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Adaptive Delta Modulation with a One-Bit Memory

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We propose a delta modulator which, at every sampling instant r, adapts its step-size (for a staircase approximation to the input signal) on the basis of a comparison between the two latest channel symbols, C_r *and* C_{r-1} . *Specifically, the ratio of the modified step-size* m_r *to the previous step size* m_{r-1} *is either* $+P$ *or* $-Q$ *depending on whether* C_r *and* C_{r-1} *are equal or not. (We recall that, in delta modulation,* C_r *represents the polarity of the difference, at the sampling instant r, between the input signal* X_r *and the latest staircase approximation to it,* Y_{r-1} .*)*

A simulation of the delta modulator with a band-limited speech input has revealed that $PQ = 1$ *and* $P \simeq 1.5$ *represent optimal adaptation characteristics, on the basis of signal-to-error ratios, over an important range of sampling frequencies; and that at* 60 kHz, *delta modulation with these adaptation parameters compares favorably with 7-bit logarithmic* PCM, *which reproduces speech with good telephone quality. We present several graphical results from this simulation, and include an evaluation of the effect of independent channel errors on the adaptive delta modulator.*

We proceed to suggest a heuristic theory of the delta modulator which explains the optimality of the condition $PQ = 1$, *and develops an upper bound of 2 for the optimum value of* P.

We conclude with a summary of results from a video simulation which revealed that aforementioned optima for P *and* Q *apply to a video signal*

as well; with these optimum parameters, a useful delta-modulator output was obtained at 10 MHz *operation.*

The results of this paper reaffirm the utility of delta modulation as a simple alternative to PCM, *particularly in systems that operate at relatively low bit-rates.*

I. INTRODUCTION

Linear (or unadaptive) delta modulators, which work with a fixed step-size for the "staircase" approximation to an input signal, have the following basic limitation. Small values of the step size introduce slope-overload distortion during bursts of large signal slope; large values of the step-size accentuate the granular noise during periods of small signal slope; and, even when the step-size is optimized, the performance of these modulators will be satisfactory only at sampling frequencies that may be undesirably high. Equivalently, one encounters important ranges of operating frequency in which the performance of conventional delta modulation falls short of the standards attainable by conventional PCM or by d-level differential PCM, of which delta modulation is a special case ($d = 2$).

With a view to employing delta modulation (which is inherently a very simple signal-processing strategy) at such relatively low operating frequencies, several types of adaptive delta modulation have been proposed.[1–3] In these schemes, the step size is changed in accordance with the time-varying slope characteristics of the input signal, as per a predetermined adaptation strategy. Such adaptation or "companding" can be either at a syllabic rate (long-term) or instantaneous (short-term).

Typical of syllabic-companding delta modulators are recently developed schemes for reproducing telephone quality speech at operating frequencies of the order of 50 kHz.[1,5,6] These systems are characterized by "continuous" adaptation of the step magnitude. Instantaneous compandors, on the other hand, usually incorporate discrete adaptations, and illustrative schemes for speech, television and Gaussian signals are given in Abate[2] and, for speech transmission, in Winkler.[3] Abate shows the capabilities of linear and exponential adaptation for speech transmission, but gives quantitative results only for specific, finite, step-size dictionaries. Likewise, Winkler's work on "High Information Delta Modulation," while providing a conceptual basis for our paper, bypasses the question of optimal adaptation. We consider in this paper, although only for a sub-class of possible schemes, the problem of optimizing the adaptation logic.

We ought to refer here to the paper entitled "Statistical Delta Modulation" by Bello, and others.[4] Philosophically, this paper treats the problem of optimizing delta modulation with a generality that exceeds the scope of our work. However, the analysis of the cited paper does not have explicit bearing on the design philosophy for the very specific, but practically important, problem of providing a time-invariant logic for step-size adaptation. The purpose of our paper is to treat the latter problem for the important case of a one-bit memory.

We begin by defining our adaptation scheme (Section II), and go on to present results from a computer simulation of the delta modulator with a speech input (Section III). The results refer to the optimization of the adaptation logic, to a comparison of the optimal delta modulator with PCM, and to an assessment of the effect of channel errors on the delta modulator. We then present a heuristic theory (Section IV) for the delta modulator and seek to explain the optimal adaptation parameters that emerged from the speech simulation. Finally, we illustrate parallel results from a video simulation (Section V) and attempt a general assessment of adaptive delta modulators (Section VI).

II. DESCRIPTION OF THE ADAPTIVE DELTA MODULATOR

In this section, we define the delta modulator with exponential adaptation and a one-bit memory, and indicate its basic performance by illustrating its response to a constant input.

2.1 *The Adaptation Logic*

The delta modulator of this paper uses instantaneous, exponential adaptation in the sense that the step-size is changed at every sampling instant by a specific factor—more precisely, by one of two specific factors. Furthermore, the adaptation logic incorporates a one-bit memory in that the immediately past channel symbol C_{r-1} is stored, and is compared with the incoming bit C_r for a decision on the new step-size m_r. Specifically, if the previous step-size is denoted by m_{r-1}, the adaptation will be of the form

$$\begin{aligned} m_r &= P \cdot m_{r-1} \quad \text{if} \quad C_r = C_{r-1}\ ; \\ m_r &= -Q \cdot m_{r-1} \quad \text{if} \quad C_r \neq C_{r-1}\ . \end{aligned} \tag{1}$$

In this paper, we assume that P and Q are time-invariant, and note that in delta modulation, the following identity is usually assumed by definition:*

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* See Ref. 7 for an example where requirement (2) is waived.

$$\text{sgn } m_r = C_r = \text{sgn } (X_r - Y_{r-1}) \tag{2}$$

where X_r and Y_{r-1} represent the amplitude of the input signal and that of the latest staircase approximation to it, respectively, at the sampling instant r. The sampling interval in question would be a suitably small fraction of the Nyquist interval for X. A block diagram of the modulator appears in Fig. 1.

