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Coupled Line Equations with Random Coupling

By J. A. MORRISON and J. McKENNA

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The coupled line equations for two modes traveling in the same direction are considered. The covariances of the mode transfer functions are calculated in the case of random coupling. Exact results are obtained when the coupling is a function of a finite state Markov chain, and also when the coupling is white noise. Perturbation results are obtained in the case of weak, zero mean, wide sense stationary coupling. It is also shown that perturbation results are valid in the case of strong coupling, if the correlation length is short.

I. INTRODUCTION

We consider the coupled line equations for two modes traveling in the forward direction,[1]

$$\frac{dI_0}{dz} + \Gamma_0 I_0(z) = jc(z)I_1(z),$$
$$\frac{dI_1}{dz} + \Gamma_1 I_1(z) = jc(z)I_0(z), \tag{1}$$

subject to the initial conditions

$$I_0(0) = i_0\ ,\ I_1(0) = i_1\ . \tag{2}$$

The time dependence $\exp(2\pi jft)$ has been suppressed. Here $c(z)$ is the real coupling coefficient, which we take to be a random function of z, and the loss and phase constants are

$$\Gamma_0 = \alpha_0 + j\beta_0\ ,\ \Gamma_1 = \alpha_1 + j\beta_1\ . \tag{3}$$

These equations provide an approximate description of a variety of physical systems[2,3,4] such as optical fibers[5,6] and metal waveguides.[7,8] Typically, with the choice $i_0 = 1$, $i_1 = 0$, $I_0(z)$ represents a desired mode launched at $z = 0$, and $I_1(z)$ an undesired spurious mode.

Rowe and Young[1,9] have studied these equations with $c(z)$ taken to be white noise. When $c(z)$ is white noise, equations (1) are symbolic, and Rowe and Young interpret them by a limiting process, and with the aid of a matrix technique obtain exact expressions for the means and covariance functions of the solutions. With the aid of these solutions they studied pulse distortion caused by the coupling, and provide additional verification of Personick's[10] interesting result that in some cases *increasing* the random coupling can improve the impulse response.

This article has three main purposes. The *first* is to show that there is another class of stochastic coupling coefficients for which the means and covariances of the solutions of (1) can be calculated exactly. These involve Markov processes which can assume only a finite number of values. We set up the equations satisfied by the covariance functions and means for a general coupling of this type in Section II, and solve them in a special case in Section III.

The *second* purpose of this paper is to show, with the use of Ito integrals and the Ito calculus, that the interpretation of equations (1) by Rowe and Young when $c(z)$ is white noise is consistent with the interpretation due to Stratonovich.[11] This is the subject matter of Section IV.

The *third* purpose is to apply a perturbation technique developed by Papanicolaou and Keller[12] to the coupled line equations. It is shown that this method can be applied not only to the case of weak coupling, but also to the case of strong coupling if the correlation length is short. This is the subject of Section V.

In the remainder of Section I, we define the quantities of interest, introduce notation, and discuss some of the previously mentioned points at greater length.

The differential loss and phase constants are given by

$$\Delta\Gamma = \Gamma_0 - \Gamma_1 = \Delta\alpha + j\Delta\beta. \tag{4}$$

We write the coupling coefficient in the form

$$c(z) = CN(z), \tag{5}$$

where $N(z)$ is dimensionless. Following Rowe and Young[1,9] we assume that $\Delta\alpha$ is independent of the frequency f, but that $\Delta\beta$ and C are odd functions of f. We further assume that C is an odd function of $\Delta\beta$. Also, let

$$g_0(z, \Delta\beta) = e^{\Gamma_0 z} I_0(z), \; g_1(z, \Delta\beta) = e^{\Gamma_0 z} I_1(z). \tag{6}$$

Note that our $g_0(z, \Delta\beta)$ corresponds to Rowe and Young's $G_0(z)$ and our $g_1(z, \Delta\beta)$ to their $\mathbf{G}_1(z)$. Then, from (1), (4), and (5),

$$\begin{aligned} \frac{dg_0}{dz} &= jC(\Delta\beta)N(z)g_1(z, \Delta\beta) \\ \frac{dg_1}{dz} &= \Delta\Gamma g_1(z, \Delta\beta) + jC(\Delta\beta)N(z)g_0(z, \Delta\beta), \end{aligned} \tag{7}$$

with initial conditions

$$g_0(0, \Delta\beta) = i_0 , \quad g_1(0, \Delta\beta) = i_1 . \tag{8}$$

Now define the correlation functions

$$R_{k\ell}(z) = \langle g_k(z, \Delta\beta + \sigma)g^*_\ell(z, \Delta\beta)\rangle, \quad (k, \ell = 0, 1), \tag{9}$$

where * denotes complex conjugate and $\langle\ \rangle$ denotes stochastic average. Rowe and Young[1,9] have calculated $R_{k\ell}(z)$ exactly for the case of white noise coupling, with spectral density D_0 ,

$$\langle N(z)N(\zeta)\rangle = D_0\delta(z - \zeta), \tag{10}$$

in the case of initial values $i_0 = 1$ and $i_1 = 0$, corresponding to the signal and spurious modes respectively. They have also calculated the average of the squared envelope of the impulse response, which involves a double integral with respect to $\Delta\beta$ and σ of $R_{00}(z)$. The calculation was exact for frequency-independent coupling[1] (i.e., $C = c_0 \operatorname{sgn} \Delta\beta$), and approximate for moderate fractional bandwidths for frequency-dependent coupling.[9]

In this paper we will be concerned with the calculation of $R_{k\ell}(z)$ only, and will not consider the time domain statistics. We note that the average powers in the two modes are given by

$$P_0(z) = e^{-2\alpha_0 z}R_{00}(z)\,|_{\sigma=0}, \quad P_1(z) = e^{-2\alpha_0 z}R_{11}(z)\,|_{\sigma=0}. \tag{11}$$

We first consider the case

$$N(z) = F(M(z), z), \tag{12}$$

where $M(z)$ is a finite state Markov chain[13] which has, in general, a nonstationary transition mechanism. Using the results of a paper by Morrison,[14] we obtain a system of ordinary differential equations, with prescribed initial conditions, for calculating $R_{k\ell}(z)$. If the process $M(z)$ has a stationary transition mechanism, and F is a function of $M(z)$ alone, these equations have constant coefficients. In the particular case in which $N(z) = T(z)$, where $T(z)$ is the random telegraph process,[15] we obtain the Laplace transforms of $R_{k\ell}$ explicitly.

