Eventual Stability for Lipschitz Functional Differential Systems

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In this paper it is established that for Lipschitz functional differential systems, the eventual uniform asymptotic stability of the origin is preserved under absolutely diminishing perturbations.

I. INTRODUCTION AND NOTATION

In two recent papers, A. Strauss and J. A. Yorke have investigated "eventual" stability properties for systems of ordinary differential equations.[1,2] In particular, they have shown that for Lipschitz systems, diminishing perturbations preserve eventual uniform asymptotic stability.[1] It is the purpose of this paper to extend a somewhat weaker form of this result to functional differential systems. Namely, it will be shown that for Lipschitz functional differential systems, the eventual uniform asymptotic stability of the origin is preserved under absolutely diminishing perturbations.

The following notation will be used in this paper: E^n is the space of n-vectors, and for x in E^n, $|x|$ denotes any vector norm. For a given number $\tau > 0$, C denotes the linear space of continuous functions mapping the interval $[-\tau, 0]$ into E^n, and for ϕ in C, $\|\phi\| = \sup |\phi(\theta)|$, $-\tau \leqq \theta \leqq 0$. For $H > 0$, C_H denotes the set of ϕ in C for which $\|\phi\| < H$. For any continuous function $x(u)$ whose domain is $-\tau \leqq u \leqq a$, $a \geqq 0$, and whose range is in E^n, and any fixed t, $0 \leqq t \leqq a$, the symbol x_t will denote the function $x_t(\theta) = x(t + \theta)$, $-\tau \leqq \theta \leqq 0$; that is, x_t is in C, and is that segment of the function $x(u)$ defined by letting u range in the interval $t - \tau \leqq u \leqq t$.

Let $F(t, \phi)$ be a function defined on $D_H = [0, \infty) \times C_H$ into E^n, and let $\dot{x}(t)$ denote the right hand derivative of $x(u)$ at $u = t$. Consider the functional differential system

$$\dot{x}(t) = F(t, x_t). \tag{1}$$

Let (s, ϕ) be in D_H. A function $x(s, \phi)(t)$ is said to be a solution of equation (1) with initial function ϕ at $t = s$ if there exists a number $b > 0$ such that

(*i*) for $t \in [s, s + b)$, $x_t(s, \phi)$ is defined and in C_H;
(*ii*) $x_s(s, \phi) = \phi$; and
(*iii*) $x(s, \phi)(t)$ satisfies equation (1) for $s \leqq t < s + b$.

$x(s, \phi)(t)$ is unique if every other solution with the same initial function ϕ at $t = s$ agrees with $x(s, \phi)(t)$ in their common domain of definition.

If F is continuous on D_H, then for every (s, ϕ) in D_H there is at least one solution of equation (1) with initial function ϕ at $t = s$.[3] If, moreover, F is Lipschitzian in ϕ, that is, there is a constant L such that for every ϕ_1, ϕ_2 in C_H

$$| F(t, \phi_1) - F(t, \phi_2) | \leqq L \, ||\phi_1 - \phi_2|| \tag{2}$$

for $t \geqq 0$, then there is only one such solution. Generally, under such assumptions, one can only expect solutions to exist over a finite interval.

II. PRELIMINARIES

We now define the stability concepts to be used herein. These definitions are stated for equation (1) in which it is assumed that for some H, $0 < H \leqq \infty$, F is continuous and Lipschitzian on D_H.

Definition 1: The origin is *eventually uniformly stable* (EvUS) if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ and $\alpha = \alpha(\epsilon) \geqq 0$ such that $|| x_t(s, \phi) || < \epsilon$ for all $|| \phi || < \delta$ and $t \geqq s \geqq \alpha$. It is *uniformly stable* (US) if one can choose $\alpha(\epsilon) \equiv 0$.

Definition 2: The origin is *eventually uniformly attracting* (EvUA) if there exists constants $\eta > 0$ and $\beta \geqq 0$, and if for every $\epsilon > 0$ there exists a $T = T(\epsilon) > 0$ such that $|| x_t(s, \phi) || < \epsilon$ for $|| \phi || < \eta$, $s \geqq \beta$, and $t \geqq s + T$. It is *uniformly attracting* (UA) if one can choose $\beta = 0$.