2.2 *Simple Bounds on P and Q*

The crucial parameters of our delta modulator are the time-invariant adaptation constants P and Q. The smallest and largest allowable step-sizes are other important parameters, but we assume that their design can be treated as an independent problem; and we mention at suitable points in the paper the considerations which influence such design. We now proceed, therefore, to state two simple bounds on the adaptation parameters P and Q:

(*i*) In order to adapt to the signal during slope overload, it is necessary that

$$P > 1. \tag{3}$$

(*ii*) In order to converge to a constant input signal during a purely "hunting" situation ($m_r = -Qm_{r-1}$ with probability 1), it is necessary that

$$Q < 1. \tag{4}$$

Notice that $P = Q = 1$ represents (conventional) linear delta modulation.

The adaptation logic of Section 2.1 represents the simplest nontrivial form of discrete exponential adaptation, and the performance of this scheme will be an important lower bound for that of an "n-bit" strategy

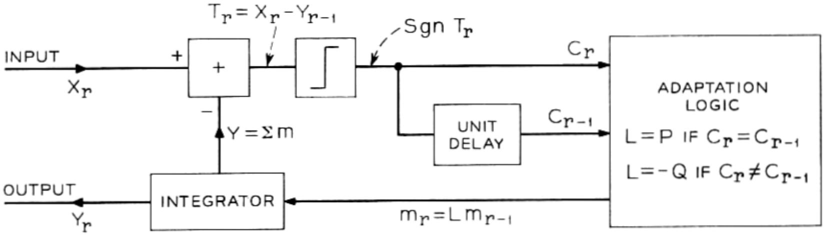

Fig. 1.— Schematic diagram of the Adaptive Delta Modulator.

($n > 2$) in which the step-size m_r is some optimal function of C_r, $C_{r-1}, \cdots, C_{r-n+1}$ and of the previous x step-sizes $m_{r-1}, m_{r-2}, \cdots, m_{r-x}$.[8]

2.3 *Step Response of the Delta Modulator*

Figure 2 shows the approximation of a step function by our adaptive delta modulator for a typical case of $P = 1.50$ and $Q = 0.66$. (These will emerge as optimum parameters later in the paper.) Step inputs of 9, 10 and 12 units have been considered for illustration, with a smallest step-size of 1 for the delta modulator.

The dependence of the "hunting" or "oscillating" characteristics on the actual magnitude of the step input is clear. We also see that during hunting, the step-size does not always assume the smallest possible value. This is an inherent feature of our adaptation logic, and emphasizes the need to make the smallest step-size as small as is practicable so that the in-band component of the noise due to hunting with nonminimal step-sizes will be tolerably low.

III. PERFORMANCE WITH A SPEECH INPUT

We describe in this section several results from a simulation of the adaptive delta modulator of Section II with a speech input. In particular, we highlight the optimization of the adaptation parameters P and

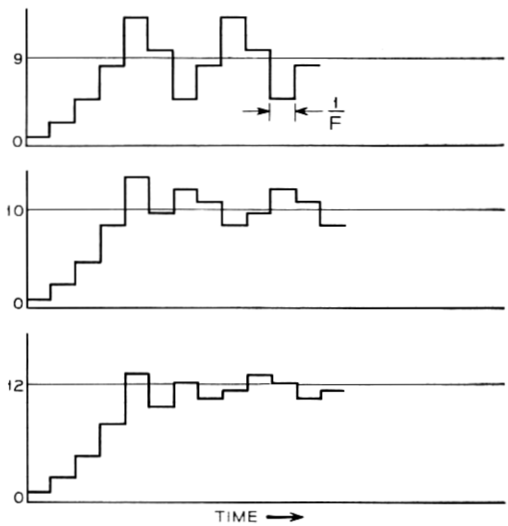

Fig. 2 — Step response of the Delta Modulator.

Q, and the relative performance of adaptive delta modulation (abbreviated henceforth as ADM) and of logarithmic PCM.

3.1 *Description of the Simulator*

The input speech signal used for the simulation was a male utterance of "Have you seen Bill?", bandlimited to 3.3 kHz, and sampled for the simulation at 20, 40, and 60 kHz. The sentence is illustrative in that it includes sounds which are known to be susceptible to slope-overload distortion.

In the computer simulation, the peak-to-peak range of the 12-bit speech signal was 4096 units. The configuration of the step-size dictionary was not predetermined, and the changes of the step-size were allowed to follow the exponential adaptation rule (*i*) of Section II. The simulation started with an initial step-size magnitude of 1 unit and it may be mentioned that step-size magnitudes as large as 380 units were typically encountered in the simulation. A histogram of utilized step-sizes for the typical case of $P = 1.50$, $Q = 0.66$ is illustrated in Fig. 3, and represents mean step magnitudes of the order of 30 units. For the special case of $P = Q = 1.0$, the constant step-size was selected to maximize a performance criterion to be defined presently. The step-size so optimized was approximately 80, 60 and 45 units for sampling frequencies of 20, 40 and 60 kHz respectively.

The simulation used an ideal integrator in the feedback loop of the delta modulator;* it also incorporated a nonrecursive low pass filter using a Fourier kernel, which was designed to have a 40 dB attenuation from 3 kHz to 3.3 kHz. Practical low pass filters may have to be sloppy in comparison, but the sharp filter was included in the simulation for a correct assessment of the modulator performance, and for comparison with Nyquist-rate PCM.