Next we investigate the case of white noise coupling by means of the Ito calculus.[16] This approach differs from that used by Rowe and Young,[1,9] who consider the line as the limit of discrete sections, of vanishing length, of uncoupled ideal lines, with discrete mode converters at the end of each section. The quantities $R_{k\ell}(z)$ are calculated exactly for white noise coupling, in agreement with the results of Rowe and Young.[1,9]

Finally, we turn to the asymptotic calculation, for long lines, of the quantities $R_{k\ell}(z)$, in the case of weak coupling, weak attenuation, and narrow fractional bandwidth. The coupling is assumed to have zero mean and to be wide sense stationary, so that $c(z)$ is given by (5), where

$$\langle N(z) \rangle = 0, \quad \langle N(z)N(\zeta) \rangle = \rho(z - \zeta). \tag{13}$$

The asymptotic equations for $R_{k\ell}(z)$ are determined, first for the nonresonance case, and then for the resonance case, in which the differential phase constant is small. There is an alternate interpretation of the resonance case, for which the correlation length is short. It turns out that for white noise coupling (which has zero correlation length), corresponding to (10), the asymptotic equations for $R_{00}(z)$ and $R_{11}(z)$ are exact, in both the nonresonance and resonance cases, and those for $R_{01}(z)$ and $R_{10}(z)$ are exact in the resonance case.

II. MARKOV CHAIN COUPLING

We begin by writing down the equations satisfied by the quantities

$$r_{k\ell}(z) = g_k(z, \Delta\beta + \sigma)g_\ell^*(z, \Delta\beta), \quad (k, \ell = 0, 1). \tag{14}$$

Let

$$C_0 = C(\Delta\beta), \quad C_\sigma = C(\Delta\beta + \sigma). \tag{15}$$

Then, from (4) and (7), for sufficiently smooth $N(z)$, it follows that

$$\frac{dr_{00}}{dz} = jN(z)(C_\sigma r_{10} - C_0 r_{01}), \tag{16}$$

$$\frac{dr_{01}}{dz} = jN(z)(C_\sigma r_{11} - C_0 r_{00}) + \Delta\Gamma^* r_{01} , \tag{17}$$

$$\frac{dr_{10}}{dz} = jN(z)(C_\sigma r_{00} - C_0 r_{11}) + (\Delta\Gamma + j\sigma)r_{10} , \tag{18}$$

$$\frac{dr_{11}}{dz} = jN(z)(C_\sigma r_{01} - C_0 r_{10}) + (\Delta\Gamma^* + \Delta\Gamma + j\sigma)r_{11} . \tag{19}$$

These equations do not hold when $N(z)$ is white noise, as will be discussed later. From (8) the initial conditions are

$$r_{k\ell}(0) = i_k i_\ell^*, \quad (k, \ell = 0, 1). \tag{20}$$

We now consider the case of coupling corresponding to (12), and first state some properties of the finite state Markov chain.[13] The sample functions $M(z)$ are defined on the half line $0 \leqq z < \infty$, have right-continuous paths, and can take on only a finite number N of distinct values $a_p(p = 1, \cdots, N)$. An initial probability distribution is given:

$$\mathrm{X}_p = \text{Prob}\,\{M(0) = a_p\}, \quad (p = 1, \cdots, N), \tag{21}$$

where $\mathrm{X}_p > 0$ and

$$\sum_{p=1}^{N} \mathrm{X}_p = 1. \tag{22}$$

The transition probabilities are defined, for $0 \leqq x \leqq y$, by

$$P_{pq}(x, y) = \text{Prob}\,\{M(y) = a_q \mid M(x) = a_p\}, \qquad (p, q = 1, \cdots, N). \tag{23}$$

We consider only those processes which can be defined by means of a continuous, bounded infinitesimal generator. Thus, we assume given an $N \times N$ matrix function

$$\boldsymbol{\tau}(z) = (\tau_{pq}(z)), \tag{24}$$

satisfying the conditions

$$\tau_{pq}(z) \geqq 0, \quad p \neq q, \quad \tau_{pp}(z) \leqq 0, \quad (p, q = 1, \cdots, N), \tag{25}$$

and

$$\sum_{q=1}^{N} \tau_{pq}(z) = 0, \qquad (p = 1, \cdots, N). \tag{26}$$

Then[13], for $\delta z \to 0+$, $(p, q = 1, \cdots, N)$,

$$P_{pp}(z, z + \delta z) = 1 + \tau_{pp}(z)\, \delta z + o(\delta z), \tag{27}$$

$$P_{pq}(z, z + \delta z) = \tau_{pq}(z)\, \delta z + o(\delta z), \qquad p \neq q. \tag{28}$$

If the matrix $\boldsymbol{\tau}$ is constant then the process $M(z)$ is said to have a stationary transition mechanism.

Stochastic matrix differential equations of the form

$$\frac{d\mathbf{W}}{dz} = \mathbf{A}(M(z), z)\mathbf{W}(z), \qquad \mathbf{W}(0) = \boldsymbol{\gamma}[M(0)], \tag{29}$$

where $\mathbf{W}$ and $\boldsymbol{\gamma}$ are $n \times m$ matrices, and $\mathbf{A}$ is an $n \times n$ matrix, have been considered by Morrison.[14] Equations were obtained for calculating the stochastic average $\langle \mathbf{W}(z) \rangle$, in the case that $\mathbf{A}$ and $\boldsymbol{\gamma}$ are real, but it is easily verified that these equations are still valid for complex valued $\mathbf{A}$ and $\boldsymbol{\gamma}$. This may be shown by writing the system (29) in real form, and then combining in complex form the equations for calculating the stochastic averages of the real and imaginary parts of $\mathbf{W}(z)$.

Let $\mathbf{E}_N$ denote the row vector with all N elements equal to 1, and let

$$\mathrm{X} = (\mathrm{X}_1 , \cdots , \mathrm{X}_N) \tag{30}$$

be the row vector of initial probabilities given by (21). Note, from (22) and (26), that

$$\mathbf{E}_N \mathrm{X}^t = 1, \quad \mathbf{E}_N \boldsymbol{\tau}^t(z) \equiv \mathbf{0}, \tag{31}$$

where t denotes transpose. Let

$$\langle r_{k\ell}(z) \rangle_p = \langle r_{k\ell}(z) \mid M(z) = a_p \rangle \text{ Prob } \{M(z) = a_p\}, \tag{32}$$

$(k, \ell = 0, 1)$, $(p = 1, \cdots, N)$, and introduce the column vectors

$$\mathbf{R}_{k\ell}(z) = \text{col } (\langle r_{k\ell}(z) \rangle_1 , \cdots , \langle r_{k\ell}(z) \rangle_N), \quad (k, \ell = 0, 1). \tag{33}$$

