Theorems on the Analysis of Nonlinear Transistor Networks*

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(Manuscript received August 19, 1969)

This paper reports on further results concerning nonlinear equations of the form $F(x) + Ax = B$, in which $F(\cdot)$ is a "diagonal nonlinear mapping" of real Euclidean n-space E^n into itself, A is a real $n \times n$ matrix, and B is an element of E^n. Such equations play a central role in the dc analysis of transistor networks, the computation of the transient response of transistor networks, and the numerical solution of certain nonlinear partial-differential equations.

Here a nonuniqueness result, which focuses attention on a simple special property of transistor-type nonlinearities, is proved; this result shows that under certain conditions the equation $F(x) + Ax = B$ has at least two solutions for some $B \in E^n$. The result proves that some earlier conditions for the existence of a unique solution cannot be improved by taking into account more information concerning the nonlinearities, and therefore makes more clear that the set of matrices denoted in earlier work by P_0 plays a very basic role in the theory of nonlinear transistor networks. In addition, some material concerned with the convergence of algorithms for computing the solution of the equation $F(x) + Ax = B$ is presented, and some theorems are proved which provide more of a theoretical basis for the efficient computation of the transient response of transistor networks. In particular, the following proposition is proved.

If the dc equations of a certain general type of transistor network possess at most one solution for all $B \in E^n$ for "the original set of α's as well as for an arbitrary set of not-larger α's", then the nonlinear equations encountered at each time step in the use of certain implicit numerical integration algorithms possess a unique solution for all values of the step size, and hence then for all step-size values it is possible to carry out the algorithms.

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* The material of this paper was presented at the Advanced Study Institute on Network Theory (sponsored by the N.A.T.O.; Knokke, Belgium; September 1-12, 1969).

I. INTRODUCTION AND DISCUSSION OF RESULTS

References 1 and 2 present some results concerning the equation

$$F(x) + Ax = B, \tag{1}$$

in which, with n an arbitrary positive integer, A is a real $n \times n$ matrix, B is an element of real Euclidean n-space E^n, and $F(\cdot)$ is a mapping of E^n into E^n defined by the condition that* for all $x = (x_1, x_2, \cdots, x_n)^{tr} \in E^n$,

$$F(x) = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]^{tr} \tag{2}$$

with each $f_i(\cdot)$ a strictly monotone increasing mapping of E^1 into itself. Equation (1) plays a central role in the dc analysis of transistor networks,** the transient analysis of transistor networks (see Section 1.4), and the numerical solution of certain nonlinear partial differential equations.

In Ref. 1 it is proved that there exists a unique solution x of equation (1) for each strictly monotone increasing mapping $F(\cdot)$ of E^n onto E^n (that is, for each set of strictly monotone increasing mappings $f_i(\cdot)$ of E^1 onto itself) and each $B \in E^n$ if and only if A is a member of the set P_0 of real $n \times n$ matrices with all principal minors nonnegative. It is also proved in Ref. 1 that equation (1) possesses a unique solution x for each continuous monotone nondecreasing mapping $F(\cdot)$ of E^n into E^n (that is, for each set of continuous monotone nondecreasing mappings of E^1 into E^1) and each $B \in E^n$ if A belongs to the set P of all real $n \times n$ matrices with all principal minors positive†. A direct modification of the existence proof given in Ref. 1, as indicated in Ref. 2, shows that equation (1) possesses a unique solution for each strictly monotone increasing mapping $F(\cdot)$ of E^n onto $(\alpha_1, \beta_1) \times (\alpha_2, \beta_2) \times \cdots \times (\alpha_n, \beta_n)$ with each α_i and β_i elements of the extended real line‡ (real line) such that $\alpha_i < \beta_i$ and each $B \in E^n$ if (and only if) $A \in P_0$ and $\det A \neq 0$. Some network theoretic implications of these and related results are discussed in Refs. 1 and 2, where the matter of determining whether or not $A \in P_0$ or $A \in P$ is considered in some detail.

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* Throughout the paper the superscript tr denotes transpose.

** See Ref. 1 for a derivation of the equation within the context of the transistor dc-analysis problem.

† There are some interesting applications of this result in the study of numerical methods for solving certain nonlinear partial-differential equations, in which A has nonpositive off-diagonal terms and is irreducibly diagonally dominant.[3]

‡ The numbers α_i and β_i are members of the *extended real line* if $-\infty \leqq \alpha_i \leqq \infty$ and $-\infty \leqq \beta_i \leqq \infty$.

This paper presents a proof of a nonuniqueness result. The proof focuses attention on a simple special property of transistor-type nonlinearities. The result shows that under certain conditions equation (1) has at least two solutions for some $B \in E^n$. In addition, the paper presents some material concerned with the convergence of algorithms for computing the solution of equation (1) and proves some theorems which provide more of a theoretical basis for the efficient computation of the transient response of transistor networks. The remaining portion of Section I is concerned with a detailed discussion of the results and their significance.

1.1 *An Application of the Nonuniqueness Theorem*

The standard Ebers–Moll transistor model, which is widely used, gives rise to functions $f_i(\cdot)$ which, while continuous and strictly monotone increasing, are mappings of E^1 onto open semi-infinite intervals. For such $f_i(\cdot)$, the results stated above assert that the equation (1) possesses at most one solution x for each $B \in E^n$ if $A \in P_0$; and if $A \in P_0$ and det $A \neq 0$, then equation (1) possesses a solution for each $B \in E^n$. Since, as indicated in Ref. 1, $A = T^{-1}G$ with T a nonsingular matrix which takes into account the forward and reverse transistor α's, and G is the short circuit conductance matrix of the linear portion of the network, the condition that det A not vanish is equivalent to the rather weak assumption that the linear portion of the network possess an open-circuit resistance matrix.