3.2 *Definition of a Signal-to-Noise Ratio G*

The basic purpose of the simulator was to study the performance of the delta modulator as a function of the adaptation parameters P and Q, and the sampling frequency F. The quality criterion which was adapted was an "objective signal-to-noise ratio" G, which was defined as the ratio of the power of the signal X_r to that of the error $E_r = X_r - Y_r$, averaged over the duration of the speech sample.

It is seen that no distinction was made between overload distortion and hunting noise in defining G. In adaptive delta modulation, instan-

* See Section 3.9 for a reference to the utility of leaky integrators for delta modulation in the presence of channel errors.

taneous companding is expected to render long bursts of one particular type of distortion very improbable; the total error power, defined as the summation of E_r^2 over the overload and hunting phases, was therefore adopted as a good measure of performance. As a matter of fact, in the absence of a better criterion, the same measure has been assumed in this paper for the nonadaptive case as well; and the credibility of the procedure has been borne out by the observation of a good correlation between the subjectively assessed quality of representative speech reproductions and the corresponding values of G.

3.3 *Stability of the Modulator*

Preliminary studies of stability revealed the significance of the product PQ, and the adaptation was seen to be inherently unstable (that is, resulting in a step-size oscillation between limits that were independent of the input) if PQ exceeded $(1 + \epsilon)$ where ϵ is positive, and much smaller than unity. Further studies of performance therefore

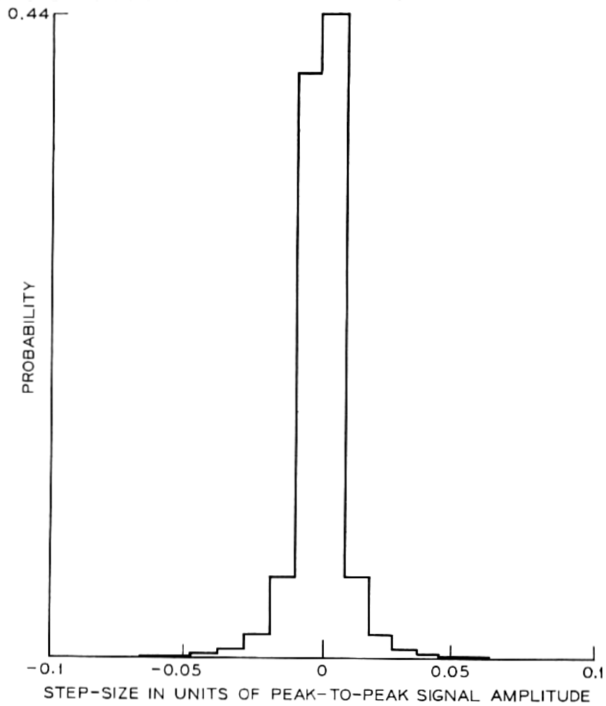

Fig. 3 — Histogram of utilized step sizes in the speech simulation. ($F = 60$ kHz)

assumed as stability condition of the form

$$PQ \leq 1. \tag{5}$$

The signal-to-noise ratio G was then studied as a function of allowable values of PQ, of P and of the sampling frequency F.

3.4 *The Dependence of G on PQ*

Using a typical value of $P = 1.6$, Fig. 4 shows the behavior of G as a function of PQ, with F as a parameter. The value of $G = -\infty$ (dB) at $PQ = 1.1$ represents an example of unstable adaptation, and the monotonic rise of G with PQ in its stable range is evident; in conjunction with the condition (5) in Section 3.3, it follows that

$$PQ = 1 \tag{6}$$

represents an optimal condition for all F; this conclusion was verified to be independent of the value of P.

Notice that (6) also represents a very desirable condition from the point of view of implementation. This is because the reciprocity of P and Q facilitates the use of a compact step-size dictionary. Finally, note that condition (6) is obviously satisfied in conventional delta modulation.

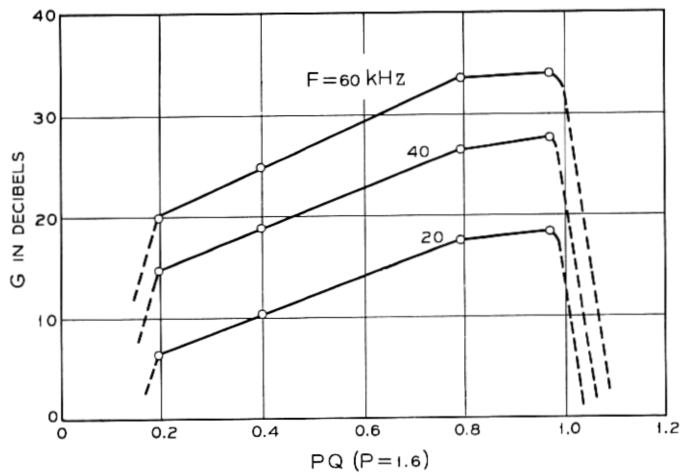

Fig. 4 — Results of the speech simulation: signal-to-error ratios as functions of (PQ).

