Pull-In Range of a Phase-Locked Loop With a Binary Phase Comparator

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We develop a method for calculating the pull-in range of a phase-locked loop with a binary phase comparator and an arbitrary loop filter. Complete numerical results are presented for loop filters of the phase-lag and low-pass types. The problem of stability is also considered, and it is proved that with these loop filters no steady-state phase jitter can exist after frequency acquistion has been achieved.

I. INTRODUCTION

The phase-locked loop (PLL) is an important element of many modern communication and control systems. A PLL block diagram is shown in Fig. 1. The input $v_1(\omega_0 t + \theta_1)$ is a narrow-band signal with carrier frequency ω_0 and phase $\theta_1(t)$. This phase is compared with the phase $\theta_2(t)$ of the voltage-controlled oscillator (VCO) in the phase comparator (PC). The PC output $f(\phi)$, where $\phi = \theta_1 - \theta_2$, is filtered by the loop filter $H(p)$ and applied to the VCO control terminal.

Depending on the values of the PLL parameters, the phase error ϕ can be kept small even with input phase modulation. Thus with $\theta_1(t) = \Omega t + \theta_{10}$, which represents a constant input frequency offset, the system can produce a synchronized signal $v_2(t)$ with frequency $\omega_0 + \Omega$. This synchronization capability leads to PLL applications in carrier extraction,[1] frequency synthesis,[2] narrow-band filtering,[3] FM demodulation,[3] timing extraction in PCM and data transmission systems,[4] etc.

In this paper, we examine the acquisition, or pull-in, range of a PLL with a binary PC. We present numerical results for the special case of a second-order PLL with either low-pass or phase-lag loop filter.

The PC characteristic considered here is the binary curve shown in Fig. 2. It is of interest in at least three situations. First, since many synchronization systems are designed to operate with very small

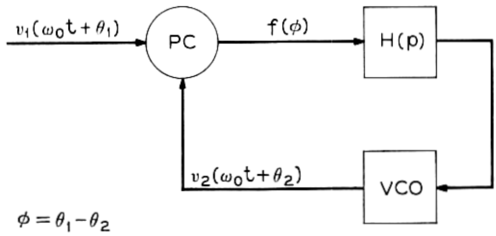

Fig. 1—PLL block diagram.

phase errors, dynamic range limitations in the PC circuitry often produce severe saturation effects. The characteristic of Fig. 2 corresponds to the extreme case of vanishing linear range near zero phase error. However, it is a useful approximation for systems with small but nonzero dynamic range for the purpose of studying pull-in performance. Second, a binary PC can be easily implemented with logic circuits. The resulting characteristic differs from the ideal of Fig. 2 by exhibiting small hysteresis about the zeros at $\phi = n\pi$, but this hysteresis has no effect on the pull-in range achieved. Finally, J. J. Stiffler[5] has shown that for a first-order PLL with additive white gaussian noise and no frequency offset, the cross-correlation type PC which minimizes $\Pr \{ |\phi| > \phi_0 \}$ for all ϕ_0 has the characteristic of Fig. 2. Although such a square-wave correlation function is unrealizable, this result suggests that PLLs employing other types of PC having this characteristic are worthy of consideration. In addition, similar "bang-bang" control characteristics are known to be optimum for PLL acquisition.[6]

II. CALCULATION OF PULL-IN RANGE

The phase model corresponding to the PLL block diagram in Fig. 1 is shown in Fig. 3. We assume that the gain of the loop filter $H(p)$ is unity at DC. The input-signal frequency differs from the VCO center frequency by Ω rad/s:

$$\theta_1(t) = \Omega t + \theta_{10}. \tag{1}$$

It is convenient to normalize the detuning Ω to the dc loop gain* α:

$$\gamma = \Omega/\alpha. \tag{2}$$

* Since no gain can be defined for the binary PC being considered, the symbol α does not represent the usual small-signal loop gain.

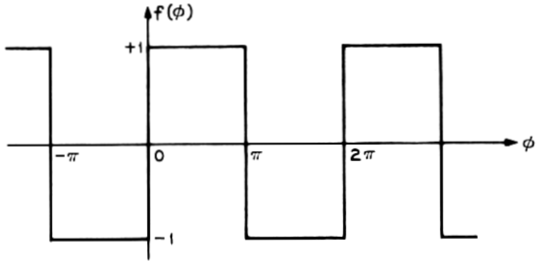

Fig. 2—Phase comparator characteristic.

