the one condition

$$R_pC_3 = R_3C_s\ ; \tag{10a}$$

that is, the two first-order factors of the denominator are equal. Inserting R_p from equation (1b) in (10a) yields

$$R_1R_2C_3 = R_3R_sC_s\ , \tag{10b}$$

which is the same condition as equation (8). Dividing all numerators in Y by $(sR_pC_3 + 1)$ and inserting $R_3 = (R_1R_2C_3/R_sC_s)$ from (10b) yields the second-order Y-matrix

$$Y = \begin{bmatrix} \dfrac{s^2R_1R_2C_1C_2\dfrac{C_3}{C_s} + s(R_sC_1 + R_2C_3) + 1}{R_s(sR_pC_3 + 1)} & -\dfrac{s^2R_1R_2C_1C_2\dfrac{C_3}{C_s} + 1}{R_s(sR_pC_3 + 1)} \\ -\dfrac{s^2R_1R_2C_1C_2\dfrac{C_3}{C_s} + 1}{R_s(sR_pC_3 + 1)} & \dfrac{s^2R_1R_2C_1C_2\dfrac{C_3}{C_s} + s(R_sC_2 + R_1C_3) + 1}{R_s(sR_pC_3 + 1)} \end{bmatrix}. \tag{11}$$

Again the pole-zero cancellation is guaranteed by only one constraint leaving five parameters of the twin-T at our disposal. Before we consider an application, let us look at some special cases for the constraints (7) or (10b).

IV. SPECIAL CASES

We fulfill the condition (7) or (10b) by selecting one or more of the remaining five free parameters of the twin-T in a special way. This will simplify sometimes the equations for $T(s)$ and Y. We shall list the following six special cases.

(*i*) $$R_1 = R_2 = 2R_3 = R; \qquad C_1 = C_2 = \frac{C_3}{2} = C.$$

With this choice of values the conditions (7) and (10b) are satisfied simultaneously. So we shall obtain a second order $T(s)$ with zeros on the imaginary axis and a Y-matrix of second order.

From (9c) we get

$$T(s) = \frac{s^2 + \dfrac{1}{R^2C^2}}{s^2 + \dfrac{4}{RC}s + \dfrac{1}{R^2C^2}},$$

and from (11) we get

$$Y = \begin{bmatrix} \dfrac{s^2R^2C^2 + 4RCs + 1}{2R(sRC + 1)} & -\dfrac{s^2R^2C^2 + 1}{2R(sRC + 1)} \\ -\dfrac{s^2R^2C^2 + 1}{2R(sRC + 1)} & \dfrac{s^2R^2C^2 + 4RCs + 1}{2R(sRC + 1)} \end{bmatrix}.$$

This symmetrical twin-T with the two free parameters R and C has been used in the filters described in Refs. 1 and 2.

(*ii*) $R_2 = \rho R_1$; $R_3 = \dfrac{\rho}{1 + \rho} R_1$; $C_2 = \dfrac{C}{\rho}$; $C_3 = C_1 \dfrac{\rho + 1}{\rho}$; $\rho \geqq 0.$

The conditions (7) and (10b) are again satisfied simultaneously. Thus we get a $T(s)$ of second degree with zeros on the imaginary axis. From (9c) we obtain

$$T(s) = \frac{s^2 + \left(\dfrac{1}{R_1C_1}\right)}{s^2 + 2\dfrac{\rho + 1}{\rho}\dfrac{1}{R_1C_1}s + \left(\dfrac{1}{R_1C_1}\right)^2}.$$

This so-called potentially symmetrical twin-T has been used in Ref. 5.

(*iii*) $R_1 = R_3 = R; \quad C_1 = C_3 = C.$

With this choice of values only condition (7) is satisfied. We, therefore, expect a $T(s)$ of second degree. Since condition (10a) is not fulfilled, the Y-matrix will remain of third-degree. In this case we get from (5) with (7)

$$T(s) = \frac{s^2 + \dfrac{1}{RC}\dfrac{R}{R_2}s + \dfrac{1}{RCR_2C_2}}{s^2 + \dfrac{1}{RC}\left(2\dfrac{R}{R_2} + 1 + \dfrac{RC}{R_2C_2}\right)s + \dfrac{1}{RCR_2C_2}}.$$

Obviously, only zeros in the open left-half plane are possible.