3.5 *The Dependence of G on P*

Assuming the optimal reciprocity condition (6), the variation of G with P was investigated, and the results are given in Fig. 5. The "flat"

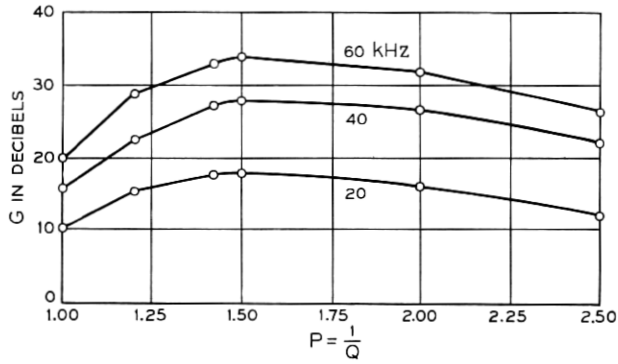

Fig. 5 — Results of the speech simulation: signal-to-error ratios as functions of P.

nature of the $G - P$ curves in the region of their maxima is noteworthy,* and the fact that the optimum value

$$P_{\text{opt}} \simeq 1.5 \tag{7}$$

is nearly independent of F is quite striking. Furthermore, the improvement that the optimized adaptive delta modulator affords, over the conventional system ($P = 1$), is seen to be an increasing function of F; and for 60 kHz operation, the gain exceeds 10 dB.

3.6 *A Note on Implementation*

The step-sizes in our simulation were real-valued quantities which changed according to equation (1). In a practical implementation, it may be preferable to work with integer-valued step-sizes; or, equivalently, to employ a suitably discretized step-size dictionary; and to avoid the actual operation of analogue multiplication. Such multiplication could pose significant practical problems. For example, the values of the multipliers P and Q may be subject to random perturbations about their design values, and these fluctuations may be independent at the encoder and at the decoder. Preliminary simulations that incorporated such imperfect multipliers suggest that the attendant deterioration of delta-modulator performance may well justify a mandatory

* For a corresponding observation with the adaptation logic described in Ref. 8, see Fig. 12 in that reference.

use of a discretized step-size dictionary. The design procedure for such a dictionary is seen to be greatly facilitated by virtue of the reciprocity condition for PQ, and the broad optima for P. The criteria for selecting the minimum and maximum step-sizes have been mentioned elsewhere in this paper, and the intermediate (discrete) step-sizes can be chosen to fit the optimum condition (7) as closely as possible, through the range of the dictionary. A further simplification will result if the slightly suboptimal value $P = 2$ is adopted as a uniform adaptation parameter.

3.7 *Subjective Performance*

Formal subjective tests of performance have not been carried out. However, the optimum ADM ($P = 1.50$, $Q = 0.66$) achieves very good telephone quality at 60 kHz, and the degradation at 40 kHz is very small. The ADM deteriorates in quality at 20 kHz operation, though most of the intelligibility of speech is still preserved.

3.8 *Comparison of* ADM *and Logarithmic* PCM

Table I shows the objective signal-to-noise ratio G for the optimum ADM at $F = 20$, 40 and 60 kHz; and, for n-bit logarithmic PCM at the Nyquist rate, the three values of n which provide correspondingly equal values of G. The PCM figures are due to the theory of Smith,[9] and represent average values over the significant range ($100 < \mu < 1000$) of his logarithmic-companding parameter μ. Furthermore, the PCM figures refer to the "strong-signal" or "full-load" case ($C \rightarrow 0$) in Smith's theory; inasmuch as our delta modulator could handle arbitrarily strong signals, according to equation (1), the "full-load" values for PCM performance were adopted as meaningful measures for our comparison.

It is generally accepted that 7-bit log-PCM represents a good quality of speech reproduction. It would therefore appear, from Table I, that a sampling frequency in the range of 40 to 60 kHz would be a critical figure for the employment of instantaneously companding ADM to reproduce telephone quality speech. This is an important conclusion of this paper, and follows a similar claim for a syllabic-companding delta modulator for speech at 56 kHz operation.[1]

Figure 6 replots the results of Table I, depicting G as a function of

Table I — Comparison with Logarithmic PCM

ADM sampling rate: F(kHz)	20	40	60
ADM Performance: G(dB)	18	28	34
Equivalent log-PCM bits: n	4.7	6.3	7.3

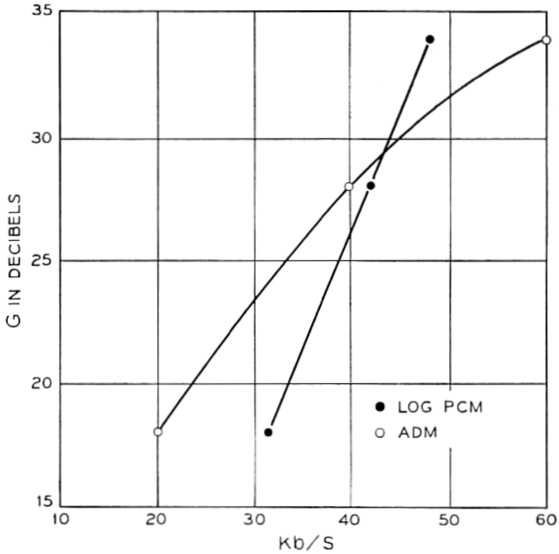

Fig. 6 — Comparison of Adaptive Delta Modulation and Logarithmic PCM: signal-to-error ratios as functions of bit-rate.

the bit-rate (product of the sampling frequency and the number of bits per channel symbol). The bit-rate is equal to the product (6600.n) for n-bit log-PCM at the Nyquist rate, and is equal to the sampling frequency F for ADM. The crossover of the curves in Fig. 6 at about 40 KBPS is significant.* It suggests an important, though narrow, range of usable bit-rates where ADM, which was conceived originally only for its simplicity, can actually excel conventional log-PCM for speech transmission.

3.9 *The Effect of Independent Channel Errors*

We conclude the section on ADM simulation with speech by presenting a qualitative discussion of the effect of independent channel errors on the performance of the delta modulator.

When such errors were first allowed in the simulation, deterioration of ADM performance was observed at error rates as low as 1 in 10^5. This was expected because of the inherent susceptibility of ADM to channel errors; every such error will have the effect of producing a long sequence of erroneous or suboptimal step-sizes which integrate in the output.