Then, from (9) and (14),

$$R_{k\ell}(z) = \langle r_{k\ell}(z) \rangle = \mathbf{E}_N \mathbf{R}_{k\ell}(z), \quad (k, \ell = 0, 1). \tag{34}$$

We define the $N \times N$ diagonal matrix $\mathbf{D}(z)$ by

$$\mathbf{D}(z) = \text{diag } [F(a_p , z)], \tag{35}$$

and denote the unit matrix of order N by $\mathbf{I}_N$. Then, for the system (16) through (19), with $N(z)$ given by (12), it is found that[14]

$$\frac{d\mathbf{R}_{00}}{dz} = j\mathbf{D}(z)(C_\sigma \mathbf{R}_{10} - C_0 \mathbf{R}_{01}) + \boldsymbol{\tau}^t(z)\mathbf{R}_{00} , \tag{36}$$

$$\frac{d\mathbf{R}_{01}}{dz} = j\mathbf{D}(z)(C_\sigma \mathbf{R}_{11} - C_0 \mathbf{R}_{00}) + [\Delta\Gamma^* \mathbf{I}_N + \boldsymbol{\tau}^t(z)]\mathbf{R}_{01} , \tag{37}$$

$$\frac{d\mathbf{R}_{10}}{dz} = j\mathbf{D}(z)(C_\sigma \mathbf{R}_{00} - C_0 \mathbf{R}_{11}) + [(\Delta\Gamma + j\sigma)\mathbf{I}_N + \boldsymbol{\tau}^t(z)]\mathbf{R}_{10} , \tag{38}$$

$$\frac{d\mathbf{R}_{11}}{dz} = j\mathbf{D}(z)(C_\sigma \mathbf{R}_{01} - C_0 \mathbf{R}_{10}) + [(\Delta\Gamma^* + \Delta\Gamma + j\sigma)\mathbf{I}_N + \boldsymbol{\tau}^t(z)]\mathbf{R}_{11} . \tag{39}$$

From (20), (21), (30), (32) and (33), the initial conditions are

$$\mathbf{R}_{k\ell}(0) = i_k i_\ell^* \mathrm{X}^t, \quad (k, \ell = 0, 1). \tag{40}$$

If the process $M(z)$ has a stationary transition mechanism, and if the coupling coefficient (5) depends on z through $M(z)$ only, then the differential equations (36) through (39) have constant coefficients. We consider a particular example of this in more detail in the next section.

III. RANDOM TELEGRAPH COUPLING

We now consider the particular case when $N(z) = T(z)$, the random telegraph process,[15] which is one of the simplest finite state Markov chains. It is an ensemble of square wave functions $\{T(z)\}$, such that each sample function $T(z)$ can assume only the values ± 1. For fixed z, a sample function chosen at random will equal $+1$ or -1 with equal probability. The probability $p(n, z)$ of a given sample function changing sign n times in an interval of length z is given by the Poisson process

$$p(n, z) = \frac{(bz)^n}{n!} e^{-bz}, \qquad (n = 0, 1, 2, \cdots), \tag{41}$$

where b is the average number of changes per unit length. The process $T(z)$ has zero mean and is wide sense stationary,

$$\langle T(z) \rangle = 0, \quad \langle T(z)T(\zeta) \rangle = \exp\{-2b \mid z - \zeta \mid\}. \tag{42}$$

The states of the process $T(z)$ are $a_1 = 1$ and $a_2 = -1$, and the vector of initial probabilities is

$$\mathbf{X} = (\tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{1}{2}\mathbf{E}_2 = \tfrac{1}{2}\mathbf{E}. \tag{43}$$

The process has a stationary transition mechanism, with infinitesimal generator

$$\boldsymbol{\tau} = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix} = \boldsymbol{\tau}^t. \tag{44}$$

Since $N(z) = T(z)$, we have, from (12),

$$F(a_p, z) = a_p = (-1)^{p-1}, \quad (p = 1, 2). \tag{45}$$

Hence, from (35),

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{46}$$

Note that

$$\mathbf{D}^2 = \mathbf{I}_2 = \mathbf{I}, \quad \mathbf{E}\boldsymbol{\tau}^t = 0, \quad \mathbf{E}\mathbf{D}\boldsymbol{\tau}^t = -2b\mathbf{E}\mathbf{D}. \tag{47}$$

Let

$$Q_{k\ell}(z) = \mathbf{EDR}_{k\ell}(z), \quad (k, \ell = 0, 1). \tag{48}$$

Then, multiplying equations (36) and (39) by **E**, and equations (37) and (38) by **ED**, we obtain, from (34), (47), and (48),

$$\frac{dR_{00}}{dz} = j(C_\sigma Q_{10} - C_0 Q_{01}), \tag{49}$$

$$\frac{dR_{11}}{dz} = j(C_\sigma Q_{01} - C_0 Q_{10}) + (\Delta\Gamma^* + \Delta\Gamma + j\sigma)R_{11} , \tag{50}$$

and

$$\frac{dQ_{01}}{dz} = j(C_\sigma R_{11} - C_0 R_{00}) + (\Delta\Gamma^* - 2b)Q_{01} , \tag{51}$$

$$\frac{dQ_{10}}{dz} = j(C_\sigma R_{00} - C_0 R_{11}) + (\Delta\Gamma + j\sigma - 2b)Q_{10} . \tag{52}$$

Similarly, multiplying equations (37) and (38) by **E**, and equations (36) and (39) by **ED**, we obtain

$$\frac{dR_{01}}{dz} = j(C_\sigma Q_{11} - C_0 Q_{00}) + \Delta\Gamma^* R_{01} , \tag{53}$$

$$\frac{dR_{10}}{dz} = j(C_\sigma Q_{00} - C_0 Q_{11}) + (\Delta\Gamma + j\sigma)R_{10} , \tag{54}$$

and

$$\frac{dQ_{00}}{dz} = j(C_\sigma R_{10} - C_0 R_{01}) - 2bQ_{00} , \tag{55}$$

$$\frac{dQ_{11}}{dz} = j(C_\sigma R_{01} - C_0 R_{10}) + (\Delta\Gamma^* + \Delta\Gamma + j\sigma - 2b)Q_{11} . \tag{56}$$

From (34), (40), (43), (46) and (48), the initial conditions are

$$R_{k\ell}(0) = i_k i_\ell^* , \quad Q_{k\ell}(0) = 0, \quad (k, \ell = 0, 1). \tag{57}$$

Equations (49) through (56), subject to (57), are solved by means of Laplace transforms in Appendix A. The Laplace transforms $\hat{R}_{00}(s)$ and $\hat{R}_{11}(s)$ of $R_{00}(z)$ and $R_{11}(z)$ are given by (128) and (129), where $\Delta(s)$ is given by (130), and d_1 , d_2 and η are defined in (125). Note that d_1 and d_2 are linear in the Laplace transform parameter s. Thus $\Delta(s)$ is a quartic in s, and we denote the roots of $\Delta(s) = 0$ by $s_k = s_k(\Delta\beta, \sigma)$, $k = 1, 2, 3, 4$. Inversion of the Laplace transforms in (128) and (129) leads to

$$R_{00}(z) = \sum_{k=1}^{4} A_{0k}(\Delta\beta, \sigma) \exp [s_k(\Delta\beta, \sigma)z] \tag{58}$$

and

$$R_{11}(z) = \sum_{k=1}^{4} A_{1k}(\Delta\beta, \sigma) \exp [s_k(\Delta\beta, \sigma)z], \tag{59}$$

the coefficients A_{0k} and A_{1k} being the residues at $s = s_k$ of the expressions in (128) and (129) respectively.