(*iv*) $R_2C_2 = R_3(C_1 + C_2) = R_1(C_2 + C_3).$

This choice satisfies only (7) and yields the transfer function

$$T(s) = \frac{s^2 + \dfrac{1}{R_2C_3}s + \dfrac{1}{R_2^2C_2^2}\left(1 + \dfrac{C_2}{C_1}\right)}{s^2 + \left[\dfrac{1}{R_2C_3} + \dfrac{2}{R_2C_2}\left(1 + \dfrac{C_2}{C_1}\right)\right]s + \dfrac{1}{R_2^2C_2^2}\left(1 + \dfrac{C_2}{C_1}\right)},$$

where again no right-half-plane zeros are possible.

(*v*) $R_1C_1 = R_2C_2 = R_3C_3$.

Again this special choice satisfies only constraint (7). Equations (5) and (7) yield

$$T(s) = \frac{s^2 + \dfrac{1}{R_1C_1}\left(\dfrac{R_3}{R_1} + \dfrac{R_3}{R_2} - 1\right)s + \dfrac{1}{R_1^2C_1^2}}{s^2 + \dfrac{1}{R_1C_1}\left(\dfrac{R_3}{R_1} + \dfrac{R_3}{R_2} + \dfrac{R_1}{R_3} + \dfrac{R_1}{R_2}\right)s + \dfrac{1}{R_1^2C_1^2}}.$$

Obviously this transfer function is capable of realizing right-half-plane zeros.

(*vi*) $$R_1 = R_3 = R; \qquad C_2 = \frac{R}{R + R_2}\sqrt{C_1C_3}.$$

This choice satisfies (7) and yields the transfer function

$$T(s) = \frac{s^2 + \dfrac{R(C_1 + \sqrt{C_1C_3}) + R_2(C_1 - C_3)}{RR_2C_3(C_1 + \sqrt{C_1C_3})}s + \dfrac{(R + R_2)(C_1 + \sqrt{C_1C_3})}{R^2R_2C_1(C_3 + \sqrt{C_1C_3})\sqrt{C_1C_3}}}{s^2 + \left[\dfrac{R(C_1 + \sqrt{C_1C_3}) + R_2(C_1 - C_3)}{RR_2C_3(C_1 + \sqrt{C_1C_3})} + \dfrac{(R + R_2)(C_1 + \sqrt{C_1C_3})}{RR_2C_1\sqrt{C_1C_3}}\right]s + \dfrac{(R + R_2)(C_1 + \sqrt{C_1C_3})}{R^2R_2C_1(C_3 + \sqrt{C_1C_3})\sqrt{C_1C_3}}},$$

which is capable of realizing right-half-plane zeros. An application for the general second-order twin-T will be demonstrated in the following section.

V. FREQUENCY-EMPHASIZING NETWORKS (FENS) WITH GENERAL TWIN-TS

The so-called medium-selectivity FEN (MSFEN)[2] is shown in Fig. 3. Its transfer function is

$$\frac{V_{\text{out}}}{V_{\text{in}}} = T_0(s) = -\frac{R_F}{R_G}\frac{1}{1 + \mu\beta T(s)}, \tag{12}$$

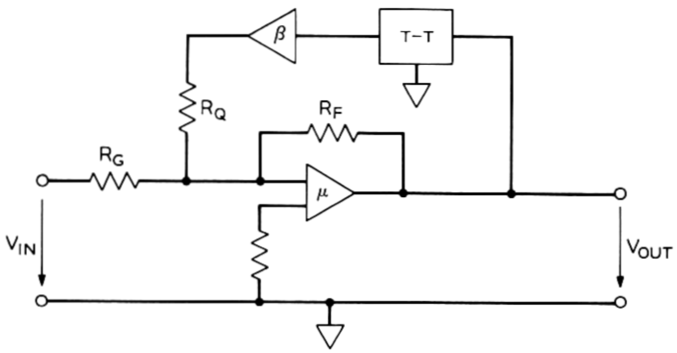

Fig. 3—A frequency-emphasizing network.