It is natural to ask whether the use of more-detailed information concerning the nonlinearities of the transistor model would enable us to make assertions concerning the existence of a unique solution of equation (1) for all $B \in E^n$ under weaker assumptions on A. In particular, can the condition that A belong to P_0 be relaxed? The first result proved in this paper, Theorem 1 of Section II, shows that if the $f_i(\cdot)$ are exponential nonlinearities of the type associated with the Ebers–Moll model, then the condition that A belong to P_0 cannot be replaced by a weaker condition. More explicitly, in Section II a set $\mathfrak{F}_0^n$ of mappings of E^n into E^n is defined, and $\mathfrak{F}_0^n$ contains all of the mappings $F(\cdot)$ that correspond to Ebers–Moll type $f_i(\cdot)$'s. It is proved there that if $A \notin P_0$, then for *any* $F(\cdot) \in \mathfrak{F}_0^n$, there is a $B \in E^n$ such that equation (1) possesses at least two solutions. In fact, it is proved that if $A \notin P_0$ and if δ is an arbitrary positive number, then for any $F(\cdot) \in \mathfrak{F}_0^n$, there is a $B \in E^n$ such that equation (1) possesses two solutions such that the distance in E^n between the two solutions is δ.

Thus Theorem 1 together with the earlier results mentioned above concerning existence of solutions show that the set of matrices P_0 plays a quite fundamental role in the theory of nonlinear transistor networks.

1.2 *An Algorithm for Computing the Solution of Equation (1)*

Several results which assert that $A \in P_0$ under certain conditions on the transistor α's and the short-circuit conductance matrix of the linear portion of the network are proved in Refs. 1 and 2. In particular, Ref. 1 proves that $A \in P$, and hence that $A \in P_0$, if $A = P^{-1}Q$ with P and Q real $n \times n$ matrices such that for all $j = 1, 2, \cdots, n$

$$p_{jj} > \sum_{i \neq j} | p_{ij} | \quad \text{and} \quad q_{jj} > \sum_{i \neq j} | q_{ij} |.^*$$

Theorem 2 of Section II shows that a relatively simple and entirely constructive algorithm can be used to generate a sequence $x^{(0)}, x^{(1)}, \cdots$ of elements of E^n that converges to the unique solution of (1) if $A = P^{-1}Q$ with P and Q as defined above and each $f_i(\cdot)$ is a continuous (but not necessarily differentiable) monotone nondecreasing mapping of E^1 into E^1.**

1.3 *Palais' Theorem, Existence of Solutions of Equation (1), and Algorithms for Computing the Solution of Equation (1)*

Reference 1 gives two existence proofs concerning equation (1). One proof, the more basic of the two, is based on first principles and employs an inductive argument in which, with k an arbitrary positive integer less than n, the existence proposition is assumed to be true with n replaced by k and it is proved that then the proposition is true with n replaced by $(k + 1)$. The second proof uses a theorem of R. S. Palais and requires that the $f_i(\cdot)$ be continuously differentiable throughout E^1. More explicitly, Palais' theorem† asserts that if $R(\cdot)$ is a continuously differentiable mapping of E^n into itself with values $R(q)$ for $q \in E^n$, then $R(\cdot)$ is a diffeomorphism‡ of E^n onto itself if and only if

(*i*) $\det J_q \neq 0$ for all $q \in E^n$, in which J_q is the Jacobian matrix of $R(\cdot)$ with respect to q, and
(*ii*) $\| R(q) \| \to \infty$ as $\| q \| \to \infty$.††

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* It is proved also that $A \in P_0$ if $A = P^{-1}Q$ with $p_{jj} > \sum_{i \neq j} | p_{ij} |$ and $q_{jj} \geqq \sum_{i \neq j} | q_{ij} |$ for all j.

** A related result given in Ref. 4 is not directly applicable here because of assumptions made in Ref. 4 concerning the existence and boundedness of a certain Jacobian matrix.

† See Ref. 5 and the appendix of Ref. 6.

‡ A diffeomorphism of E^n onto itself is a continuously differentiable mapping of E^n into E^n which possesses a continuously differentiable inverse.

†† Here $\| \cdot \|$ denotes any norm on E^n.

And the second proof of Ref. 1 shows that, with

$$R(q) = F(q) + Aq$$

for all $q \in E^n$, the two conditions (*i*) and (*ii*) are met when $A \in P_0$ and each $f_i(\cdot)$ is a continuously differentiable strictly-monotone-increasing function which maps E^1 onto E^1 and whose slope is positive throughout E^1.*

There are some problems which arise in connection with, for example, the numerical solution of certain nonlinear partial-differential equations** in which one encounters an equation of the form (1) with $A \in P_0$ and $\det A \neq 0$, but with functions $f_i(\cdot)$ which, while continuously differentiable, are monotone nondecreasing (rather than strictly monotone increasing) mappings of E^1 into E^1. We can prove that even in such cases equation (1) possesses a unique solution for each $B \in E^n$ as follows. Here the Jacobian matrix of $F(q)+Aq$ exists and is of the form $D(q)+A$ in which $D(q)$ is a diagonal matrix with nonnegative diagonal elements. Since $A \in P_0$ and $\det A \neq 0$, we have[2] $\det [D(q) + A] \neq 0$ for all $q \in E^n$. An immediate application of Theorem 3 of Section II shows that $\| F(q) + Aq \| \to \infty$ as $\| q \| \to \infty$.† Therefore, by Palais' theorem, $F(x) + Ax = B$ possesses a unique solution for each B.