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* Crossovers of this type are indicated in Ref. 2 for television signals, and in Ref. 10, for Gaussian signals with an integrated spectrum.

In order to reduce the noise-memory of the ADM, the ideal integrator in the ADM simulator of Section 3.1 was replaced by a counterpart that had a finite time constant of the order of 10 to 20 sampling intervals. Furthermore, to mitigate the instability arising out of incorrect step-size adaptation in a noisy situation, the maximum allowable step-size was limited to a suitable value* (that would not introduce noticeable slope overload in the noiseless case). As a result of these refinements, the tolerable probability of channel errors was raised from about 1 in 10^5 to a figure of the order of 1 in 10^4. In fact, the intelligibility of ADM speech was still very much preserved at error rates of the order of 1 in 10^3, but the quality of the output was affected by "clicks" that were introduced by the channel errors.

An additional parameter that has a potential for enhancing the noise-resistance of ADM would be the length of the bit-memory in the adaptation scheme. The simple adaptation used in this paper has a minimal, one-bit memory and a suitably longer memory could indeed decrease the noise-susceptibility of ADM by a useful factor. In the ultimate analysis, however, it should be clear that such noise-susceptibility is a general limitation of all classes of ADM, because of the integrator employed in these systems; and this observation will be a very important factor in the assessment of adaptive delta modulation with reference to PCM for use on specific communication channels.

It may not be out of place to comment on the effect of transmitter-receiver mistracking on delta-modulator performance. In general, "mistracking" would characterize a situation where the step-size sequence in the receiver tracks that at the transmitter only in polarities and adaptation ratios—as determined by the transmitted binary sequence—but not in actual step-size magnitudes. Typically, this can be a result of some kind of an asynchronous operation. Thus, for example, the receiver may be switched on at a random instant in time, with the transmitter already in operation; the step-size in the decorder will then be different, in general, from that in the transmitter at that time instant. It would appear, now, that the effect of such mistracking would be akin to that of a random channel error occurring at the time instant in question; for, as in the case of such an error, the "decoding failure" due to asynchronous operation can be traced to a single point in time, although it propagates in the decoder output in the form of a long sequence of suboptimal step-sizes. In other words, we expect that the

* Specifically, the maximum step-size was limited to $0.05D$, where D was the dynamic range of the input speech; in the original simulation of the noiseless case, step-sizes as large as $0.10D$ had been encountered.

effect of mistracking—as that of a channel error—will be perceived as a transient in the decoder output, and the extent of such decoding failure will again depend, among other things, on (*i*) the time constant of the integrator employed at the decoder, and (*ii*) the maximum and minimum allowable step-sizes, which provide "locking points" for an asynchronous transmitter-receiver pair.

IV. A THEORY OF THE DELTA MODULATOR

We have mentioned in Section III that the optimal adaptation equations (6) and (7) were nearly invariant with respect to the sampling frequency. We shall see later, by virtue of the simulation in Section V, that these equations also hold good for a video input. These observations suggested the possibility of a fundamental and general explanation for the observed optima of P and Q. The purpose of this section is to provide such an explanation. Specifically, we propose a heuristic statistical model for the adaptive delta modulator, and go on to explain the reciprocity between the optimum values of P and Q. We also develop an upper bound of 2 for the optimum value of P.

4.1 *The Model*

Our statistical model is based on assumptions that are backed by computer simulation and physical appeal. We believe that the resulting theory provides a simplified, but useful, description of our delta modulator. The following are our tacit assumptions:

(*i*) The signal gradient $s_r = X_r - X_{r-1}$ is a random variable with a probability density function that is symmetrical about a mean value of zero.

(*ii*) In the optimal modulator, the "dynamic range" of the distribution of $|X_r|$, which denotes the signal magnitude, is much greater than the "dynamic range" of $|m_r| = |Y_r - Y_{r-1}|$, which denotes the random step-size in the staircase approximation to X.

(*iii*) With optimal adaptation, the probabilities of "P" type and "$-Q$" type adaptations of the step-size are equal. If we denote these probabilities by p and q respectively, we assume that

$$p = q = 0.5. \tag{8}$$

Assumption (*iii*) would appear to be the strongest. It is also the most crucial part of our model. In essence, the assumption states that, with optimal adaptation, overload and hunting situations are equally likely. In other words, the best adaptation logic is one which, by definition, is neither over-slow nor over-fast, but optimal in an average sense—as expressed in equation (8)—for the given input signal.

4.2 *The Optimum Value of PQ*

Consider the ratio $R(N)$ of the magnitude of m_{U+N} (the step size at the sampling instant $U + N$) to that of the step size m_U at the sampling instant U. Let the number of "P" type and "$-Q$" type adaptations of the step size m in the interval N be Np_0 and Nq_0 respectively, so that

$$R(N) = \frac{|m_{U+N}|}{|m_U|} = P^{Np_0} \cdot Q^{Nq_0} = (PQ^{q_0/p_0})^{Np_0}. \tag{9}$$

Note that, for the "most typical" sequence of step-sizes, as $N \to \infty$, p_0 and q_0 tend to the probabilities p and q. Furthermore, we have said that, for optimal adaptation, $p = q$. We can therefore define, for the optimal case, a "most typical" asymptotic value $R^M(\infty)$ for $R(N)$ as follows:

$$R^M_{\text{opt}}(\infty) = \lim_{N\to\infty} R^M_{\text{opt}}(N) = \lim_{N\to\infty} (P_{\text{opt}}Q_{\text{opt}})^{Np}. \tag{10}$$

We will now postulate an optimality criterion which will insist that the asymptotic ratio defined in equation (10) be finite and non-zero;* and because $Np \to \infty$ when $N \to \infty$, a necessary and sufficient condition for such stability will be given by

$$(PQ)_{\text{opt}} = 1. \tag{11}$$

Note that this condition applies only to the optimal system defined by equation (8).