When the normalized detuning γ is not too great, the VCO frequency changes toward the input frequency, and eventually the PLL synchronizes to the input signal with zero frequency error and finite phase error. The normalized lock range γ_L is the maximum detuning $|\gamma|$ for which the PLL can remain locked after synchronization has been established. Inspection of dc conditions in the phase model of Fig. 3 shows that $\gamma_L = 1$. The normalized pull-in range γ_P is the maximum $|\gamma|$ for which eventual synchronization is assured from any initial conditions in the loop filter and VCO. In general $\gamma_P < 1$ for PLLs of order higher than first. The order of a PLL is defined as one plus the number of poles in $H(p)$. Calculation of γ_P is the subject of this paper.

The method employed here is similar to that used by A. J. Goldstein[7] to calculate γ_P for a PLL with a sawtooth PC. Due to the binary nature of $f(\phi)$, the PC output waveform $f[\phi(t)]$ is piecewise constant, assuming only the values ± 1. Assume that the PLL is not synchronized to the input signal. Then $\phi(t)$ increases with time (for $\Omega > 0$), and the waveforms $\phi(t)$ and $f[\phi(t)]$ appear as shown in Fig. 4. The time origin has been selected so that $\phi(0) = 0$. The transition instants are

$$0 = t_{02} < t_{11} < t_{12} < \cdots < t_{j1} < t_{j2} < \cdots \tag{3}$$

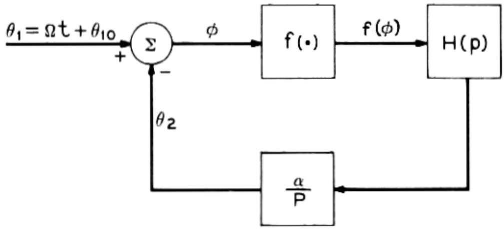

Fig. 3—Phase model of PLL.

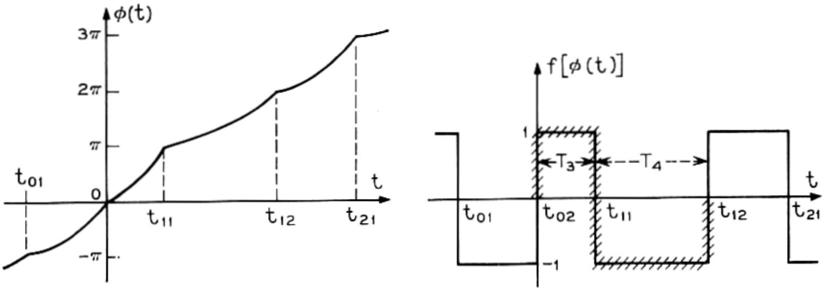

Fig. 4—Phase and PC output waveforms.

where

$$\phi(t_{j1}) = (2j - 1)\pi \text{ (negative transition)},$$
$$\phi(t_{j2}) = 2j\pi \text{(positive transition)}. \tag{4}$$

An expression can be written for $f[\phi(t)]$ by summing all of the segments corresponding to 2π increments in $\phi(t)$:

$$f[\phi(t)] = \sum_{j=-\infty}^{\infty} [u(t - t_{j-1,2}) - 2u(t - t_{j1}) + u(t - t_{j2})]. \tag{5}$$

The $j = 1$ segment is shown crosshatched in Fig. 4. Since the PLL is not synchronized to the input signal, the steady-state PC waveform $f_{ss}[\phi(t)]$ is periodic, and the transition instants can be written:

$$t_{j2} = j(T_3 + T_4),$$
$$t_{j1} = j(T_3 + T_4) - T_4 . \tag{6}$$

T_3 and T_4 are the times between transitions as indicated in Fig. 4. Using equation (6), equation (5) becomes:

$$f_{ss}[\phi(t)] = \sum_{j=-\infty}^{\infty} [u(t - [j - 1][T_3 + T_4]) - 2u(t - j[T_3 + T_4] + T_4) + u(t - j[T_3 + T_4])]. \tag{7}$$