Similarly, the Laplace transforms $\hat{R}_{01}(s)$ and $\hat{R}_{10}(s)$ of $R_{01}(z)$ and $R_{10}(z)$ are given by (138) and (139), where $E(s)$ is given by (140), and e_1 and e_2 are defined in (135). Note that e_1 and e_2 are linear in s, so that $E(s)$ is a quartic in s. We denote the roots of $E(s) = 0$ by $\mu_k = \mu_k(\Delta\beta, \sigma)$, $k = 1, 2, 3, 4$. Inversion of the Laplace transforms in (138) and (139) leads to

$$R_{01}(z) = \sum_{k=1}^{4} B_{0k}(\Delta\beta, \sigma) \exp [\mu_k(\Delta\beta, \sigma)z], \tag{60}$$

and

$$R_{10}(z) = \sum_{k=1}^{4} B_{1k}(\Delta\beta, \sigma) \exp [\mu_k(\Delta\beta, \sigma)z], \tag{61}$$

the coefficients B_{0k} and B_{1k} being the residues at $s = \mu_k$ of the expressions in (138) and (139) respectively.

IV. WHITE NOISE COUPLING

In this section we investigate the case of white noise coupling, corresponding to (5) and (10), by means of the Ito calculus. We begin by considering the vector equation

$$\frac{d\mathbf{u}}{dz} = \mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{u}\xi(z), \tag{62}$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant matrices, and $\xi(z)$ is Gaussian white noise, with zero mean and spectral density 1. According to Wonham,[17] the Ito differential equation corresponding to (62) is

$$d\mathbf{u} = (\mathbf{A} + \tfrac{1}{2}\mathbf{B}^2)\mathbf{u}dz + \mathbf{B}\mathbf{u}dw, \tag{63}$$

where w is a Wiener process. The result is stated for real equations, but it may be verified that it holds for complex $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{u}$, by considering the equations for the real and imaginary parts.

The integral equation corresponding to (63) is

$$\mathbf{u}(z) = \mathbf{u}(0) + (\mathbf{A} + \tfrac{1}{2}\mathbf{B}^2) \int_0^z \mathbf{u}(\zeta)\, d\zeta + \mathbf{B} \int_0^z \mathbf{u}(\zeta)\, dw(\zeta). \tag{64}$$

The second integral in (64) is an Ito integral, and it has the property that its stochastic average is zero.[18] Thus,

$$\langle \mathbf{u}(z) \rangle = \langle \mathbf{u}(0) \rangle + (\mathbf{A} + \tfrac{1}{2}\mathbf{B}^2) \int_0^z \langle \mathbf{u}(\zeta) \rangle\, d\zeta, \tag{65}$$

or, in differential form,

$$\frac{d}{dz} \langle \mathbf{u}(z) \rangle = (\mathbf{A} + \tfrac{1}{2}\mathbf{B}^2) \langle \mathbf{u}(z) \rangle. \tag{66}$$

We will also make use of the Ito calculus for the product of differentials,[16] namely

$$(dw)^2 = dz, \quad dwdz = 0, \quad (dz)^2 = 0. \tag{67}$$

We now consider the line equations for white noise coupling, so that, from (5) and (10),

$$c(z) = C\sqrt{D_0}\,\xi(z). \tag{68}$$

Hence, from (1), (62), and (66),

$$\frac{d}{dz} \langle I_0(z) \rangle = -(\Gamma_0 + \tfrac{1}{2}C^2 D_0) \langle I_0(z) \rangle, \tag{69}$$

and

$$\frac{d}{dz} \langle I_1(z) \rangle = -(\Gamma_1 + \tfrac{1}{2}C^2 D_0) \langle I_1(z) \rangle. \tag{70}$$

These equations agree with those obtained by Rowe and Young.[1,9] From (2), (69), and (70),

$$\langle I_\ell(z) \rangle = i_\ell \exp\left[-(\Gamma_\ell + \tfrac{1}{2}C^2 D_0)z\right], \quad (\ell = 0, 1). \tag{71}$$

Also, from (5), (7), (62), (63), and (68),

$$\begin{aligned} dg_0 &= -\tfrac{1}{2}C^2 D_0 g_0\, dz + jC\sqrt{D_0}\, g_1\, dw, \\ dg_1 &= (\Delta\Gamma - \tfrac{1}{2}C^2 D_0) g_1\, dz + jC\sqrt{D_0}\, g_0\, dw. \end{aligned} \tag{72}$$

But, from (14),

$$\begin{aligned} dr_{k\ell}(z) = {} & dg_k(z, \Delta\beta + \sigma) g_\ell^*(z, \Delta\beta) + g_k(z, \Delta\beta + \sigma)\, dg_\ell^*(z, \Delta\beta) \\ & + dg_k(z, \Delta\beta + \sigma)\, dg_\ell^*(z, \Delta\beta). \end{aligned} \tag{73}$$

Hence, from (4), (14), (15), (72), and (73), making use of the relationships in (67),

$$\begin{aligned} dr_{00} &= D_0[C_0C_\sigma r_{11} - \tfrac{1}{2}(C_0^2 + C_\sigma^2)r_{00}]\,dz \\ &\quad + j\sqrt{D_0}\,(C_\sigma r_{10} - C_0 r_{01})\,dw, \end{aligned} \tag{74}$$

$$\begin{aligned} dr_{01} &= [\Delta\Gamma^* - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0]r_{01}\,dz + C_0C_\sigma D_0 r_{10}\,dz \\ &\quad + j\sqrt{D_0}\,(C_\sigma r_{11} - C_0 r_{00})\,dw, \end{aligned} \tag{75}$$

$$\begin{aligned} dr_{10} &= [(\Delta\Gamma + j\sigma) - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0]r_{10}\,dz + C_0C_\sigma D_0 r_{01}\,dz \\ &\quad + j\sqrt{D_0}\,(C_\sigma r_{00} - C_0 r_{11})\,dw, \end{aligned} \tag{76}$$

$$\begin{aligned} dr_{11} &= [(\Delta\Gamma^* + \Delta\Gamma + j\sigma) - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0]r_{11}\,dz + C_0C_\sigma D_0 r_{00}\,dz \\ &\quad + j\sqrt{D_0}\,(C_\sigma r_{01} - C_0 r_{10})\,dw. \end{aligned} \tag{77}$$