The next two sections of this article are devoted to the derivation of a lower bound on the optimum value of Q_{opt}. By virtue of equation (11) such a bound on Q_{opt} will implicate a reciprocal bound on P_{opt}.

4.3 *Minimization of Mean Square Error*

We will adopt minimum mean square error as a criterion of optimality, and employ the notation

$$\text{Min}\,\langle E_r^2\rangle \to \text{Min}\,\langle (X_r - Y_r)^2\rangle \tag{12}$$

$$\to \text{Min}\,\langle (X_{r-1} + s_r - Y_{r-1} - m_r)^2\rangle$$

$$\to \text{Min}\,\langle (E_{r-1} + s_r - L_r m_{r-1})^2\rangle \tag{13}$$

$$\to \underset{L_r}{\text{Min}} \left\langle \left(\frac{E_{r-1} + s_r}{m_{r-1}} - L_r\right)^2 \cdot m_{r-1}^2 \right\rangle.$$

* Clearly, the idea is to prevent the tendency of the step-size m either to increase beyond bounds or to decay; and the formulation in equation (10) provides a tractable way of expressing this idea.

The method of optimization that will be adopted in the sequel is equivalent to carrying out the above minimization for every specific value of m_{r-1}. Therefore, we may write

$$\text{Min}\,\langle E_r^2\rangle \to \underset{L_r}{\text{Min}}\left\langle\left(\frac{E_{r-1}-s_r}{m_{r-1}}-L_r\right)^2\right\rangle. \tag{14}$$

We note that, given the polarity of the adaptation parameter L_r, the magnitude of L_r is time-invariant, and hence that $|L_r|$ is independent of E, s, or m. Therefore, it can be seen that the minimization of $\langle E_r^2\rangle$ is equivalent to the following optimization of L_r:

$$\text{Min}\,\langle E_r^2\rangle \to \left[L_{\text{opt}} = \left\langle\frac{E_{r-1}+s_r}{m_{r-1}}\right\rangle, \quad \text{given sgn}\quad (L_r)\right]^*. \tag{15}$$

[The above optimization of L_r has the following physical meaning. In the optimal system, the step size m_r at every sampling instant is designed so that, on the average, the resulting value of Y_r tends to that of the input X_r. In other words, the value of m_r attempts to compensate, at every sampling instant, for the corresponding "lag" of the staircase signal, as expressed by the quantity $X_r - Y_{r-1}$.

This random lag $(X_r - Y_{r-1})$ has two distinct components. The first component is given by the random error E_{r-1} (an overload or undershoot) arising out of the "instantaneous" suboptimality of the previous step m_{r-1}; the second component of the lag is the signal gradient s_r, which is the amount by which the signal X will have deviated after the delta-modulator integrated its previous step m_{r-1}. Our optimization procedure is tantamount to estimating the expected value of the sum of these two components of the lag—E_{r-1} and s_r—with respect to the value of m_{r-1}.]

4.4 *An Upper Bound for* $P_{\text{opt}} = 1/Q_{\text{opt}}$

As mentioned earlier, in view of the reciprocity that has been developed for the values of P_{opt} and Q_{opt}, we can now restrict the optimization procedure to that of optimizing the value of Q on the basis of equation (15):

$$-Q_{\text{opt}} = \left\langle\frac{E_{r-1}+s_r}{m_{r-1}}\right\rangle, \quad \text{given that sgn}\quad (L_r) = -1; \tag{16}$$

$$= \left\langle\frac{E_{r-1}}{m_{r-1}}\right\rangle + \left\langle\frac{s_r}{m_{r-1}}\right\rangle, \quad \text{given that sgn}\quad (L_r) = -1; \tag{17}$$

* We have utilized the well known statistical result: If A is a random variable and B is a parameter that is statistically independent of A, the expectation $\langle(A-B)^2\rangle$ has a minimum at $B_{\text{opt}} = \langle A\rangle$.

$$= Q_1 + Q_2 , \quad \text{given that sgn} \quad (L_r) = -1 \tag{18}$$

where Q_1 and Q_2 obviously refer to conditional expectations of the ratios in equation (17).

Figure 7 depicts a situation where sgn (L_r) is negative, and illustrates the random variables in (17). The problem will be to evaluate Q_1 and Q_2 with reference to Fig. 7. Notice at the outset that in the figure

$$\left.\begin{aligned} E_{r-1} &< 0 \\ m_{r-1} &> 0 \end{aligned}\right\}. \tag{19}$$

In what follows, we will denote the probability density functions of E_{r-1} , s_r , and of the signal amplitude X_{r-1} by $f_E(\ )$, $f_s(\ )$, and $f_X(\ )$ respectively.