From equation (1) and Fig. 3, the phase error in steady state is:

$$\phi_{ss}(t) = \Omega t + \phi_0 - f_{ss}[\phi(t)] * \mathcal{L}^{-1}\left\{\frac{\alpha H(p)}{p}\right\}, \tag{8}$$

where ϕ_0 is some constant. Since $f_{ss}[\phi(t)]$ is composed only of step functions, the last term in equation (8) can be written as a sum of delayed functions $g(t)$, where

$$\mathcal{L}\{g(t)\} = G(p) = \frac{H(p)}{p^2}. \tag{9}$$

The general expression for $\phi_{ss}(t)$ is then

$$\phi_{ss}(t) = \Omega t + \phi_0 - \alpha \sum_{j=-\infty}^{\infty} [g(t - [j - 1][T_3 + T_4]) - 2g(t - j[T_3 + T_4] + T_4) + g(t - j[T_3 + T_4])]. \tag{10}$$

From equation (4) and Fig. 4, we have the following conditions on $\phi_{ss}(t)$:

$$\begin{aligned} \phi_{ss}(0) &= 0, \\ \phi_{ss}(T_3) &= \pi, \\ \phi_{ss}(T_3 + T_4) &= 2\pi. \end{aligned} \tag{11}$$

These conditions are sufficient to determine the unknown constants ϕ_0, T_3, and T_4 in equation (10). Thus equations (10) and (11) together define the relationship between the normalized detuning γ and the loop-filter parameters [through $g(t)$] which must be satisfied for the PLL to be out-of-lock in the steady state. Then clearly γ_P is the minimum value of γ for which these equations possess a solution.

III. RESULTS FOR SECOND-ORDER PLL

In this section, the method derived above is applied to the second-order PLL with loop filter

$$H(p) = \frac{1 + T_2 p}{1 + T_1 p}. \tag{12}$$

This is a phase-lag filter for $0 < T_2 < T_1$. It becomes a simple low-pass filter for $T_2 = 0$, and setting $T_2 = T_1$ reduces the PLL to first order. The corresponding $g(t)$, from equation (9), is

$$g(t) = [t - (T_1 - T_2)(1 - \exp(-t/T_1))]u(t). \tag{13}$$

In Appendix A, equation (10) is rewritten using equation (13) and is evaluated at $t = 0$, T_3, and $T_3 + T_4$. Equation (11) is then applied, along with the normalization

$$\tau_i = \alpha T_i, \qquad i = 1, 2, 3, 4. \tag{14}$$

The equations which result are

$$\gamma = \frac{2\pi + \tau_3 - \tau_4}{\tau_3 + \tau_4}, \tag{15}$$

$$4(\tau_1 - \tau_2)(\tau_3 + \tau_4) = [\tau_4(\tau_3 + \pi) + \tau_3(\tau_4 - \pi)]\left[\coth \frac{\tau_3}{2\tau_1} + \coth \frac{\tau_4}{2\tau_1}\right]. \tag{16}$$

According to the discussion following equation (11), the pull-in range is

$$\gamma_p = \min_{\tau_3, \tau_4 > 0} \frac{2\pi + \tau_3 - \tau_4}{\tau_3 + \tau_4}, \tag{17}$$

subject to the constraint equation (16). It is important to note that equation (15) is simply the dc balance equation for the PLL model of Fig. 3, and therefore holds for any $H(p)$ with unity dc gain. Since equation (15) gives γ explicitly in terms of τ_3 and τ_4, it can always be used to eliminate γ from a constraint equation corresponding to equation (16). Therefore γ_p can be calculated from equation (17) for any $H(p)$ subject to the appropriate constraint equation which relates the loop-filter parameters to the transition-time parameters τ_3 and τ_4. Hence only $\phi_{ss}(0)$ and $\phi_{ss}(T_3)$ actually had to be evaluated in Appendix A.

The method employed to evaluate γ_p for various values of τ_1 and τ_2 was to choose $\tau_3 > 0$, use equation (16) to obtain the corresponding τ_4, and calculate γ from equation (15). The τ_3, τ_4 relationship was found to be single-valued for all loop filters investigated. Examples of the behavior of γ with τ_3 are given in Fig. 5. The filter parameter r is defined as

$$r = T_2/T_1. \tag{18}$$

In all cases the curves $\gamma(\tau_3)$ are smooth and exhibit a single local minimum. This minimum is found by computing a sequence $\gamma(\tau_{3n})$, where $\tau_{3n} > \tau_{3,n-1}$. When this sequence begins to increase, the minimum γ_p has just been passed, and can be estimated accurately from the last three computed values of γ.