Definition 3: The origin is *eventually uniform-asymptotically stable* (EvUAS) if it is both EvUS and EvUA. It is *uniform-asymptotically stable* (UAS) if it is both US and UA.

The above definitions show that EvUS, EvUA, and EvUAS are weaker stability concepts than their respective Lyapunov counterparts: US, UA, and UAS. Also, it should be noted that in these definitions we do not require that the zero function be a solution of equation (1). When the origin is US, this implies that the zero function is a unique solution of equation (1) for any $s \geqq 0$. Thus, we see that EvUS (EvUAS) is a natural generalization of US (UAS) in which it is not assumed that the zero function is a solution. Finally, it is important to note that UA does not imply that the zero function is a solution (Ref. 1, example 2.8).

Definition 4: Let $V(t, \phi)$ be a function defined for (t, ϕ) in D_H. The derivative of V along solutions of equation (1) will be denoted by $\dot{V}_{(1)}[t, x_t(s, \phi)]$ and is defined to be

$$\dot{V}_{(1)}[t, x_t(s, \phi)] = \limsup_{h \to 0+} \frac{1}{h} \{V[t + h, x_{t+h}(s, \phi)] - V[t, x_t(s, \phi)]\}.$$

If F is continuous and Lipschitzian, and if the origin is EvUAS, then the existence of a Lyapunov type comparison function can be established. By following D. Wexler[4] and A. Halanay[5] one can prove the following theorem.

Theorem 1: Let F be continuous and Lipschitzian on D_H, and let the origin be EvUAS. Then there exists a number K, $0 < K < H$, and a function $V(t, \phi)$ with the properties: (i) there exists functions $a(r)$, $b(r)$ continuous, positive, and monotone increasing for $r > 0$, with $a(0) = b(0) = 0$, such that for m in $(0, K]$

$$a(\|\phi\|) \leqq V(t, \phi) \leqq b(\|\phi\|)$$

for $m \leqq \|\phi\| \leqq K$, $t \geqq d(m)$, where $d(r)$ is a continuous, nonnegative, and nonincreasing function for $r > 0$; (ii) there exists a function $c(r)$ continuous, positive, and monotone-increasing for $r > 0$, with $c(0) = 0$ such that

$$\dot{V}_{(1)}[t, x_t(s, \phi)] \leqq -c[\|x_t(s, \phi)\|]$$

for $\|\phi\| \leqq K$, $t \geqq s \geqq d(K)$; and (iii) for $0 < r \leqq \|\phi_i\| \leqq K$, $t \geqq d(K)$

$$|V(t, \phi_1) - V(t, \phi_2)| \leqq M(r)\|\phi_1 - \phi_2\|,$$

where $M(r)$ is continuous and monotone-decreasing on $(0, K]$.

III. PERTURBED EQUATION

We now prove a theorem which shows that the EvUAS of the origin of the nominal equation

$$\dot{y}(t) = F(t, y_t) \tag{N}$$

is preserved for the perturbed equation

$$\dot{x}(t) = F(t, x_t) + G(t, x_t) \tag{P}$$

when F and G satisfy certain conditions. In particular, $G(t, \phi)$ is required to be *absolutely diminishing*, that is, for every m in $(0, H)$, there exists a $\gamma_m \geqq 0$ and a function $g_m(t)$ continuous on $[\gamma_m, \infty)$ such that for $m \leqq \|\phi\| < H$, $t \geqq \gamma_m$

$$|G(t, \phi)| \leqq g_m(t) \quad \text{and} \quad I_m(t) \triangleq \int_t^{t+1} g_m(s)\, ds \to 0 \quad \text{as} \quad t \to \infty.$$

Theorem 2: Suppose that F and G are continuous and Lipschitzian on D_H, that G is absolutely diminishing, and that the origin is EvUAS for equation (N). Then the origin is EvUAS also for equation (P).

Proof: Define $J_m(t) = \sup\,[I_m(s): t - 1 \leqq s < \infty]$ for $t \geqq 1$. Since $I_m(t) \to 0$ as $t \to \infty$, this implies $J_m(t) \to 0$ monotonically as $t \to \infty$.