4.4.1 *Evaluation of* Q_1 :

Let us first note the following equivalence of events:

$$\{E = e\} \leftrightarrow \{X = Y + e\}. \tag{20}$$

Notice next the following constraint for the overshoot error E_{r-1} :

$$-m_{r-1} < E_{r-1} < 0. \tag{21}$$

In other words, allowable values of E fall in the interval $(-m_{r-1} , 0)$. We can now invoke assumption (*ii*) in Section 4.1, (which says that the "dynamic range" of the step-magnitude $| m |$ is much smaller than that of the signal amplitude $| X |$) to make the approximation

$$f_X(Y_{r-1} + e_1) \simeq f_X(Y_{r-1} + e_2) \tag{22}$$

where e_1 and e_2 are two values of E within the "small" permissible range $(-m_{r-1} , 0)$ for E. In writing (22), we have approximated $f_X(\ )$ in the "narrow" range—from $Y + e_1$ to $Y + e_2$—by a constant function. In other words, the distribution of the overshoot error can be assumed to be uniform in the allowable range of E:

$$f_E(e) = \frac{1}{m_{r-1}} ; \qquad -m_{r-1} < e < 0. \tag{23}$$

Obviously then, the expected value of the ratio of the overshoot error E_{r-1} to the step-size m_{r-1} is given by

$$Q_1 = \int_e \frac{e}{m_{r-1}} \cdot f_E(e)\, de = \int_{-m_{r-1}}^{0} \frac{e}{m_{r-1}} \cdot \frac{1}{m_{r-1}}\, de = -0.5. \tag{24}$$

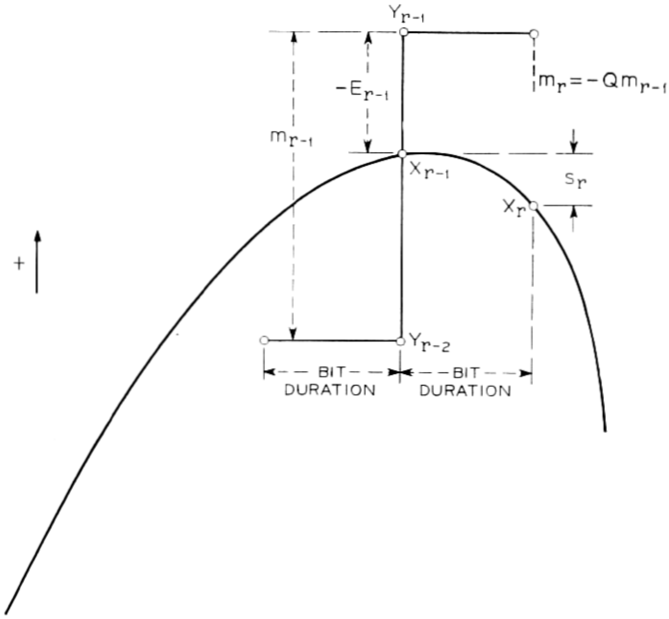

Fig. 7 — Illustration of a reversal of step-polarity.

4.4.2 *Evaluation of* Q_2 :

As a requirement for the reversal of step polarity in Fig. 7, one notes the constraint

$$E_{r-1} + s_r < 0. \tag{25}$$

Hence, allowable values of the signal gradient s_r have to lie in the range

$$-\infty < s_r < -E_{r-1} . \tag{26}$$

Notice that, by virtue of (19), the upper bound for s_r in (26) is positive.

Before proceeding to evaluate the expected value of s_r, we shall comment on the use of the unconditional density function of s_r in the ensuing analysis. With a one-bit memory, the polarity of m_{r-2} is unknown. Equivalently, it can be seen that there is no constraint on the gradient s_{r-1} analogous to that on s_r in (26). This means that, with a one-bit memory, one cannot develop any conditional distributions for the future signal gradient s_r , and the use of the unconditional density function $f_s(\ )$ will therefore be valid. Consequently, using (26) and (19) and the zero-mean assumption (*i*) for s_r in Section 4.1,

$$Q_2 = \int_x \frac{x}{m_{r-1}} \cdot f_s(x)\, dx = \frac{1}{m_{r-1}} \int_{-\infty}^{-E_{r-1}} x \cdot f_s(x)\, dx; \tag{27}$$

$$= \frac{1}{m_{r-1}} \left[\int_{-\infty}^{\infty} x f_s(x)\, dx - \int_{-E_{r-1}}^{\infty} x f_s(x)\, dx \right]; \tag{28}$$

$$= \frac{1}{m_{r-1}} [0 - \epsilon]; \qquad \epsilon > 0. \tag{29}$$

In other words, during an overshoot situation, the expected value of the future signal gradient is negative with respect to the present step m_r; and this is a consequence of a finite positive bound $-E_{r-1}$ (26) on the symmetrically distributed random variable s_r.

Utilizing equations (24) and (29) we can rewrite (18) in the form

$$-Q_{\text{opt}} = -0.5 - \delta; \qquad \delta > 0. \tag{30}$$

Finally, utilizing the simple upper bound (4) of 1 for Q_{opt}, we may write

$$0 < \delta < 0.5, \qquad 0.5 < Q_{\text{opt}} < 1.0 \tag{31}$$

and, by virtue of (11),

$$1.0 < P_{\text{opt}} = \frac{1}{Q_{\text{opt}}} < 2.0. \tag{32}$$

4.5 *Evaluation of the Theory*

Table II presents the values of important adaptation parameters obtained in a 60 kHz speech simulation of optimum delta modulation and compares them with the predictions of our theory. The comparison is good, and is particularly so with reference to the critical parameter p of assumption (*iii*).

We believe, in retrospect, that the heuristic theory of this section

TABLE II—CHARACTERISTICS OF AN OPTIMUM DELTA MODULATOR

Parameter	$(PQ)_{\text{opt}}$	P_{opt}	p	δ	Q_1
Theoretical Value	1	$1 < P_{\text{opt}} < 2$	0.50	$0 < \delta < 0.5$	−0.5
Value from Speech Simulation	1	1.5	0.47	0.12	−0.55

provides a simple understanding of adaptive delta modulation characterized by exponential adaption and a one-bit memory. The theory is still insufficient, however, and unanswered problems include an explicit derivation for the signal-to-error ratio and the question of analyzing the noise performance of adaptive delta modulation.