Curves of γ_p versus τ_1 with r as a parameter are presented in Fig. 6. Several characteristics are notable. First, $\gamma_p = 1$ for $r \geqq 0.5$ independent of τ_1. Second, as τ_1 increases with r constant, γ_p approaches an asymptotic value γ_{pa} which is a function only of r. Later we shall derive an explicit formula for γ_{pa}. The same results are presented in a different way in Fig. 7, which shows curves of constant γ_p on the τ_1, τ_2

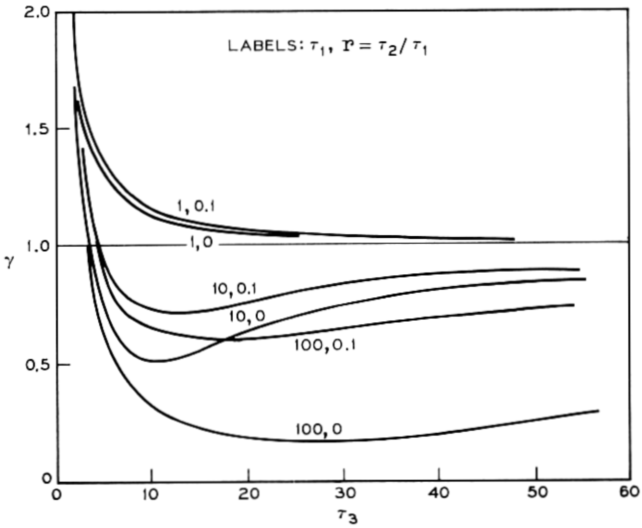

Fig. 5—γ versus τ_3.

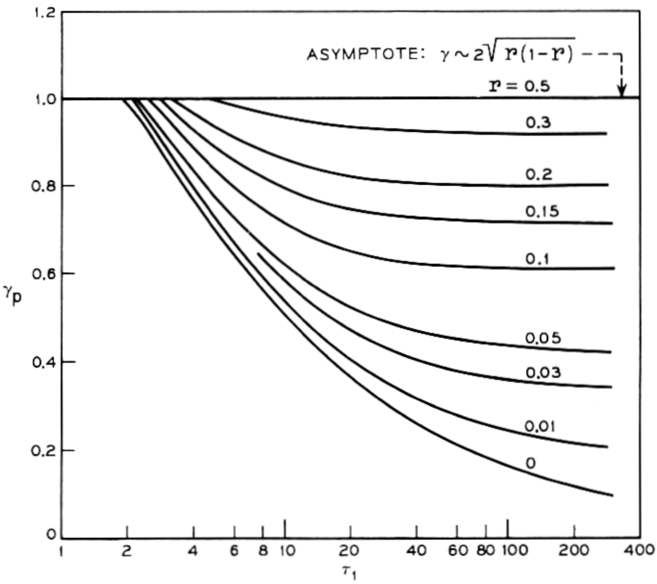

Fig. 6—γ_p versus τ_1 with parameter r.

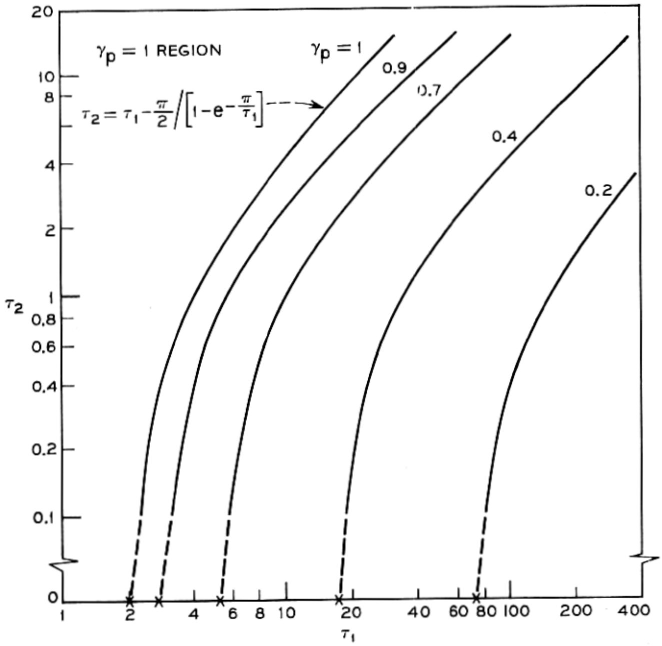

Fig. 7—τ_2 versus τ_1 with parameter γ_p.

parameter plane. Below we derive the equation for the curve in this figure corresponding to $\gamma_p = 1$.