From (9), (14), (63), and (66), it follows that

$$\frac{dR_{00}}{dz} = -\tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0R_{00} + C_0C_\sigma D_0R_{11}\ , \tag{78}$$

$$\frac{dR_{11}}{dz} = C_0C_\sigma D_0R_{00} + [(\Delta\Gamma^* + \Delta\Gamma + j\sigma) - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0]R_{11}\ , \tag{79}$$

and

$$\frac{dR_{01}}{dz} = [\Delta\Gamma^* - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0]R_{01} + C_0C_\sigma D_0R_{10} \tag{80}$$

$$\frac{dR_{10}}{dz} = C_0C_\sigma D_0R_{01} + [(\Delta\Gamma + j\sigma) - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0]R_{10}\ . \tag{81}$$

These equations for $R_{k\ell}$ are consistent with those obtained by Rowe and Young.[1,9] The initial conditions are given by (57), and the solutions of (78) through (81) are easy to write down. The characteristic roots corresponding to (78) and (79) are, from (4),

$$\lambda = (\Delta\alpha + \tfrac{1}{2}j\sigma) - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0 \pm [C_0^2C_\sigma^2D_0^2 + (\Delta\alpha + \tfrac{1}{2}j\sigma)^2]^{\frac{1}{2}}, \tag{82}$$

and those corresponding to (80) and (81) are

$$\lambda = (\Delta\alpha + \tfrac{1}{2}j\sigma) - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0 \pm [C_0^2C_\sigma^2D_0^2 - (\Delta\beta + \tfrac{1}{2}\sigma)^2]^{\frac{1}{2}}. \tag{83}$$

We now return to the random telegraph coupling and consider a case where the coupling is large but has a short correlation length. We define the correlation length of the random telegraph process to be

$$\ell_c = \frac{1}{2b}\ , \tag{84}$$

which is just the separation at which the covariance drops to $1/e$ of its peak value. This also agrees with a more general definition of the correlation length of a wide sense stationary process, due to C. L. Mallows:[19]

$$\ell_c = \lim_{\Omega \to \infty} \left\{ 2\pi \int_{-\Omega}^{\Omega} | \mathcal{S}(\omega) |^2 \, d\omega \Big/ \left| \int_{-\Omega}^{\Omega} \mathcal{S}(\omega) \, d\omega \right|^2 \right\}, \tag{85}$$

where $\mathcal{S}(\omega)$ is the spectral density function. We set

$$C_0 = \sqrt{bS_0}, \qquad C_\sigma = \sqrt{bS_\sigma}. \tag{86}$$

If we substitute (86) into the expression (130) for $\Delta(s)$, and (140) for $E(s)$, as $b \to \infty$, $\Delta(s)$ has the roots

$$s = (\Delta\alpha + \tfrac{1}{2}j\sigma) - \tfrac{1}{2}(S_0 + S_\sigma) \pm [(\Delta\alpha + \tfrac{1}{2}j\sigma)^2 + S_0 S_\sigma]^{\frac{1}{2}} + 0\left(\frac{1}{b}\right), \tag{87}$$

$$s = -2b + (\Delta\alpha + \tfrac{1}{2}j\sigma) + \tfrac{1}{2}(S_0 + S_\sigma) \pm [S_0 S_\sigma - (\Delta\beta + \tfrac{1}{2}\sigma)^2]^{\frac{1}{2}} + 0\left(\frac{1}{b}\right), \tag{88}$$

and $E(s)$ has the roots

$$\mu = (\Delta\alpha + \tfrac{1}{2}j\sigma) - \tfrac{1}{2}(S_0 + S_\sigma) \pm [S_0 S_\sigma - (\Delta\beta + \tfrac{1}{2}\sigma)^2]^{\frac{1}{2}} + 0\left(\frac{1}{b}\right), \tag{89}$$

$$\mu = -2b + (\Delta\alpha + \tfrac{1}{2}j\sigma) + \tfrac{1}{2}(S_0 + S_\sigma) \pm [S_0 S_\sigma + (\Delta\alpha + \tfrac{1}{2}j\sigma)^2]^{\frac{1}{2}} + 0\left(\frac{1}{b}\right). \tag{90}$$

With the aid of (87) through (90), it is straightforward to calculate $\lim_{b \to \infty} R_{k\ell}(z)$, $(k, \ell = 0, 1)$. These limits are just the corresponding expressions for white noise coupling, if in the expressions for white noise we set

$$C_0^2 D_0 = S_0, \quad C_\sigma^2 D_0 = S_\sigma. \tag{91}$$

But (91) is consistent with (86), in view of (10), since

$$\int_{-\infty}^{\infty} \langle T(z + \zeta) T(\zeta) \rangle \, dz = \frac{1}{b}, \tag{92}$$

from (42).

VI. PERTURBATION THEORY

We now assume the random coupling is described by a zero mean, wide sense stationary process $N(z)$, with an autocorrelation function satisfying (102), but which is otherwise arbitrary. We assume in addition that there is a characteristic length ℓ, e.g., the correlation length ℓ_c, and an ϵ, $0 < \epsilon \ll 1$, such that

$$C_0\ell = 0(\epsilon), \quad C_\sigma\ell = 0(\epsilon), \quad \Delta\alpha\ell = 0(\epsilon^2), \quad \sigma\ell = 0(\epsilon^2), \tag{93}$$

and in the first instance

$$\Delta\beta\ell = 0(1). \tag{94}$$

This can best be described as a case with weak coupling, weak attenuation, and narrow fractional bandwidth.

We consider the asymptotic calculation of $R_{k\ell}(z)$ for small ϵ. The asymptotic results are valid for $z/\ell = 0(1/\epsilon^2)$. We begin by stating, in a form which will suffice for our present purposes, some results of Papanicolaou and Keller[12] on stochastic differential equations.