V. RESULTS FROM A VIDEO SIMULATION

We devote this section to a cursory presentation of results obtained from a simulation of the ADM with a video signal in a format that may be appropriate for communication purposes. The picture frame was made up of 250 scan lines, and a resolution of about 275 picture elements per line. The picture elements were 10-bit samples; therefore, assuming a scan rate of 30 frames/second, we were employing a 20 megabit/sec (MBPS) original. The simulator used an ideal integrator in the feedback loop and incorporated a digital low pass filter with a sharp cut-off at 1 MHz.

An important finding of the simulation was that optimum values of the adaptation parameters P and Q were still nearly equal to 1.5 and 0.66, which were values encountered in the speech simulation. Furthermore, as with speech, these optima of P and Q were nearly independent of the sampling frequency. Also, the optimized ADM performed significantly better than the unadaptive ($P = Q = 1$) encoder with an optimized step-size; at 10 MHz operation, for example, the performance

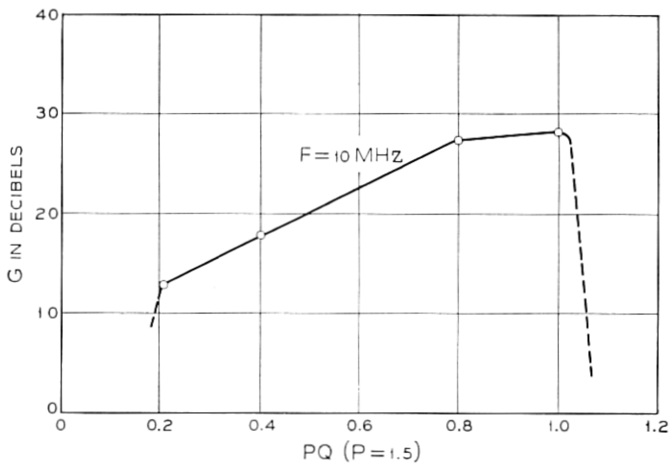

Fig. 8 — Results of the video simulation: signal-to-error ratio as a function of (PQ).

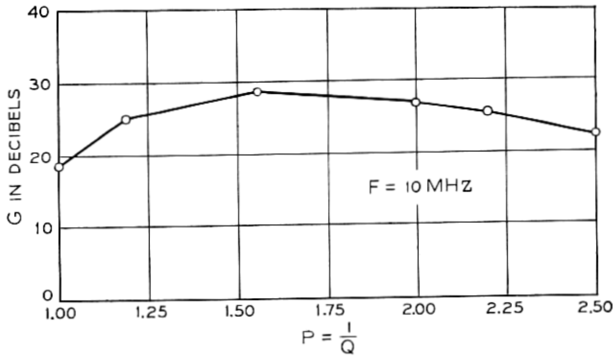

Fig. 9 — Results of the video simulation: signal-to-error ratio as a function of P.

gain, using the criterion of Section 3.2, was nearly 10 dB. We have provided, in Figs. 8 and 9, signal-to-error ratio curves that demonstrate the delta-modulator performance at 10 MHz, as a function of P and Q; the function G represents a signal-to-error ratio as averaged over the "active" or "picture" portion of the video frame.

Other sampling rates used in the simulation were 5 and 20 MHz. The performance of the modulator at 5 MHz was unsatisfactory, while the picture reproduction at 20 MHz was very acceptable. The capabilities and limitations of our scheme were best revealed in the 10 MHz simulation. In Fig. 10, we compare the output of the 10 MHz ADM, corresponding to a single frame of video input, with the 20 MBPS PCM original. The 10 MBPS ADM picture can be said to constitute a useful output; but it is not indistinguishable from the original. One notices, for example, the inadequate reproduction of the stripes on the dress of the subject.* This is attributable to the inability of the coder to follow sudden changes of input signal level; and would manifest, in the ADM version of a moving scene, as a corresponding twinkle.

The processing of moving scenes as well as the accumulation of subjective performance measures, were topics that were beyond the scope of our simulation. But such studies represent important prerequisites for a correct assessment of our delta modulator for general video application.

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* Interested readers may obtain glossy prints of Fig. 10 from the author at Bell Telephone Laboratories, Murray Hill, New Jersey.

VI. CONCLUSION

We have presented a very simple form of discrete adaptive delta modulation, characterized by the use of a one-bit memory and by exponential adaptations of the step-size. We have discussed optimization procedures for such a device, and demonstrated the applicability of the modulator to audio- and video-signal reproduction at practically useful operating frequencies, such as 60 kHz for audio and 10 MHz for video. It is well known that conventional (linear) delta modulators are inefficient at such frequencies. Though our ADM can be practically important in its own right, we reiterate that the performance of our adaptation logic is to be regarded as a lower bound on the performance of more sophisticated schemes[7]—in particular, of adaptations that employ more than a one-bit memory, or of those which exploit very specific statistics of the signal to be encoded.

We have also afforded, in this paper, a comparative evaluation of adaptive delta modulation and of PCM in the contexts of Fig. 6 (for speech signals) and Fig. 10 (for video signals). It is an important conclusion from the aforecited illustrations—and from Fig. 15 in Ref. 2—that there are ranges of bit-rates, in both speech and picture systems, where ADM performance is competitive with that of PCM; this constitutes a nontrivial observation in that the original conception of delta

Fig. 10 — Results of a video simulation: (a) 20 MBPS PCM original (b) 10 MBPS ADM output.

modulation was very much that of an inferior, though useful, alternative to PCM. The noise-susceptibility of delta modulation could however delimit its utility for specific noisy channels. On the other hand, a simple adaptive delta modulator would appear to have an edge over conventional/differential PCM in systems characterized by relatively noise-protected channels, in low bit-rate applications, and in systems where simplicity of implementation is a critical matter.

VII. ACKNOWLEDGMENT

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