Let $0 < \epsilon \leqq K$, choose $\|\phi\| < \delta(\epsilon) = b^{-1}[a(\epsilon)/2]$, and pick $\theta = \theta(\epsilon) \geqq 0$ and such that

$$2LM(\delta)J_\delta(t) < \min\,[a(\epsilon), c(\delta)] \tag{3}$$

for $t \geqq \theta$, where L is the Lipschitz constant associated with F. Then for $t \geqq s \geqq \alpha(\epsilon) = \max\,[1, \theta(\epsilon), d(\delta)]$, $\|x_t(s, \phi)\| < \epsilon$. Suppose not, that is, for some $t \geqq s$, $\|x_t(s, \phi)\| = \epsilon$. Let q be the first t-value greater than s for which $\|x_q(s, \phi)\| = \epsilon$, and let p be the last t-value less than q for which $\|x_p(s, \phi)\| = \delta$. Then

$$\delta \leqq \|x_t(s, \phi)\| \leqq \epsilon, \qquad p \leqq t \leqq q. \tag{4}$$

For t in an interval on which $x(s, \phi)(t)$ exists, we evaluate

$$\begin{aligned}\dot{V}_{(P)}[t, x_t(s, \phi)] \leqq\ & \dot{V}_{(N)}[t, x_t(s, \phi)] \\ & + \limsup_{h \to 0^+} \frac{1}{h}\,(V\{t + h, x_{t+h}[t, x_t(s, \phi)]\} \\ & \qquad - V\{t + h, y_{t+h}[t, x_t(s, \phi)]\}) \\ \leqq\ & -c[\|x_t(s, \phi)\|] \\ & + \limsup_{h \to 0^+} \frac{M}{h}\,\{\|x_{t+h}[t, x_t(s, \phi)] \\ & \qquad - y_{t+h}[t, x_t(s, \phi)]\|\}\end{aligned}$$

where the function V is as described in Theorem 1. By assuming—with no loss of generality—that $L > 1$, we obtain[5] from the above inequality

$$\dot{V}_{(P)}[t, x_t(s, \phi)] \leqq -c[\|x_t(s, \phi)\|] + LM\,|G[t, x_t(s, \phi)]|.$$

Employing the absolute diminishing character of G and equation (4), we obtain by integrating the above from p to q

$$a(\epsilon) \leqq b(\delta) - (q - p)c(\delta) + LM \int_p^q g_\delta(t)\, dt. \tag{5}$$

Using the easily shown fact that

$$\int_u^t g_m(s)\, ds \leqq \int_{u-1}^t I_m(s)\, ds, \qquad t \geqq u \geqq 1,$$

and equations (3) and (5), we see that

$$a(\epsilon) \leqq b(\delta) - (q - p)c(\delta) + LM(q - p + 1)J_\delta(p)$$
$$< b(\delta) + a(\epsilon)/2 = a(\epsilon).$$

Hence, we arrive at a contradiction which shows that the origin is EvUS.

Let $\eta = \delta(K)$, $\beta = \alpha(K)$, and

$$T(\epsilon) = \alpha(\epsilon) + 2[LMJ_\delta(1) + b(K)]/c(\delta). \tag{6}$$

Consider $s \geqq \beta$ and $\|\phi\| < \eta$. Thus, $x(s, \phi)(t)$ exists for all $t \geqq s$. Moreover, since the origin is EvUS, to prove EvUA it is sufficient to show the existence of a u, $s + \alpha \leqq u \leqq s + T$, such that $\| x_u(s, \phi) \| < \delta(\epsilon)$. Assume the contrary, that is,

$$\delta \leqq \| x_t(s, \phi) \| \leqq K, \; s + \alpha \leqq t \leqq s + T.$$

Employing the same procedure as above, we arrive at the estimate

$$a(\delta) < b(K) - (T - \alpha)c(\delta) + ML(T - \alpha + 1)J_\delta(s + \alpha).$$

Using the monotonicity of J_δ and equations (3) and (6), we compute

$$a(\delta) < b(K) - \frac{c(\delta)}{2}(T - \alpha) + MLJ_\delta(1) = 0.$$

This contradiction then completes the proof of this theorem.

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