Thus, consider the linear vector stochastic differential equation

$$\frac{d\mathbf{w}}{dz} = [\epsilon\mathbf{A}_1(z) + \epsilon^2\mathbf{A}_2]w(z), \qquad \mathbf{w}(0) = \mathbf{w}_0, \tag{95}$$

where $\mathbf{A}_1(z)$ is a randon matrix with zero mean,

$$\langle\mathbf{A}_1(z)\rangle \equiv 0, \tag{96}$$

$\mathbf{A}_2$ is a constant nonstochastic matrix, and $\mathbf{w}_0$ is a nonstochastic vector. Define

$$\overline{\mathbf{A}_1\mathbf{A}_1} = \lim_{Z\to\infty}\left[\frac{1}{Z}\int_0^Z\int_0^z \langle\mathbf{A}_1(z)\mathbf{A}_1(\zeta)\rangle\, d\zeta\, dz\right]. \tag{97}$$

Then asymptotically, for $0 < \epsilon \ll 1$,

$$\frac{d}{dz}\langle\mathbf{w}(z)\rangle \approx \epsilon^2(\overline{\mathbf{A}_1\mathbf{A}_1} + \mathbf{A}_2)\langle\mathbf{w}(z)\rangle, \qquad \langle\mathbf{w}(0)\rangle = \mathbf{w}_0. \tag{98}$$

We now apply the above result to equations (16) through (19) for $r_{k\ell}(z)$, subject to the assumptions (93) and (94), where $\Delta\Gamma$ is given by (4). Thus, we let

$$\mathbf{w} = (r_{00}, r_{11}, e^{i\Delta\beta z}r_{01}, e^{-i\Delta\beta z}r_{10})^t, \tag{99}$$

where t denotes transpose. Then,

$$\epsilon\mathbf{A}_1(z) = jN(z)\begin{bmatrix} 0 & 0 & -C_0e^{-i\Delta\beta z} & C_\sigma e^{i\Delta\beta z} \\ 0 & 0 & C_\sigma e^{-i\Delta\beta z} & -C_0e^{i\Delta\beta z} \\ -C_0e^{i\Delta\beta z} & C_\sigma e^{i\Delta\beta z} & 0 & 0 \\ C_\sigma e^{-i\Delta\beta z} & -C_0e^{-i\Delta\beta z} & 0 & 0 \end{bmatrix}, \tag{100}$$

and

$$\epsilon^2 \mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (2\Delta\alpha + j\sigma) & 0 & 0 \\ 0 & 0 & \Delta\alpha & 0 \\ 0 & 0 & 0 & (\Delta\alpha + j\sigma) \end{bmatrix}. \tag{101}$$

Note that $\mathbf{A}_1(z)$ as given by (100) satisfies (96), by virtue of (13). The calculation of $\overline{\mathbf{A}_1\mathbf{A}_1}$, defined by (97), is carried out in Appendix B, under the assumption that

$$\lim_{Z\to\infty} \left[\frac{1}{Z} \int_0^Z \zeta \mid \rho(\zeta) \mid d\zeta \right] = 0, \tag{102}$$

where $\rho(\zeta)$ is given by (13). Let

$$S(\omega) = \int_0^\infty e^{j\omega\zeta} \rho(\zeta)\, d\zeta. \tag{103}$$

Since, from (9) and (14), $R_{k\ell}(z) = \langle r_{k\ell}(z) \rangle$, it follows from (98), (99), (101), (148), and (149) that, asymptotically,

$$\frac{dR_{00}}{dz} \approx -[C_0^2 S(-\Delta\beta) + C_\sigma^2 S(\Delta\beta)]R_{00} + C_0 C_\sigma [S(\Delta\beta) + S(-\Delta\beta)]R_{11}\,, \tag{104}$$

$$\frac{dR_{11}}{dz} \approx C_0 C_\sigma [S(\Delta\beta) + S(-\Delta\beta)]R_{00} + [(2\Delta\alpha + j\sigma) - C_0^2 S(\Delta\beta) - C_\sigma^2 S(-\Delta\beta)]R_{11}\,, \tag{105}$$

and, from (150), that

$$\frac{dR_{01}}{dz} \approx [(\Delta\alpha - j\Delta\beta) - (C_0^2 + C_\sigma^2) S(\Delta\beta)]R_{01}\,, \tag{106}$$

$$\frac{dR_{10}}{dz} \approx [(\Delta\alpha + j\Delta\beta + j\sigma) - (C_0^2 + C_\sigma^2) S(-\Delta\beta)]R_{10}\,. \tag{107}$$

The initial conditions are given by (57).

Now,

$$\int_0^\infty \cos(\omega\zeta)\, \delta(\zeta)\, d\zeta = \frac{1}{2} \int_{-\infty}^\infty \cos(\omega\zeta)\, \delta(\zeta)\, d\zeta = \tfrac{1}{2}, \tag{108}$$

$$\int_0^\infty \sin(\omega\zeta)\, \delta(\zeta)\, d\zeta = 0.$$

Hence, from (103), for $\rho(\zeta) = D_0\delta(\zeta)$ we have $S(\omega) \equiv 1/2(D_0)$. Note that the asymptotic equations (104) and (105) for R_{00} and R_{11} are, in fact, exact for white noise, as is seen from (4), (78) and (79). However, from (57), (106) and (107), for $S(\omega) = 1/2(D_0)$, we have asymptotically

$$R_{01}(z) \approx i_0 i_1^* \exp\{[\Delta\Gamma^* - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0]z\}, \tag{109}$$

$$R_{10}(z) \approx i_1 i_0^* \exp\{[(\Delta\Gamma + j\sigma) - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0]z\}. \tag{110}$$

It is readily verified that these results are asymptotically consistent for $0 \leqq z/\ell \leqq 0(1/\epsilon^2)$, to lowest order in ϵ, with the exact equations (80) and (81) for white noise, under the assumptions in (93) and (94). The characteristic roots corresponding to (80) and (81) are, from (83),

$$\lambda = \pm j(\Delta\beta + \tfrac{1}{2}\sigma) + (\Delta\alpha + \tfrac{1}{2}j\sigma) - \tfrac{1}{2}(C_0^2 + C_\sigma^2)D_0 + \frac{0(\epsilon^4)}{\ell}. \tag{111}$$

The above results are valid under the assumptions (93) and (94). Suppose now that instead of (94) we have

$$(\Delta\beta)\ell = 0(\epsilon^2), \tag{112}$$

which we refer to as the resonance case. Note if we set $b = b_0/\epsilon^2$ and $\ell = \ell_c = 1/2b$, the correlation length, the random telegraph case considered at the end of Section IV is a special example of this resonance case. The resonance case can thus correspond to large coupling, moderate attenuation and fractional bandwidth, and small correlation length. Corresponding to (16) through (19), we take

$$\mathbf{w} = (r_{00}, r_{11}, r_{01}, r_{10})^t, \tag{113}$$

so that, from (93), (95), and (112), we now have

$$\epsilon\mathbf{A}_1(z) = jN(z)\begin{bmatrix} 0 & 0 & -C_0 & C_\sigma \\ 0 & 0 & C_\sigma & -C_0 \\ -C_0 & C_\sigma & 0 & 0 \\ C_\sigma & -C_0 & 0 & 0 \end{bmatrix}, \tag{114}$$

and

$$\epsilon^2\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (\Delta\Gamma^* + \Delta\Gamma + j\sigma) & 0 & 0 \\ 0 & 0 & \Delta\Gamma^* & 0 \\ 0 & 0 & 0 & (\Delta\Gamma + j\sigma) \end{bmatrix}. \tag{115}$$