Theorem 3 is of use not only in connection with the proof given in the preceding paragraph; it also plays a key role in showing that there is an algorithm which generates a sequence of elements of E^n $x^{(0)}, x^{(1)}, \cdots$ that converges to the unique solution of $F(x) + Ax = B$ whenever each $f_i(\cdot)$ is twice continuously differentiable on E^1 and the conditions on A and $F(\cdot)$ of the preceding paragraph are satisfied.‡

More generally, if $R(\cdot)$ is any twice-differentiable mapping of E^n into itself such that conditions (*i*) and (*ii*) of Palais' theorem are satisfied, then, with $R^{-1}(\cdot)$ the continuously-differentiable inverse of $R(\cdot)$, $x = R^{-1}(\theta)$ satisfies $R(x) = \theta$ in which θ is the zero element of E^n, and there are steepest decent as well as Newton-type algorithms each of

* The reasons that two proofs were presented in Ref. 1, with the second proof a proof of a somewhat weaker result, are that the arguments needed for the application of Palais' theorem had already been developed in Ref. 1 and used for other purposes there, and it was felt desirable to indicate an alternative approach to essentially the same problem.

** The writer is indebted to J. McKenna and E. Wasserstrom for bringing this fact to his attention.

† More explicitly, Theorem 3 shows that there is a vector $C \in E^n$ such that $\| F(q) + Aq + C \| \to \infty$ as $\| q \| \to \infty$, which is equivalent to the statement concerning $\| F(q) + Aq \|$ made above.

‡ The differentiability assumption here is introduced as a matter of convenience, and is certainly satisfied when the $f_i(\cdot)$ are Ebers–Moll exponential-type nonlinearities.

which generates a sequence in E^n that converges to x. To show this, let[7] $f(y) = R(y) \|^2$ for all $y \in E^n$ in which $\| \cdot \|$ denotes the usual Euclidean norm (that is, the square-root of the sum of squares). Since condition (*i*) of Palais' theorem is satisfied, the gradient ∇f of $f(\cdot)$ satisfies $(\nabla f)(y) \neq \theta$ unless $f(y) = 0$,* and since condition (*ii*) of Palais' theorem is satisfied the set $S = \{y \in E^n : f(y) \leqq f(x^{(0)}\}$ is bounded for any $x^{(0)} \in E^n$. Therefore we may appeal to, for example, the theorem of page 43 of Ref. 7 according to which for any $x^{(0)} \in E^n$, for any member of a certain class of mappings $\varphi(\cdot)$ of S into E^n, and for suitably chosen constants γ_0, $\gamma_1, \cdots$, the sequence $x^{(0)}, x^{(1)}, \cdots$ defined by

$$x^{(k+1)} = x^{(k)} + \gamma_k \varphi(x^{(k)}) \quad \text{for all} \quad k \geqq 0$$

belongs to S and is such that $\| R(x^{(k)} \| \to 0$ as $k \to \infty$. However, since $R^{-1}(\cdot)$ exists and is continuous,† it follows from

$$x^{(k)} = R^{-1}[R(x^{(k)})] \quad \text{for all} \quad k \geqq 0$$

and the fact that $R(x^{(k)}) \to \theta$ as $k \to \infty$, that $\lim_{k\to\infty} x^{(k)}$ exists and

$$\lim_{k\to\infty} x^{(k)} = R^{-1}(\theta),$$

which means that $x = \lim_{k\to\infty} x^{(k)}$.‡

1.4 *Transient Response of Transistor-Diode Networks and Implicit Numerical-Integration Formulas*

At this point we briefly consider some aspects of the manner in which the previous material bears on the important problem of providing more of a theoretical basis for numerically integrating the ordinary differential equations which govern the transient response of nonlinear transistor networks. Although we consider explicitly only networks containing transistors, diodes, and resistors, the material to be presented can be extended to take into account other types of elements as well. In addition, we shall focus attention on the use of linear multipoint integration formulas of closed (that is, of implicit) type, since such

* Here we have used the fact that $(\nabla f)(y) = 2J_y{}^{tr}R(y)$ for all $y \in E^n$.[7]

† By Palais' theorem $R(\cdot)$ is a diffeomorphism of E^n onto itself.

‡ The material of the second part of Section 1.3 was motivated by previous recent work of the writer's colleague A. Gersho who made the observation that the convergence of an algorithm for the solution of equation (1) could be shown by combining results of Ref. 1 with the approaches described by Goldstein.[7] (See the November 1969 B.S.T.J. Brief by A. Gersho.)

formulas are of considerable use in connection with the typically "stiff systems" of differential equations encountered.

A very large class of networks containing resistors, transistors, and diodes modeled in a standard manner is governed by the equation[8]

$$\frac{du}{dt} + TF[C^{-1}(u)] + (I + GR)^{-1}GC^{-1}(u) = B(t), \qquad t \geqq 0 \qquad (3)$$

where, assuming that there are q diodes and p transistors,

(*i*) $T = I_q \oplus T_1 \oplus T_2 \oplus \cdots \oplus T_p$, the direct sum of the identity matrix of order q and p 2×2 matrices T_k in which

$$T_k = \begin{Bmatrix} 1 & -\alpha_r^{(k)} \\ -\alpha_f^{(k)} & 1 \end{Bmatrix}$$

with $0 < \alpha_r^{(k)} < 1$ and $0 < \alpha_f^{(k)} < 1$ for $k = 1, 2, \cdots, p$.