where $\beta \geqq 1$ is the gain of the noninverting amplifier in the feedback loop,

$$\mu = \frac{R_F}{R_G} \tag{13}$$

is the gain of the inverting amplifier in the forward path, and $T(s)$ designates the transfer function of the unloaded twin-T. Inserting $T(s)$ from (9a), which enables realization of right-half-plane zeros, we get from (12)

$$T_0(s) = -\frac{R_F}{R_G} \frac{s^2 + \left(\frac{b}{a} - \frac{e}{d} + \frac{d}{a}\right)s + \frac{d}{ae}}{(1 + \mu\beta)\left[s^2 + \left(\frac{b}{a} - \frac{e}{d} + \frac{d}{a}\frac{1}{1 + \mu\beta}\right)s + \frac{d}{ae}\right]} \tag{14}$$

with constraint equation (7). We want to realize the following transfer function of a FEN

$$T_{FEN}(s) = -K \frac{s^2 + \frac{\omega_0}{q_z} s + \omega_0^2}{s^2 + \frac{\omega_0}{q_p} s + \omega_0^2}. \tag{15a}$$

$| T_{FEN} (j\omega) |$ has a maximum at $\omega = \omega_0$ where $| T_{FEN} (j\omega_0) | = | q_p/q_z |$. In order to get a frequency-emphasizing network

$$q_p > | q_z |. \tag{15b}$$

We immediately get from equation (15a) the following requirements for (14a):

$$\omega_0^2 = \frac{d}{ae}, \tag{16a}$$

$$\frac{\omega_0}{q_z} = \frac{b}{a} - \frac{e}{d} + \frac{d}{a}, \tag{16b}$$

$$\frac{\omega_0}{q_p} = \frac{b}{a} - \frac{e}{d} + \frac{d}{a}\frac{1}{1 + \mu\beta}, \tag{16c}$$

and from (14b):

$$\left(\frac{d}{e}\right)^2 = R_1R_3C_1C_3 . \tag{16d}$$

The last requirement $K = R_F/R_G[1/(1 + \mu\beta)]$ can always be satisfied by R_G which does not occur in (16a) through (16d).

We simplify the equations (16a) through (16d). Inserting (16b) into (16c) we get

$$\frac{d}{a} = \left(1 + \frac{1}{\mu\beta}\right)\omega_0\left(\frac{1}{q_z} - \frac{1}{q_p}\right) \triangleq x. \tag{17a}$$

Inserting (17a) in (16a), (16a) in (16d), and (17a) and (16a) in (16b) we get

$$e\omega_0^2 = x, \tag{17b}$$

$$R_1R_3C_1C_3 = a^2\omega_0^4, \tag{17c}$$

$$-\frac{b}{a} + \frac{1}{a\omega_0^2} + \frac{\omega_0}{q_z} = x. \tag{17d}$$

With a, b, d, e from (16) we obtain, finally,

$$\frac{(R_2C_2 + R_3C_1 + R_3C_2)}{R_2R_3C_1C_2} = x, \tag{18a}$$

$$(R_1C_3 + R_1C_2 + R_2C_2)\omega_0^2 = x, \tag{18b}$$

$$-\frac{(R_1 + R_2)}{R_1R_2C_3} + R_2C_2\omega_0^2 + \frac{\omega_0}{q_z} = x, \tag{18c}$$

$$1 = R_1R_2^2R_3C_1C_2^2C_3\omega_0^4, \tag{18d}$$

where (18d) has been used to simplify (18c). This nonlinear system of four equations with the seven unknowns: R_i, $i = 1, 2, 3$, C_i, $i = 1, 2, 3$, and $\mu\beta$ contained in x may be solved by choosing three unknowns in such a way that all solutions are realizable, that is they are positive real numbers. Picking R_1, R_2, C_3 and solving for R_3, C_1, C_2, and x does not give a feasible solution. We pick now R_2, C_2 and x and solve for R_1, R_3, C_1 and C_3. From (18b, c) we get

$$R_1C_3 = \frac{x}{\omega_0^2} - R_2C_2 - R_1C_2 = \frac{R_1 + R_2}{R_2}\left[\frac{1}{\frac{\omega_0}{q_z} + R_2C_2\omega_0^2 - x}\right]. \tag{19}$$