IV. FURTHER RESULTS

Let us consider the important case of very large τ_1, which corresponds to strong filtering in the PLL. Noting that

$$\lim_{x \to 0} \coth x = \frac{1}{x} \tag{19}$$

and using equation (18), equation (16) becomes for large τ_1:

$$4\tau_1(1 - r)(\tau_3 + \tau_4) = [\tau_4(\tau_3 + \pi) + \tau_3(\tau_4 - \pi)]\left[\frac{2\tau_1}{\tau_3} + \frac{2\tau_1}{\tau_4}\right]. \tag{20}$$

This simplifies to

$$\tau_4 = \frac{\pi\tau_3}{2r\tau_3 + \pi}. \tag{21}$$

Substituting equation (21) into equation (15) gives the result for γ

with τ_1 large:

$$\gamma = \frac{r\tau_3^2 + 2\pi r\tau_3 + \pi^2}{r\tau_3^2 + \pi\tau_3}. \tag{22}$$

All minima of γ must satisfy

$$\frac{d\gamma}{d\tau_3} = 0. \tag{23}$$

Applying equation (23) to equation (22) yields a single positive value of τ_3:

$$\tau_3 = \frac{\pi\left[1 + \left(\frac{1}{r} - 1\right)^{\frac{1}{2}}\right]}{1 - 2r}, \qquad r < 0.5. \tag{24}$$

Substituting equation (24) into equation (22) gives the result

$$\gamma_{pa} = \begin{cases} 2[r(1 - r)]^{\frac{1}{2}}, & 0 \leqq r < 0.5, \\ 1, & r \geqq 0.5. \end{cases} \tag{25}$$

This agrees with the numerical results in Fig. 6 and with a result obtained by M. V. Kapranov by a different method in an untranslated Russian paper.[8] The existence of only a single minimum of $\gamma(\tau_3)$ for large τ_1 supports our hypothesis of a single minimum for all τ_1 which is based on the curves in Fig. 5.

The region $\gamma_p = 1$ in Fig. 7 corresponds to τ_1, τ_2 such that $\gamma \geqq 1$ for all τ_3, τ_4. From equation (15), this implies that

$$\tau_4 \leqq \pi. \tag{26}$$

Thus from equation (16), τ_1, τ_2, and τ_3 on the $\gamma_p = 1$ boundary must satisfy

$$4(\tau_1 - \tau_2)(\tau_3 + \pi) = \pi(\tau_3 + \pi)\left[\coth\frac{\tau_3}{2\tau_1} + \coth\frac{\pi}{2\tau_1}\right]. \tag{27}$$

Examination of Fig. 5 shows that the critical γ curves approach $\gamma = 1$ from above for very large τ_3. Using

$$\lim_{x\to\infty} \coth x = 1, \tag{28}$$

equation (27) becomes

$$4(\tau_1 - \tau_2)\tau_3 = \pi\tau_3\left[1 + \coth\frac{\pi}{2\tau_1}\right]. \tag{29}$$

This reduces to the simple expression for the $\gamma_p = 1$ curve:

$$\tau_2 = \tau_1 - \frac{\pi}{2} \Big/ [1 - \exp(-\pi/\tau_1)]. \tag{30}$$

Finally, let us consider the following problem. The periodic behavior of $\phi_{ss}(t)$ assumed throughout this paper which led to equations (15) and (16) is known as a limit cycle of the second kind in the phase plane.[9] The nonexistence of such limit cycles for $|\gamma| < \gamma_p$ proves that frequency lock is eventually attained for these values of γ. Physically speaking, $\phi_{ss}(t)$ cannot increase monotonically with time as assumed in Fig. 4. However, proper synchronization of the PLL requires that phase lock also be achieved. This means that after a long enough time, the system comes to rest:

$$\lim_{t \to \infty} \phi(t) = 2n\pi, \qquad |\gamma| < \gamma_p . \tag{31}$$

Because of the gross nonlinearity $f(\phi)$ in the system considered here, it is not obvious that equation (31) will be satisfied. Specifically, it is conceivable that a series of self-sustaining overshoots in $\phi(t)$ could become established after pull-in which would produce a periodic phase jitter. This behavior corresponds to a limit cycle of the first kind in the phase plane. Although this problem is not solved in general here, a test which is valid for any $H(p)$ is applied to the phase-lag filter case in Appendix B. It is found that in this case, phase lock described by equation (31) is always achieved.