(*ii*) $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_p$, the direct sum of a diagonal matrix $R_0 = \text{diag}\,(r_1, r_2, \cdots, r_q)$ with $r_k \geqq 0$ for $k = 1, 2, \cdots, q$ and p 2×2 matrices R_k in which for all $k = 1, 2, \cdots, p$

$$R_k = \begin{Bmatrix} r_e^{(k)} + r_b^{(k)} & r_b^{(k)} \\ r_b^{(k)} & r_c^{(k)} + r_b^{(k)} \end{Bmatrix}$$

with $r_e^{(k)} \geqq 0$, $r_b^{(k)} \geqq 0$, and $r_c^{(k)} \geqq 0$. (The matrix R takes into account the presence of bulk resistance in series with the diodes and the emitter, base, and collector leads of the transistors.)

(*iii*) G is the short-circuit conductance matrix associated with the resistors of the network. (It does not take into account the bulk resistances of the semiconductor devices.)

(*iv*) $F(\cdot)$ is a mapping of $E^{(2p+q)}$ into $E^{(2p+q)}$ defined by the condition that

$$F(x) = [f_1(x_1), f_2(x_2), \cdots, f_{2p+q}(x_{2p+q})]^{tr}$$

for all $x \in E^{(2p+q)}$ with each $f_i(\cdot)$ a continuously differentiable strictly-monotone increasing mapping of E^1 into E^1.

(*v*) $C^{-1}(\cdot)$ is the inverse of the mapping $C(\cdot)$, of $E^{(2p+q)}$ into itself, defined by

$$C(x) = \text{diag}\,(c_1, c_2, \cdots, c_{2p+q})x + \text{diag}\,(\tau_1, \tau_2, \cdots, \tau_{2p+q})F(x)$$

for all $x \in E^{(2p+q)}$ with each c_i and each τ_i a positive constant.

(*vi*) $B(t)$ is a $(2p + q)$-vector which takes into account the voltage and current generators present in the network, and (*vii*) u is related to v the vector of junction voltages of the semiconductor devices through $C(v) = u$ for all $v \in E^{(2p+q)}$.

Equation (3) is equivalent to*

$$\dot{u} + f(u, t) = \theta_{(2p+q)}, \qquad t \geqq 0 \tag{4}$$

in which of course

$$f(u, t) = TF[C^{-1}(u)] + (I + GR)^{-1}GC^{-1}(u) - B(t) \tag{5}$$

and $\theta_{(2p+q)}$ is the zero vector of order $(2p + q)$.

It is well known that certain specializations of the general multipoint formula[9,10]

$$y_{n+1} = \sum_{k=0}^{r} a_k y_{n-k} + h \sum_{k=-1}^{r} b_k \tilde{y}_{n-k} \tag{6}$$

in which

$$\tilde{y}_{n-k} = -f[y_{n-k}, (n - k)h] \tag{7}$$

can be used as a basis for computing the solution of equation (4). Here h, a positive number, is the step size, the a_k and the b_k are real numbers, and of course y_n is the approximation to $u(nh)$ for $n \geqq 1$.

In the literature dealing with formulas of the type (6) in connection with systems of equations of the type (4), information concerning the location of the eigenvalues of the Jacobian matrix J_u of $f(u, t)$ with respect to u plays an important role in determining whether or not a given formula will be (in some suitable sense) stable. In particular, an assumption often made is that all of the eigenvalues of J_u lie in the strict right-half plane for all $t \geqq 0$ and all u. For $f(u, t)$ given by equation (5), we have

$$J_u = T \operatorname{diag}\left\{\frac{f_j'[g_j(u_j)]}{c_j + \tau_j f_j'[g_j(u_j)]}\right\} + (I + GR)^{-1}G \operatorname{diag}\left\{\frac{1}{c_j + \tau_j f_j'[g_j(u_j)]}\right\} \tag{8}$$

in which for $j = 1, 2, \cdots, (2p + q)$ $g_i(u_j)$ is the j^{th} component of $C^{-1}(u)$. Thus here J_u is a matrix of the form

$$TD_1 + (I + GR)^{-1}GD_2 \tag{9}$$

where D_1 and D_2 are diagonal matrices with positive diagonal elements.

* Ref. 8 shows that if $B(\cdot)$ is a continuous mapping of $[0, \infty)$ into $E^{(2p+q)}$, then for any initial condition $u^{(0)} \in E^{(2p+q)}$ there exists a unique continuous $(2p + q)$- vector-valued function $u(\cdot)$ such that $u(0) = u^{(0)}$ and (3) is satisfied for all $t > 0$.

A simple result concerning equation (9), Theorem 4 of Section II, asserts that if there exists a diagonal matrix D with positive diagonal elements such that*

(*i*) DT is strongly column-sum dominant, and
(*ii*) $D(I + GR)^{-1}G$ is weakly column-sum dominant,

then for all diagonal matrices D_1 and D_2 with positive diagonal elements, all eigenvalues of (9) lie in the strict right-half plane. This condition on T, G, and R is often satisfied.†

The subclass of numerical integration formulas (6) defined by the condition that $b_{-1} > 0$ are of considerable use[11,12,13] in applications involving the typically "stiff systems" of differential equations encountered in the analysis of nonlinear transistor networks. With $b_{-1} > 0$, y_{n+1} is defined *implicitly* through

$$y_{n+1} + hb_{-1}f(y_{n+1}, (n + 1)h) = \sum_{k=0}^{r} a_k y_{n-k} + h \sum_{k=0}^{r} b_k \tilde{y}_{n-k}$$

in which the right side depends on y_{n-k} only for $k \in \{0, 1, 2, \cdots, r\}$, and for $f(u, t)$ given by equation (5), we have

$$y_{n+1} + hb_{-1}\{TF[C^{-1}(y_{n+1})] + (I + GR)^{-1}GC^{-1}(y_{n+1})\} = q_n \quad (10)$$

in which

$$q_n = \sum_{k=0}^{r} a_k y_{n-k} + h \sum_{k=0}^{r} b_k \tilde{y}_{n-k} + hb_{-1}B[(n + 1)h].$$

Obviously, the numerical integration formula (10) makes sense only if there exists for each n a $y_{n+1} \in E^{(2p+q)}$ such that equation (10) is satisfied.