The equation on the right side of (19) yields

$$R_1 = \frac{\left(\frac{x}{\omega_0^2} - R_2C_2\right)\left(\frac{\omega_0}{q_z} + R_2C_2\omega_0^2 - x\right) - 1}{\frac{1}{R_2} + C_2\left(\frac{\omega_0}{q_z} + R_2C_2\omega_0^2 - x\right)}. \tag{20a}$$

From (19) we obtain, further,

$$C_3 = \frac{R_1 + R_2}{R_1R_2} \frac{1}{\frac{\omega_0}{q_z} + R_2C_2\omega_0^2 - x}, \tag{20b}$$

with R_1 in equation (20a).

Eliminating C_1 in (18a) and (18d) and solving for R_3 yields

$$R_3 = \frac{x - \frac{1}{R_2C_2} - (R_2C_2)^2\omega_0^4R_1C_3}{R_1R_2C_2^2C_3\omega_0^4}, \tag{20c}$$

while (18d) provides

$$C_1 = \frac{1}{R_1R_2^2C_2^2R_3C_3\omega_0^4}. \tag{20d}$$

From (17a) we obtain

$$\mu\beta = \frac{1}{\frac{x}{\omega_0\left(\frac{1}{q_z} - \frac{1}{q_p}\right)} - 1}. \tag{20e}$$

The solution for R_1, R_3, C_1, C_3 and $\mu\beta$ has to be positive. C_3 is positive if

$$\frac{\omega_0}{q_z} + R_2C_2\omega_0^2 - x > 0. \tag{21a}$$

With equation (21a) the denominator of (20a) is positive; thus the numerator also has to be positive, which yields

$$x^2 - x\left(\frac{\omega_0}{q_z} + 2\omega_0^2R_2C_2\right) + \omega_0^2 + \frac{\omega_0^3R_2C_2}{q_z} + \omega_0^4R_2^2C_2^2 < 0.$$

This is satisfied for

$$x \,\varepsilon \left[\frac{\omega_0}{2q_z} + \omega_0^2R_2C_2 - \omega_0\sqrt{\frac{1}{4q_z^2} - 1}, \frac{\omega_0}{2q_z} + \omega_0^2R_2C_2 + \omega_0\sqrt{\frac{1}{4q_z^2} - 1}\right]. \tag{21b}$$

The square root in (21b) is always real since the numerator of equation (14a) or (15a) belongs to a passive RC-two port where $q_z < \frac{1}{2}$ always holds. For the requirement $\mu\beta \geqq 0$ we get from (20e)

$$x > \omega_0\left(\frac{1}{q_z} - \frac{1}{q_p}\right) \tag{21c}$$

which, with (15b), is always satisfied as long as $|q_z| < q_p$. Finally $R_3 > 0$ yields

$$x > \frac{1}{R_2C_2} + (R_2C_2)^2R_1C_3\omega_0^4 .$$

With R_1 in equation (20a) and C_3 in equation (20b) we then obtain

$$x^2 - x\left(\frac{2}{R_2C_2} + \frac{\omega_0}{q_z}\right) + \frac{1}{R_2^2C_2^2} + \omega_0^2 + \frac{\omega_0}{q_zR_2C_2} < 0.$$

This is fulfilled for

$$x \in \left(\frac{\omega_0}{2q_z} + \frac{1}{R_2C_2} - \omega_0\sqrt{\frac{1}{4q_z^2} - 1}, \frac{\omega_0}{2q_z} + \frac{1}{R_2C_2} + \omega_0\sqrt{\frac{1}{4q_z^2} - 1}\right). \tag{21d}$$

We now have to check the compatibility of the inequalities (21a) through (21d). It can be shown easily that the upper bound in (21b) guarantees (21a), which must no longer be considered. The inequalities (21b) and (21d) have a range in common if the upper bound of (21b) is higher than the lower bound of (21d) and if the lower bound of (21b) is lower than the upper bound of (21d). This is satisfied if

$$R_2C_2 \in \left[\frac{1}{\omega_0}\left(\frac{1}{2q_z} - \sqrt{\frac{1}{4q_z^2} - 1}\right), \frac{1}{\omega_0}\left(\frac{1}{2q_z} + \sqrt{\frac{1}{4q_z^2} - 1}\right)\right]. \tag{22a}$$