Note that $\mathbf{A}_1(z)$ as given by (114) satisfies (96), by virtue of (13). The calculation of $\overline{\mathbf{A}_1\mathbf{A}_1}$, defined by (97), is carried out in Appendix B. It follows from (98), (113), (115), (152), and (153), with $R_{k\ell}(z) = \langle r_{k\ell}(z) \rangle$, that asymptotically

$$\frac{dR_{00}}{dz} \approx -(C_0^2 + C_\sigma^2)S(0)R_{00} + 2C_0C_\sigma S(0)R_{11} , \tag{116}$$

$$\frac{dR_{11}}{dz} \approx 2C_0C_\sigma S(0)R_{00} + [(\Delta\Gamma^* + \Delta\Gamma + j\sigma) - (C_0^2 + C_\sigma^2)S(0)]R_{11} , \tag{117}$$

and

$$\frac{dR_{01}}{dz} \approx [\Delta\Gamma^* - (C_0^2 + C_\sigma^2)S(0)]R_{01} + 2C_0C_\sigma S(0)R_{10} , \tag{118}$$

$$\frac{dR_{10}}{dz} \approx 2C_0C_\sigma S(0)R_{01} + [(\Delta\Gamma + j\sigma) - (C_0^2 + C_\sigma^2)S(0)]R_{10} . \tag{119}$$

Thus, in the resonance case, the asymptotic equations (116) through (119) for $R_{k\ell}(z)$, $(k, \ell = 0, 1)$, are exact for white noise, for which $S(0) = 1/2(D_0)$, as is seen from (78) through (81).

APPENDIX A

We here solve equations (49) through (56), subject to the initial conditions (57). The Laplace transform of $F(z)$ is

$$\hat{F}(s) = \int_0^\infty e^{-sz}F(z)\, dz. \tag{120}$$

Taking the Laplace transforms of equations (49) through (52), and using (57), one finds that

$$s\hat{R}_{00} + jC_0\hat{Q}_{01} - jC_\sigma\hat{Q}_{10} = i_0i_0^*, \tag{121}$$

$$(s - \Delta\Gamma^* - \Delta\Gamma - j\sigma)\hat{R}_{11} - jC_\sigma\hat{Q}_{01} + jC_0\hat{Q}_{10} = i_1i_1^*, \tag{122}$$

and

$$(s + 2b - \Delta\Gamma^*)\hat{Q}_{01} = jC_\sigma\hat{R}_{11} - jC_0\hat{R}_{00} , \tag{123}$$

$$(s + 2b - \Delta\Gamma - j\sigma)\hat{Q}_{10} = jC_\sigma\hat{R}_{00} - jC_0\hat{R}_{11} . \tag{124}$$

Let

$$d_1 = s + 2b - \Delta\Gamma^*, \quad d_2 = s + 2b - \Delta\Gamma - j\sigma,$$
$$\eta = 2\Delta\alpha + j\sigma. \tag{125}$$

Then, substituting for $\hat{Q}_{01}$ and $\hat{Q}_{10}$ from (123) and (124) into (121) and (122), it follows that

$$\left(s + \frac{C_0^2}{d_1} + \frac{C_\sigma^2}{d_2}\right)\hat{R}_{00} - C_0C_\sigma\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\hat{R}_{11} = i_0i_0^* , \tag{126}$$

$$-C_0C_\sigma\left(\frac{1}{d_1} + \frac{1}{d_2}\right)\hat{R}_{00} + \left(s - \eta + \frac{C_0^2}{d_2} + \frac{C_\sigma^2}{d_1}\right)\hat{R}_{11} = i_1i_1^* . \tag{127}$$

Hence

$$\hat{R}_{00} = \{[(s - \eta)\, d_1\, d_2 + C_0^2\, d_1 + C_\sigma^2\, d_2]i_0i_0^* + C_0C_\sigma(d_1 + d_2)i_1i_1^*\}/\Delta(s), \tag{128}$$

$$\hat{R}_{11} = \{C_0C_\sigma(d_1 + d_2)i_0i_0^* + (s\, d_1\, d_2 + C_0^2\, d_2 + C_\sigma^2\, d_1)i_1i_1^*\}/\Delta(s), \tag{129}$$

where

$$\Delta(s) \equiv [s(s - \eta)\, d_1\, d_2 + (C_0^2 + C_\sigma^2)s(d_1 + d_2) - \eta(C_\sigma^2\, d_1 + C_0^2\, d_2) + (C_\sigma^2 - C_0^2)^2]. \tag{130}$$

Similarly, taking the Laplace transforms of equations (53) through (56), and using (57),

$$(s - \Delta\Gamma^*)\hat{R}_{01} + jC_0\hat{Q}_{00} - jC_\sigma\hat{Q}_{11} = i_0i_1^* , \tag{131}$$

$$(s - \Delta\Gamma - j\sigma)\hat{R}_{10} - jC_\sigma\hat{Q}_{00} + jC_0\hat{Q}_{11} = i_1i_0^* , \tag{132}$$

and

$$(s + 2b)\hat{Q}_{00} = jC_\sigma\hat{R}_{10} - jC_0\hat{R}_{01} , \tag{133}$$

$$(s + 2b - \eta)\hat{Q}_{11} = jC_\sigma\hat{R}_{01} - jC_0\hat{R}_{10} , \tag{134}$$

where η is defined in (125). Let

$$e_1 = s + 2b, \quad e_2 = s + 2b - \eta. \tag{135}$$

Then, substituting for $\hat{Q}_{00}$ and $\hat{Q}_{11}$ from (133) and (134) into (131) and (132),

$$\left(s - \Delta\Gamma^* + \frac{C_0^2}{e_1} + \frac{C_\sigma^2}{e_2}\right)\hat{R}_{01} - C_0C_\sigma\left(\frac{1}{e_1} + \frac{1}{e_2}\right)\hat{R}_{10} = i_0i_1^* , \tag{136}$$

$$-C_0C_\sigma\left(\frac{1}{e_1} + \frac{1}{e_2}\right)\hat{R}_{01} + \left(s - \Delta\Gamma - j\sigma + \frac{C_0^2}{e_2} + \frac{C_\sigma^2}{e_1}\right)\hat{R}_{10} = i_1i_0^* . \tag{137}$$