V. CONCLUSION

A method has been presented for calculating the pull-in range γ_p of a PLL with a binary phase comparator and an arbitrary loop filter. The result is obtained as the minimum value of a function of two variables, subject to a constraint equation which relates these variables to the parameters of the loop filter. Complete numerical results for γ_p were obtained for loop filters of the phase-lag and low-pass types. Explicit formulas were given in this case for the asymptotic value of γ_p with strong loop filtering, and for the set of filter parameters which result in unity pull-in range. Finally, it was proved that no steady-state phase jitter can exist after pull-in with these loop filters.

VI. ACKNOWLEDGMENT

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APPENDIX A

Evaluation of $\phi_{ss}(t)$

Consider one period of $\phi_{ss}(t)$, and define the functions $\phi_i(t)$ as:

$$\phi_{ss}(t) = \begin{cases} \phi_1(t), & 0 \leqq t \leqq T_3 , \\ \phi_2(t), & T_3 \leqq t \leqq T_3 + T_4 . \end{cases} \tag{32}$$

From Fig. 4, $\phi_1(t)$ includes all terms in equation (10) which correspond to transitions prior to $t_{11} = T_3$. Using equation (13) in (10), $\phi_1(t)$ becomes:

$$\begin{aligned}\phi_1(t) = \Omega t + \phi_0 - \alpha \sum_{j=-\infty}^{1} [t - (j-1)(T_3 + T_4) - (T_1 - T_2)) \\ \cdot(1 - \exp[-(t - [j-1][T_3 + T_4])/T_1])] \\ + 2\alpha \sum_{j=-\infty}^{0} [t - j(T_3 + T_4) + T_4 - (T_1 - T_2)) \\ \cdot(1 - \exp[-(t - j[T_3 + T_4] + T_4)/T_1])] \\ - \alpha \sum_{j=-\infty}^{0} [t - j(T_3 + T_4) - (T_1 - T_2)) \\ \cdot(1 - \exp[-(t - j[T_3 + T_4])/T_1])].\end{aligned} \tag{33}$$

Absorbing all constant terms into ϕ_0, equation (33) becomes:

$$\begin{aligned}\phi_1(t) &= \Omega t + \phi_0' - \alpha t \\ &\quad - \alpha(T_1 - T_2)\exp(-t/T_1) \sum_{j=-\infty}^{0} [\exp[j(T_3 + T_4)/T_1] \\ &\quad - 2\exp[(j[T_3 + T_4] - T_4)/T_1] + \exp[j(T_3 + T_4)/T_1]], \\ &= \Omega t + \phi_0' - \alpha t - 2\alpha(T_1 - T_2) \\ &\quad \cdot \exp(-t/T_1) \frac{1 - \exp(-T_4/T_1)}{1 - \exp[-(T_3 + T_4)/T_1]}.\end{aligned} \tag{34}$$

ϕ_0' is eliminated from equation (34) using

$$\phi_1(0) = 0, \tag{11a}$$

which yields

$$\begin{aligned}\phi_1(t) &= (\Omega - \alpha)t + 2\alpha(T_1 - T_2) \\ &\quad \cdot \frac{1 - \exp(-T_4/T_1)}{1 - \exp[-(T_3 + T_4)/T_1]} (1 - \exp(-t/T_1)).\end{aligned} \tag{35}$$

Now requiring that

$$\phi_1(T_3) = \pi \tag{11b}$$

results in

$$\pi = (\Omega - \alpha)T_3 + 2\alpha(T_1 - T_2) \cdot \frac{(1 - \exp(-T_3/T_1))(1 - \exp(-T_4/T_1))}{1 - \exp[-(T_3 + T_4)/T_1]} \tag{36}$$

which after some manipulation becomes

$$\pi = (\Omega - \alpha)T_3 + 4\alpha(T_1 - T_2) \Big/ \left[\coth\frac{T_3}{2T_1} + \coth\frac{T_4}{2T_1}\right]. \tag{37}$$