Then the common range of (21b) and (21d) is $x \in \{$max [lower bounds of (21b) and (21c)], min [upper bounds of (21b) and (21d)]$\}$. This yields

$$x \in \left[\frac{\omega_0}{2q_z} + \omega_0^2R_2C_2 - \omega_0\sqrt{\frac{1}{4q_z^2} - 1}, \frac{\omega_0}{2q_z} + \frac{1}{R_2C_2} + \omega_0\sqrt{\frac{1}{4q_z^2} - 1}\right] \quad \text{if} \quad R_2C_2 > \frac{1}{\omega_0}, \tag{22b}$$

and

$$x \in \left[\frac{\omega_0}{2q_z} + \frac{1}{R_2C_2} - \omega_0\sqrt{\frac{1}{4q_z^2} - 1}, \frac{\omega_0}{2q_z} + \omega_0^2R_2C_2 + \omega_0\sqrt{\frac{1}{4q_z^2} - 1}\right] \quad \text{if} \quad R_2C_2 < \frac{1}{\omega_0}. \tag{22c}$$

The inequality (21c) may be satisfied if the upper limit of (22b) and (22c) respectively is larger than the limit in (21c). This yields for (22b)

$$\frac{1}{R_2C_2} > \frac{\omega_0}{2q_z} - \frac{\omega_0}{q_p} - \omega_0\sqrt{\frac{1}{4q_z^2} - 1}, \tag{22d}$$

and for (22c)

$$R_2C_2 > \frac{1}{\omega_0}\left(\frac{1}{2q_z} - \sqrt{\frac{1}{4q_z^2} - 1} - \frac{1}{q_p}\right). \tag{22e}$$

Obviously (22a) also guarantees (22e); (22a) and (22d) are compatible if the lower bound of (22a) is lower than the bound for R_2C_2 in (22d). Again, this is always satisfied as can be easily shown. Thus we get the two following sets of constraints:

Case 1.

$$R_2C_2 > \frac{1}{\omega_0}. \tag{23a}$$

$$R_2C_2 \,\varepsilon \left[\frac{1}{\omega_0}\left(\frac{1}{2q_z} - \sqrt{\frac{1}{4q_z^2} - 1}\right), \frac{1}{\omega_0}\left(\frac{1}{2q_z} + \sqrt{\frac{1}{4q_z^2} - 1}\right)\right]. \tag{23b}$$

$$x \,\varepsilon \left[\frac{\omega_0}{2q_z} + \omega_0^2 R_2C_2 - \omega_0\sqrt{\frac{1}{4q_z^2} - 1}, \frac{\omega_0}{2q_z} + \frac{1}{R_2C_2} + \omega_0\sqrt{\frac{1}{4q_z^2} - 1}\right]. \tag{23c}$$

$$x > \omega_0\left(\frac{1}{q_z} - \frac{1}{q_p}\right). \tag{23d}$$

Case 2.

$$R_2C_2 < \frac{1}{\omega_0}. \tag{24a}$$

$$R_2C_2 \,\varepsilon \left[\frac{1}{\omega_0}\left(\frac{1}{2q_z} - \sqrt{\frac{1}{4q_z^2} - 1}\right), \frac{1}{\omega_0}\left(\frac{1}{2q_z} + \sqrt{\frac{1}{4q_z^2} - 1}\right)\right]. \tag{24b}$$

$$x \,\varepsilon \left[\frac{\omega_0}{2q_z} + \frac{1}{R_2C_2} - \omega_0\sqrt{\frac{1}{4q_z^2} - 1}, \frac{\omega_0}{2q_z} + \omega_0^2 R_2C_2 + \omega_0\sqrt{\frac{1}{4q_z^2} - 1}\right]. \tag{24c}$$

$$x > \omega_0\left(\frac{1}{q_z} - \frac{1}{q_p}\right). \tag{24d}$$

Equations (20a) through (20e) and the constraints (23) and (24) represent the solution to the design of an FEN. For a given q_p the gain $\mu\beta$ should become as small as possible. This enlarges the bandwidth of the amplifier and provides a more stable gain. As equation (20e) shows, $\mu\beta$ becomes small if x is chosen as large as possible.