From (136) and (137) it follows that

$$\hat{R}_{01} = \{[(s - \Delta\Gamma - j\sigma)e_1e_2 + C_0^2e_1 + C_\sigma^2e_2]i_0i_1^* + C_0C_\sigma(e_1 + e_2)i_1i_0^*\}/E(s), \tag{138}$$

and

$$\hat{R}_{10} = \{C_0 C_\sigma (e_1 + e_2) i_0 i_1^* + [(s - \Delta\Gamma^*) e_1 e_2 + C_0^2 e_2 + C_\sigma^2 e_1] i_1 i_0^*\}/E(s), \tag{139}$$

where

$$\begin{aligned} E(s) \equiv {} & (s - \Delta\Gamma - j\sigma)(s - \Delta\Gamma^*) e_1 e_2 + (C_0^2 + C_\sigma^2) s (e_1 + e_2) \\ & - [C_0^2 \Delta\Gamma^* + C_\sigma^2 (\Delta\Gamma + j\sigma)] e_1 - [C_0^2 (\Delta\Gamma + j\sigma) + C_\sigma^2 \Delta\Gamma^*] e_2 \\ & + (C_\sigma^2 - C_0^2)^2. \end{aligned} \tag{140}$$

APPENDIX B

We here carry out the calculation of $\overline{\mathbf{A}_1 \mathbf{A}_1}$, defined by (97). Corresponding to (100) we have, from (13),

$$\epsilon^2 \langle \mathbf{A}_1(z) \mathbf{A}_1(\zeta) \rangle = -\rho(z - \zeta) \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}, \tag{141}$$

where

$$\mathbf{F} = \begin{bmatrix} [C_0^2 e^{-i\Delta\beta(z-\zeta)} + C_\sigma^2 e^{i\Delta\beta(z-\zeta)}] & -2C_0 C_\sigma \cos \Delta\beta(z - \zeta) \\ -2C_0 C_\sigma \cos \Delta\beta(z - \zeta) & [C_0^2 e^{i\Delta\beta(z-\zeta)} + C_\sigma^2 e^{-i\Delta\beta(z-\zeta)}] \end{bmatrix}, \tag{142}$$

and

$$\mathbf{G} = \begin{bmatrix} (C_0^2 + C_\sigma^2) e^{i\Delta\beta(z-\zeta)} & -2C_0 C_\sigma e^{i\Delta\beta(z+\zeta)} \\ -2C_0 C_\sigma e^{-i\Delta\beta(z+\zeta)} & (C_0^2 + C_\sigma^2) e^{-i\Delta\beta(z-\zeta)} \end{bmatrix}. \tag{143}$$

Now,

$$\begin{aligned} & \int_0^Z \int_0^z e^{i\omega(z-\zeta)} \rho(z - \zeta) \, d\zeta \, dz \\ & \quad = \int_0^Z \int_0^z e^{i\omega\zeta} \rho(\zeta) \, d\zeta \, dz = \int_0^Z (Z - \zeta) e^{i\omega\zeta} \rho(\zeta) \, d\zeta. \end{aligned} \tag{144}$$

Hence, for real ω, from (102) and (103),

$$\lim_{Z \to \infty} \left[\frac{1}{Z} \int_0^Z \int_0^z e^{i\omega(z-\zeta)} \rho(z - \zeta) \, d\zeta \, dz \right] = S(\omega). \tag{145}$$

Also,

$$\begin{aligned} \int_0^Z \int_0^z e^{i\omega(z+\zeta)} \rho(z - \zeta) \, d\zeta \, dz & = \int_0^Z e^{2i\omega z} \int_0^z e^{-i\omega\zeta} \rho(\zeta) \, d\zeta \, dz \\ & = \frac{1}{2j\omega} \int_0^Z (e^{2i\omega Z} e^{-i\omega\zeta} - e^{i\omega\zeta}) \rho(\zeta) \, d\zeta. \end{aligned} \tag{146}$$

Hence, for real $\omega \neq 0$,

$$\lim_{Z\to\infty}\left[\frac{1}{Z}\int_0^Z\int_0^z e^{j\omega(z+\zeta)}\rho(z-\zeta)\,d\zeta\,dz\right] = 0. \tag{147}$$

Thus, from (97), (141) through (143), (145), and (147),

$$\epsilon^2\overline{\mathbf{A}_1\mathbf{A}_1} = \begin{bmatrix}\mathbf{X}(\Delta\beta) & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}(\Delta\beta)\end{bmatrix}, \qquad (\Delta\beta)\ell = 0(1), \tag{148}$$

where

$$\mathbf{X}(\Delta\beta) = \begin{bmatrix}-[C_0^2S(-\Delta\beta) + C_\sigma^2S(\Delta\beta)] & C_0C_\sigma[S(\Delta\beta) + S(-\Delta\beta)]\\ C_0C_\sigma[S(\Delta\beta) + S(-\Delta\beta)] & -[C_0^2S(\Delta\beta) + C_\sigma^2S(-\Delta\beta)]\end{bmatrix}, \tag{149}$$

and

$$\mathbf{Y}(\Delta\beta) = (C_0^2 + C_\sigma^2)\begin{bmatrix}S(\Delta\beta) & 0\\ 0 & S(-\Delta\beta)\end{bmatrix}. \tag{150}$$

Now consider $\mathbf{A}_1(z)$ as given by (114). Then, from (13),

$$\epsilon^2\langle\mathbf{A}_1(z)\mathbf{A}_1(\zeta)\rangle = -\rho(z-\zeta)\begin{bmatrix}\mathbf{H} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}\end{bmatrix}, \tag{151}$$

where

$$\mathbf{H} = \begin{bmatrix}(C_0^2 + C_\sigma^2) & -2C_0C_\sigma\\ -2C_0C_\sigma & (C_0^2 + C_\sigma^2)\end{bmatrix}. \tag{152}$$

Hence, from (97), (145), (151), and (152),

$$\epsilon^2\overline{\mathbf{A}_1\mathbf{A}_1} = -S(0)\begin{bmatrix}\mathbf{H} & \mathbf{0}\\ \mathbf{0} & \mathbf{H}\end{bmatrix}, \qquad (\Delta\beta)\ell = 0(\epsilon^2). \tag{153}$$

Note, from (148) and (150), the nonuniform behavior of $\epsilon^2\overline{\mathbf{A}_1\mathbf{A}_1}$ in the neighborhood of resonance, which arises from the discontinuity of the limit in (147) at $\omega = 0$.

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