Next, $\phi_2(t)$ is obtained from equation (35) by adding the term in equation (10) which corresponds to the transition time $t_{11} = T_3$:

$$\phi_2(t) = (\Omega - \alpha)t + 2\alpha(T_1 - T_2) \cdot \frac{1 - \exp(-T_4/T_1)}{1 - \exp[-(T_3 + T_4)/T_1]}(1 - \exp(-t/T_1)) + 2\alpha(t - T_3) - 2\alpha(T_1 - T_2)(1 - \exp[-(t - T_3)/T_1]). \tag{38}$$

Requiring that

$$\phi_2(T_3 + T_4) = 2\pi \tag{11c}$$

and performing some simple manipulation leads to the result

$$2\pi = \Omega(T_3 + T_4) - \alpha(T_3 - T_4). \tag{39}$$

Equations (37) and (39) can be normalized by letting

$$\gamma = \Omega/\alpha, \tag{2}$$

$$\tau_i = \alpha T_i, \qquad i = 1, 2, 3, 4. \tag{14}$$

Equation (39) becomes

$$\gamma = \frac{2\pi + \tau_3 - \tau_4}{\tau_3 + \tau_4} \tag{15}$$

which is equation (15) in Section III. Equation (37) becomes

$$\gamma = 1 + \frac{1}{\tau_3}\left\{\pi - 4(\tau_1 - \tau_2) \Big/ \left[\coth\frac{\tau_3}{2\tau_1} + \coth\frac{\tau_4}{2\tau_1}\right]\right\}. \tag{40}$$

Eliminating γ between equations (15) and (40) gives the constraint equation (16).

APPENDIX B

The Phase Jitter Problem

In this appendix it is proved that whenever the second-order PLL studied here achieves frequency lock, it also achieves phase lock:

$$\lim_{t\to\infty} \phi(t) = 2n\pi, \qquad |\gamma| < \gamma_p . \tag{31}$$

It can be shown that for this system any steady-state phase jitter $\phi(t)$ is confined within the $\pm\pi$ neighborhood of some lock point $\phi = 2n\pi$. This is a result of the periodicity of the phase-plane geometry in the ϕ direction. However, since this proof requires a rather lengthy description of the properties of the phase-plane trajectories, it will be omitted. Below we prove that equation (31) holds when the phase error remains within such a $\pm\pi$ neighborhood of a lock point.

The technique used previously to calculate γ_p can also be employed here. Assume that in the steady state, a periodic phase jitter $\phi(t)$ exists. Then the waveform $f[\phi(t)]$ is binary and periodic, and the phase error is again given by equation (8). Since we are now considering a phase jitter within $\pm\pi$ of $\phi = 2n\pi$, the requirements on $\phi_{ss}(t)$ are:

$$\phi_{ss}(0) = \phi_{ss}(T_3) = \phi_{ss}(T_3 + T_4) = 2n\pi. \tag{41}$$

Proceeding as in Appendix A, we obtain the equations

$$\gamma = \frac{\tau_3 - \tau_4}{\tau_3 + \tau_4}, \tag{42}$$

$$2(\tau_1 - \tau_2)(\tau_3 + \tau_4) = \tau_3\tau_4\left[\coth\frac{\tau_3}{2\tau_1} + \coth\frac{\tau_4}{2\tau_1}\right]. \tag{43}$$

These equations may be written directly from equations (15) and (16) by replacing π with 0. Below it is demonstrated that $\tau_3 = \tau_4 = 0$ is the only solution of these equations when $|\gamma| < 1$, which proves equation (31).

Using $|\gamma| < 1$ in equation (42) yields

$$\tau_3 , \tau_4 > 0, \tag{44}$$

so that only equation (43) must be considered. Using r of equation (18) and dividing by $\tau_3\tau_4$ gives

$$2\tau_1(1 - r)\left(\frac{1}{\tau_3} + \frac{1}{\tau_4}\right) = \coth\frac{\tau_3}{2\tau_1} + \coth\frac{\tau_4}{2\tau_1}. \tag{45}$$

Defining

$$x_j = \frac{\tau_j}{2\tau_1}, \qquad j = 3, 4, \tag{46}$$

equation (45) becomes

$$(1 - r)\left(\frac{1}{x_3} + \frac{1}{x_4}\right) = \coth x_3 + \coth x_4 . \tag{47}$$

Recalling that $r \geqq 0$, we have for $x > 0$:

$$\coth x > \frac{1}{x} \geqq \frac{1 - r}{x}. \tag{48}$$

Thus the only possible nonnegative solution of equation (47) is

$$x_3 = x_4 = 0 \tag{49}$$

which is the desired result.

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