G. Malek[8] has written a computer program which provides solutions to equations (20a) through (20e) satisfying the constraints (23) and

(24). The user may specify the spread in the values of the resistors and the capacitors that can be tolerated. The program will give him a solution with the given spread and with a minimum value of the gain $\mu\beta$.

With G. Malek's program the following FENs have been designed.
(*i*) Given: $\omega_0 = 5 \times 10^4$ 1/second; $q_p = 60$; tolerable spread in the values of the *R*s and *C*s is 1:9.5; $\mu\beta$ should be as small as possible. The result is

$$R_1 = 16.821K \quad R_2 = 50.00K \quad R_3 = 22.556K \quad \mu\beta = 11.4$$

$$C_1 = 546pF \quad C_2 = 320pF \quad C_3 = 3010pF \quad q_z = 0.3.$$

(*ii*) Given: $\omega_0 = 5 \times 10^4$ 1/second; $q_p = 60$; tolerable spread 1:4.5; and again $\mu\beta$ as small as possible. The result is

$$R_1 = 25.6K \quad R_2 = 50.0K \quad R_3 = 16.5K \quad \mu\beta = 13.4$$

$$C_1 = 647pF \quad C_2 = 375pF \quad C_3 = 1674pF \quad q_z = 0.3.$$

(*iii*) Given: $\omega_0 = 5 \times 10^4$ 1/second; $q_p = 500$; tolerable spread 1:3; and $\mu\beta$ as small as possible. The result is

$$R_1 = 41.4K \quad R_2 = 50.0K \quad R_3 = 22.5K \quad \mu\beta = 20.16$$

$$C_1 = 413pF \quad C_2 = 370pF \quad C_3 = 1208pF \quad q_z = 0.25.$$

A great variery of further solutions may be found by G. Malek's program. If we would realize the three FENs with potentially symmetrical twin-Ts with the same spread in values, we would require a gain $\mu\beta = 133$ in the first case, $\mu\beta = 153$ in the second, and $\mu\beta = 1499$ in the third case. Thus the gain in the new circuits is a factor of 10 to 75 times smaller, resulting in increased bandwidth and stability.

The deviation of q_p with respect to temperature has been measured by G. Malek[8]. In the temperature range of 10°C to 70°C, q_p of the new circuit changed by 9 percent in comparison to 19.5 percent in the case of a potentially symmetrical twin-T.

VI. CONCLUSIONS

Conditions have been derived which guarantee that the transfer function or the *Y*-matrix of a general twin-T is of second degree in *s*. As a result, five parameters of the twin-T are at our disposal. Since this number is larger than in the commonly used approaches, more effective use can be made of a twin-T. It has been demonstrated in the case of an MSFEN that the gain required to realize a given pole *Q* may be reduced by a factor up to 70, while the stability of the pole may typically be improved by a factor of 2. The topological structure of the MSFEN, however, is the same as in the building blocks presently used.

APPENDIX

Given the two polynomials $N_0(s)$ and $N_1(s)$ where the degree of N_1 does not exceed that of N_0. We wish to find the largest common divider of N_0 and N_1 using Euclid's algorithm.[7] This algorithm is described below by the (a) equations, while the (b) equations are only used for the proof.

We form

$$\frac{N_0(s)}{N_1(s)} = A_1(s) + \frac{N_2(s)}{N_1(s)} \tag{25a}$$

or

$$N_0(s) = A_1(s)N_1(s) + N_2(s). \tag{25b}$$

where the degree of N_2 is smaller than the degree of N_0.

A common divider of N_0 and N_1 is a zero which both have in common. At those zeros, N_0 and N_1 vanish; thus N_2 in equation (25b) also must vanish. So the common zero is also contained in N_1 and N_2. Thus we continue by

$$\frac{N_1(s)}{N_2(s)} = A_2(s) + \frac{N_3(s)}{N_2(s)} \tag{26a}$$

or

$$N_1(s) = A_2(s)N_2(s) + N_3(s). \tag{26b}$$

With the same reasoning as before, the common zero is contained in N_2 and N_3 where the degree of N_3 is smaller than the degree of N_1. Continuing in the same manner we get