Let $x_{n+1} = C^{-1}(y_{n+1})$ for each n. Then equation (10) possesses a unique solution y_{n+1} if and only if there exists a unique $x_{n+1} \in E^{(2p+q)}$ such that

$$C(x_{n+1}) + hb_{-1}[TF(x_{n+1}) + (I + GR)^{-1}Gx_{n+1}] = q_n . \quad (11)$$

Since $C(x_{n+1}) = cx_{n+1} + \tau F(x_{n+1})$, in which

$$c = \text{diag}\,(c_1, c_2, \cdots, c_{2p+q})$$

and

$$\tau = \text{diag}\,(\tau_1, \tau_2, \cdots, \tau_{2p+q}),$$

* The terms "strongly-column-sum dominant" and "weakly-column-sum dominant" are reasonably standard. However they are defined in Section II.
† See Ref. 8 for examples.

equation (11) is equivalent to

$$[\tau + hb_{-1}T]F(x_{n+1}) + [c + hb_{-1}(I + GR)^{-1}G]x_{n+1} = q_n . \qquad (12)$$

The matrices τ and c are both diagonal with positive diagonal elements. Thus it is clear that for all positive h

$$\det [\tau + hb_{-1}T] \neq 0$$

and

$$\det [c + hb_{-1}(I + GR)^{-1}G] \neq 0.^*$$

For all sufficiently small positive h

$$[\tau + hb_{-1}T]^{-1}[c + hb_{-1}(I + GR)^{-1}G] \in P_0 .^\dagger$$

Consequently†† for all sufficiently small $h > 0$, equation (12) possesses a unique solution for each q_n .‡ However, our interest in equation (12) is primarily in connection with "large-h" algorithms.

Suppose that $\det G \neq 0$ and that $T^{-1}G \in P_0$ for all possible combinations of α_r and α_f ($0 < \alpha_r < 1, 0 < \alpha_f < 1$) for each transistor (see Ref. 1 for examples). Then, according to Theorem 6 of Section II, for any particular T and R

$$[\tau + hb_{-1}T]^{-1}[c + hb_{-1}(I + GR)^{-1}G] \in P_0$$

for all $h > 0$, and hence equation (10) possesses a unique solution y_{n+1} for all positive values of h.

An important and general proposition concerning (10) is as follows. Suppose that

$$T^{-1}[(I + GR)^{-1}G] \in P_0 \qquad (13)$$

and that condition (13) is satisfied whenever $\alpha_r^{(k)}$ and $\alpha_f^{(k)}$ are replaced with positive constants $\delta_r^{(k)}$ and $\delta_f^{(k)}$, respectively, such that $\delta_r^{(k)} \leqq \alpha_r^{(k)}$ and $\delta_f^{(k)} \leqq \alpha_f^{(k)}$ for $k = 1, 2, \cdots, p$. In other words, assuming that $F(\cdot)$ is as defined in this section and that $F(\cdot) \in \mathfrak{F}_0^{(2p+q)}$ (see Definition 1 of Section 2.1), suppose that the dc equation

$$F(x) + T^{-1}[(I + GR)^{-1}G]x = B$$

possesses at most one solution x for each $B \in E^{(2p+q)}$ for "the original set of α's as well as for an arbitrary set of *not-larger* α's." Then an

* Here we have used the fact that $(I + GR)^{-1}G$ is positive semidefinite.
† See Section 1.2.
†† See Section 1.3.
‡ Alternatively, this conclusion could have been obtained by applying the contraction-mapping fixed-point principle to (10), in view of the fact that each of the elements of J_u is bounded on $u \in E^{(p+q)}$ and $t \in [0, \infty)$.

immediate application of Theorem 5 of Section II shows that

$$[\tau + hb_{-1}T]^{-1}[c + hb_{-1}(I + GR)^{-1}G] \in P_0$$

for all $h > 0$, and hence that equation (10) possesses a unique solution y_{n+1} for all $h > 0$ and all $q_n \in E^{(2p+q)}$.

II. THEOREMS, PROOFS, AND SOME DISCUSSION

Throughout this section,

(*i*) n is an arbitrary positive integer,

(*ii*) P_0 denotes the set of all real $n \times n$ matrices M such that all principal minors of M are nonnegative,

(*iii*) real Euclidean n-space is denoted by E^n, and θ is the zero element of E^n,

(*iv*) v^{tr} denotes the transpose of the row vector $v = (v_1, v_2, \cdots, v_n)$,

(*v*) $\| v \|$ denotes $(\sum_{j=1}^{n} v_j^2)^{1/2}$ for all $v \in E^n$,

(*vi*) if D is a real diagonal matrix, then $D > 0$ ($D \geqq 0$) means that the diagonal elements of D are positive (nonnegative),

(*vii*) I_q denotes the identity matrix of order q, and I denotes the identity matrix of order determined by the context in which the symbol is used, and

(*viii*) we shall say that a real $n \times n$ matrix M is strongly (weakly) column-sum dominant if and only if for $j = 1, 2, \cdots, n$

$$m_{jj} > (\geqq) \sum_{i \neq j} | m_{ij} |.$$

2.1 *Definition 1*

For each positive integer n, let $\mathfrak{F}_0^n$ denote that collection of mappings of E^n into itself defined by: $F \in \mathfrak{F}_0^n$ if and only if there exist for $j = 1, 2, \cdots, n$, continuous functions $f_j(\cdot)$ mapping E^1 into E^1 such that for each $x = (x_1, x_2, \cdots, x_n)^{tr} \in E^n$, $F(x) = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]^{tr}$, and

$$(i) \qquad \inf_{\alpha \in (-\infty, \infty)} [f_j(\alpha + \beta) - f_j(\alpha)] = 0$$

$$(ii) \qquad \sup_{\alpha \in (-\infty, \infty)} [f_j(\alpha + \beta) - f_j(\alpha)] = +\infty$$

for all $\beta > 0$ and all $j = 1, 2, \cdots, n$.