$$\frac{N_{\nu-1}(s)}{N_\nu(s)} = A_\nu(s) + \frac{N_{\nu+1}(s)}{N_\nu(s)} \tag{27a}$$

or

$$N_{\nu-1}(s) = A_\nu(s)N_\nu(s) + N_{\nu+1}(s). \tag{27b}$$

If $N_{\nu+1}(s) \equiv 0$ then N_ν is the largest common divider. This can be seen, considering that $N_{\nu-1}$ and N_ν have the same common divider as N_0 and N_1. For $N_{\nu+1} \equiv 0$ we get from (27b)

$$N_{\nu-1}(s) = A_\nu(s)N_\nu(s), \tag{28}$$

which shows that the largest divider N_ν is also contained in $N_{\nu-1}$.

If we want the common divider to be of the first degree in s then we continue dividing as above until an $N_\nu(s)$ of the first degree and a rest of zero degree remains. Now we put the rest $N_{\nu+1}(s) \equiv 0$ from which we derive the one condition for a common factor of the first degree in N_0 and N_1.

We apply this technique to $T(s)$ in equation (2), where

$$N_0(s) = as^3 + (b + d)s^2 + (c + e)s + 1,$$

$$N_1(s) = as^3 + bs^2 + cs + 1.$$

We ask for the one condition for which a common divider of the first degree occurs. Applying the algorithm, we get

Step 1: $$N_2 = +(ds^2 + es) = +ds\left(s + \frac{e}{d}\right).$$

Step 2: $$N_3 = \left(b - \frac{ea}{d}\right)s^2 + cs + 1.$$

At this point we may use a shortcut, which is always possible and which saves a considerable amount of time. We are to find the common divider of N_2 and N_3. Since N_2 is of the second degree we may factor it as shown above. One of the roots of the first order must be the common divider; ds cannot be a root of $N_3(s)$, thus

$$D(s) = s + \frac{e}{d} \tag{29}$$

must be the divider we are after. Dividing N_3 by $s + e/d$ we get the following rest which we set equal to zero

$$d(d^2 + e^2b) - e(d^2c + e^2a) = 0. \tag{30}$$

Equation (6) is the constraint guaranteeing that $D(s)$ is the common divider of $N_0(s)$ and $N_1(s)$. Dividing $N_0(s)$ and $N_1(s)$ by $D(s)$ and considering (30) we get

$$N_0(s) = \left(s + \frac{e}{d}\right)\left(as^2 + \left(b + d - \frac{ea}{d}\right)s + \frac{d}{e}\right), \tag{31a}$$

$$N_1(s) = \left(s + \frac{e}{d}\right)\left(as^2 + \left(b - \frac{e}{d}\,a\right)s + \frac{d}{e}\right), \tag{31b}$$

and

$$T(s) = \frac{N_1(s)}{N_0(s)} = \frac{as^2 + \left(b - \frac{e}{d}a\right)s + \frac{d}{e}}{as^2 + \left(b + d - \frac{ea}{d}\right)s + \frac{d}{e}}. \tag{31c}$$

REFERENCES

1. Sallen, R. P., and Key, E. L., "A Practical Method of Designing RC-Active Filters," IRE Trans., *CT-2*, No. 1 (March 1955), pp. 74–85.
2. Moschytz, G. S., "FEN Filter Design Using Tantalum and Silicon Integrated Circuits," Proc. IEEE, *58*, No. 4 (April 1970), pp. 550–556.
3. Seshu, S., and Reed, M. B., *Linear Graphs and Electrical Networks*, Reading, Massachusetts: Addison-Wesley, 1961.
4. Moschytz, G. S., Class notes on linear active networks. Lecture 10, BTL out-of-hours course (Spring 1970).
5. Moschytz, G. S., "A General Approach to Twin-T Design and Its Application to Hybrid Integrated Linear Active Networks," B.S.T.J., *49*, No. 6 (July–August 1970), pp. 1105–1149.
6. Lueder, E., "Precision Tuning of Hybrid RC-Active Filters with Nonideal Amplifiers," unpublished work.
7. Weber, H., *Lehrbuch der Algebra, 1*, New York: Chelsea Publishing Comp., 1898.
8. Malek, G., unpublished work.