2.2 *Theorem 1*

Let $F \in \mathfrak{F}_0^n$, let A be a real $n \times n$ matrix such that $A \notin P_0$, and let δ be a positive constant. Then there exist $B \in E^n$, $x \in E^n$, and $y \in E^n$

such that

$$(i) \qquad F(x) + Ax = B,$$

$$(ii) \qquad F(y) + Ay = B,$$

and

$$(iii) \qquad \| x - y \| = \delta.$$

2.3 *Proof of Theorem 1*

Since $A \notin P_0$, there exists[2] a real diagonal matrix $D > 0$ such that $\det (D + A) = 0$. Thus there exists a $x^* \in E^n$ such that $\| x^* \| = \delta$ and $(D + A)x^* = \theta$.

Since $F \in \mathfrak{F}_0^n$, there exists a $x \in E^n$ such that

$$f_i(x_i) - f_i(x_i - x_i^*) = x_i^* d_i$$

for all $j = 1, 2, \cdots, n$ in which d_i is the j^{th} diagonal element of D. Let

$$B = F(x) + Ax,$$

and let $y = x - x^*$. Then $A(x - y) = Ax^* = -Dx^*$, and

$$F(x) - F(y) + A(x - y) = \theta. \quad \square$$

2.4 *Remarks Concerning Theorem 1*

If, as in the case of standard transistor models,

$$f_i(x_i) = e^{\lambda_i x_i} - 1$$

or

$$f_i(x_i) = 1 - e^{-\lambda_i x_i}$$

with $\lambda_i > 0$, we have, respectively,

$$f_i(\alpha + \beta) - f_i(\alpha) = e^{\lambda_i \alpha}(e^{\lambda_i \beta} - 1)$$

or

$$f_i(\alpha + \beta) - f_i(\alpha) = e^{-\lambda_i \alpha} (1 - e^{-\lambda_i \beta})$$

and it is clear that for either type of function conditions (*i*) and (*ii*) of Definition 1 are satisfied.

In Ref. 1 it is proved that if $F(\cdot) \in \mathfrak{M}$ the set of all $F(\cdot)$ of the form (2) with each $f_i(\cdot)$ a strictly monotone increasing mapping of E^1 into E^1, and if $A \in P_0$, then equation (1) possesses at most one solution.

Thus, using Theorem 1, we see that for each $F(\cdot) \in \mathfrak{M} \cap \mathfrak{F}_0^n$ there exists at most one solution of $F(x) + Ax = B$ for each $B \in E^n$ if and only if $A \in P_0$. Similarly, with $\mathfrak{M}_0$ the set of all $F(\cdot)$ of the form (2) with each $f_i(\cdot)$ a strictly monotone increasing mapping of E^1 onto (α_i, β_i) with each α_i and β_i such that $-\infty \leqq \alpha_i < \beta_i \leqq \infty$, if $F(\cdot) \in \mathfrak{M}_0 \cap \mathfrak{F}_0^n$ and $\det A \neq 0$, then there exists a unique solution x of $F(x) + Ax = B$ for each $B \in E^n$ if and only if $A \in P_0$. (The "if" part of this statement is proved in Ref. 2.) A parallel development can be carried out for equations of the form $AF(x) + x = B$ with A a real $n \times n$ matrix, $F(\cdot) \in \mathfrak{M}_0 \cap \mathfrak{F}_0^n$, and $B \in E^n$. More explicitly, we can prove that if $F(\cdot) \in \mathfrak{M}_0 \cap \mathfrak{F}_0^n$, then there exists a unique solution x of $AF(x) + x = B$ for each $B \in E^n$ if and only if $A \in P_0$.

There may be a temptation to conjecture that whenever $F(\cdot) \in \mathfrak{M} \cap \mathfrak{F}_0^n$ and $A \notin P_0$ then the equation $F(x) + Ax = B$ does not possess a solution for some $B \in E^n$. The conjecture is false. In fact, with $n = 2$, $f_1(x_1) = e^{x_1}$, $f_2(x_2) = e^{x_2}$, and

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we have a situation in which (it is easy to show that) there exists a solution for all $B \in E^2$. Of course here for some choices of B the solution is not unique.

2.5 *Theorem 2*

Let P and Q denote real $n \times n$ matrices such that

$$p_{jj} > \sum_{i \neq j} | p_{ij} | \quad \text{and} \quad q_{jj} > \sum_{i \neq j} | q_{ij} |$$

for all $j = 1, 2, \cdots, n$. For $j = 1, 2, \cdots, n$ let $f_j(\cdot)$ denote a continuous monotone nondecreasing (but not necessarily differentiable) mapping of E^1 into itself, and let $F(x) = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]^{tr}$ for all $x \in E^n$. Then for each $R \in E^n$, there exists a unique $x \in E^n$ such that

$$PF(x) + Qx = R,$$

and, for any $y_0 \in E^n$, x is the limit of the sequence $x^{(0)}, x^{(1)}, \cdots$ defined by

$$y^{(n)} = D_P F(x^{(n)}) + D_Q x^{(n)}$$

$$y^{(n+1)} + (P - D_P)F(x^{(n)}) + (Q - D_Q)x^{(n)} = R$$

for $n \geqq 0$, in which D_P and D_Q are diagonal matrices whose diagonal elements coincide with those of P and Q, respectively.

2.6 *Proof of Theorem 2*

Since the continuous mapping $[D_P F(\cdot) + D_Q]$ of E^n into E^n possesses an inverse $[D_P F(\cdot) + D_Q]^{-1}$, the equation

$$PF(x) + Qx = R$$

possesses a unique solution x if and only if $y = D_P F(x) + D_Q x$ is the unique solution of

$$y + \tilde{P}F[(D_P F(\cdot) + D_Q)^{-1} y] + \tilde{Q}[(D_P F(\cdot) + D_Q)^{-1} y] = R$$

in which $\tilde{P} = (P - D_P)$ and $\tilde{Q} = (Q - D_Q)$.

Therefore, by Banach's contraction-mapping fixed-point theorem, it suffices to show that with the metric $\rho(y, z) = \sum_{i=1}^{n} | y_i - z_i |$, the operator H defined by

$$H(y) = \tilde{P}F[(D_P F(\cdot) + D_Q)^{-1} y] + \tilde{Q}[(D_P F(\cdot) + D_Q)^{-1} y]$$

for all $y \in E^n$, is a contraction mapping of E^n into itself. We show this as follows. Let $y \in E^n$ and $z \in E^n$. Using the fact that

$$\alpha = d_{Qj}[(d_{Pj} f_j(\cdot) + d_{Qj})^{-1} \alpha] + d_{Pj} f_j[(d_{Pj} f_j(\cdot) + d_{Qj})^{-1} \alpha]$$

for all real α and all $j = 1, 2, \cdots, n$, in which d_{Pj} and d_{Qj} is the j^{th} diagonal element of D_P and D_Q, respectively, it is a simple matter to verify that for all j:

$$f_j[(d_{Pj} f_j(\cdot) + d_{Qj})^{-1} y_j] - f_j[(d_{Pj} f_j(\cdot) + d_{Qj})^{-1} z_j] = \frac{r_j}{d_{Qj} + d_{Pj} r_j} (y_j - z_j),$$

and

$$(d_{Pj} f_j(\cdot) + d_{Qj})^{-1} y_j - (d_{Pj} f_j(\cdot) + d_{Qj})^{-1} z_j = \frac{1}{d_{Qj} + d_{Pj} r_j} (y_j - z_j)$$

in which $r_j = 1$ if $y_j = z_j$, and, if $y_j \neq z_j$,

$$r_j = \frac{f_j[(d_{Pj} f_j(\cdot) + d_{Qj})^{-1} y_j] - f_j[(d_{Pj} f_j(\cdot) + d_{Qj})^{-1} z_j]}{(d_{Pj} f_j(\cdot) + d_{Qj})^{-1} y_j - (d_{Pj} f_j(\cdot) + d_{Qj})^{-1} z_j}.$$

Thus

$$H(y) - H(z) = \tilde{P} \operatorname{diag} \left\{ \frac{r_j}{d_{Qj} + d_{Pj} r_j} \right\} (y - z) + \tilde{Q} \operatorname{diag} \left\{ \frac{1}{d_{Qj} + d_{Pj} r_j} \right\} (y - z)$$

in which $r_j \geqq 0$. Therefore

$$\rho(H(y), H(z)) \leqq \max_j \left(\frac{\sigma_{Qj} + \sigma_{Pj} r_j}{d_{Qj} + d_{Pj} r_j}\right) \rho(y, z)$$

in which $\sigma_{Qj} = \sum_{i \neq j} | q_{ij} |$ and $\sigma_{Pj} = \sum_{i \neq j} | p_{ij} |$. Since $\sigma_{Qj} < d_{Qj}$ and $\sigma_{Pj} < d_{Pj}$ for all j, there exists a positive constant $\beta < 1$ such that

$$\max_j \left(\frac{\sigma_{Qj} + \sigma_{Pj} r_j}{d_{Qj} + d_{Pj} r_j}\right) \leqq \beta$$

for all $r_j \geqq 0$. □

2.7 *Theorem 3*

If $A \in P_0$ *and* det $A \neq 0$, *if for each* $j = 1, 2, \cdots, n$: $f_j(\cdot)$ *is a continuous mapping of* E^1 *into itself such that*

$$f_j(x_j) = 0 \quad \text{for all } x_j$$

or

$$f_j(x_j) > 0 \quad \text{for all } x_j > c$$

and

$$f_j(x_j) < 0 \quad \text{for all } x_j < -c$$

for some $c \geqq 0$, *then, with* $F(x) = [f_1(x_1), f_2(x_2), \cdots, f_n(x_n)]^{tr}$ *for all* $x \in E^n$,

$$|| F(x) + Ax || \to \infty \quad \text{as} \quad || x || \to \infty.$$

2.8 *Proof of Theorem 3*

We note that

$$|| F(x) + Ax || \to \infty \quad \text{as} \quad || x || \to \infty$$

if and only if

$$|| A^{-1}F(x) + x || \to \infty \quad \text{as} \quad || x || \to \infty.$$

With $M = A^{-1}$, let

$$MF(x) + x = q. \tag{14}$$

Since $A \in P_0$, we have $M \in P_0$.[1] Since $M \in P_0$, we have[1] for any $y \in E^n$ and $y \neq \theta$

$$y_k(My)_k \geqq 0$$

for some index k such that $y_k \neq 0$.

Suppose that $F(x) \neq \theta$. Then there exists an index k_1 such that

$$f_{k_1}(x_{k_1})[MF(x)]_{k_1} \geqq 0$$

with $f_{k_1}(x_{k_1}) \neq 0$. Thus, using (14),

$$f_{k_1}(x_{k_1})[MF(x)]_{k_1} + f_{k_1}(x_{k_1})x_{k_1} = f_{k_1}(x_{k_1})q_{k_1}$$

and

$$f_{k_1}(x_{k_1})x_{k_1} \leqq f_{k_1}(x_{k_1})q_{k_1} .$$

Either $x_{k_1} \in [-c, c]$ or not. If not, then $f_{k_1}(x_{k_1})x_{k_1} > 0$ and $| x_{k_1} | \leqq | q_{k_1} |$. Therefore for some index k_1 , $| x_{k_1} | \leqq \delta_1 \triangleq \max (c, | q_{k_1} |)$, whether or not $F(x) = \theta$.

Let $M^{(k_1)}$ denote the matrix obtained from M by deleting the k_1 row and column, and let $M_{(k_1)}$ denote the k_1 column of M with the k_1 entry removed. Similarly, let $x_{(k_1)}$, $q_{(k_1)}$ and $F_{(k_1)}(x_{(k_1)})$ denote the $(n - 1)$-vectors obtained from x, q, and $F(x)$, respectively, by removing the k_1 entry. Then

$$M^{(k_1)}F_{(k_1)}(x_{(k_1)}) + x_{(k_1)} = q_{(k_1)} - M_{(k_1)}f_{k_1}(x_{k_1}).$$

Since $M^{(k_1)} \in P_0$, we can repeat the argument given above. Thus there exists an index k_2 , different from k_1 , such that

$$| x_{k_2} | \leqq \delta_2 \triangleq \max (c, | q_{(k_1, k_2)} |)$$

in which

$$| q_{(k_1, k_2)} | = \max_{|x_{k_1}| \leqq \delta_1} | [q_{(k_1)} - M^{(k_1)}f_{k_1}(x_{k_1})]_{l_2} |$$

and l_2 is the index of the component of $x_{(k_1)}$ that corresponds to the k_2 component of x. By continuing in this manner we can determine positive constants δ_1 , δ_2 , $\cdots$, δ_n depending only on q, F, M, and c such that, with $\delta = \max_j \{\delta_j\}$,

$$| x_j | \leqq \delta \quad \text{for all} \quad j = 1, 2, \cdots , n$$

and each δ_i depends on q such that for any positive constant α, there exists a constant $\beta_i(\alpha)$ with the property that $\delta_i \leqq \beta_i(\alpha)$ provided that $|| q || \leqq \alpha$. Therefore for any $\alpha > 0$ there is a $\beta(\alpha)$ such that $|| x || \leqq \beta(\alpha)$ whenever $|| q || \leqq \alpha$, which implies that $|| q || \rightarrow \infty$ as $|| x || \rightarrow \infty$. □

2.9 *Theorem 4*

Let P and Q denote real $n \times n$ matrices with P strongly column-sum dominant. Suppose that there exists a real diagonal matrix $D > 0$ such *that DP is strongly column-sum dominant and DQ is weakly column-sum dominant. Then for all real diagonal matrices $D_1 > 0$ and $D_2 > 0$, all eigenvalues of $(PD_1 + QD_2)$ lie in the strict (that is, open) right-half plane.*

2.10 *Proof of Theorem 4*

Since the strict right-half plane contains all of the eigenvalues of P, there exists choices of $D_1 > 0$ and $D_2 > 0$ such that every eigenvalue of $(PD_1 + QD_2)$ lies in the strict right-half plane. Thus it suffices to show that $(PD_1 + QD_2)$ does not possess an eigenvalue on the boundary of the complex plane for all $D_1 > 0$ and all $D_2 > 0$. In other words, it suffices to show that (with $i = (-1)^{\frac{1}{2}}$)

$$PD_1 + QD_2 + i\omega I \tag{15}$$

is nonsingular for all $D_1 > 0$, all $D_2 > 0$, and all real constants ω.

Suppose that (15) is singular for some ω and some $D_1 > 0$ and some $D_2 > 0$. Then $(DPD_1 + DQD_2 + i\omega D)$ is singular. But DPD_1 is strongly column-sum dominant and DQD_2 is weakly column-sum dominant. Thus $M = (DPD_1 + DQD_2)$ is strongly column-sum dominant, and, since

$$| m_{jj} + i\omega d_j | > \sum_{i \neq j} | m_{ij} |$$

for all j, in which d_j is the j^{th} diagonal element of D, it follows that $\det (M + i\omega D) \neq 0$, which is a contradiction. □

2.11 *Definition 2*

With q and p nonnegative integers such that $(p + q) > 0$, let $\mathfrak{J}$ denote the set of all matrices M such that $M = I_q \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_p$ with

$$M_k = \begin{bmatrix} 1 & -\propto_r^{(k)} \\ -\propto_f^{(k)} & 1 \end{bmatrix}$$

and

$$0 < \propto_r^{(k)} < 1$$

$$0 < \propto_f^{(k)} < 1$$

for all $k = 1, 2, \cdots, p$.*

* As suggested, if $q = 0$, then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_p$, while if $p = 0$, then $M = I_q$.

2.12 *Definition 3*

With q and p nonnegative integers such that $(p + q) > 0$, let $\mathfrak{J}(\alpha)$ denote the set of all matrices M such that $M = I_q \oplus M_1 \oplus M_2 \oplus \cdots \oplus M_p$ with

$$M_k = \begin{bmatrix} 1 & -\delta_r^{(k)} \\ -\delta_f^{(k)} & 1 \end{bmatrix}$$

and

$$0 < \delta_r^{(k)} \leqq \alpha_r^{(k)}$$

$$0 < \delta_f^{(k)} \leqq \alpha_f^{(k)}$$

for all $k = 1, 2, \cdots, p$.*

2.13 *Theorem 5*

Let $T \in \mathfrak{J}$, let H be a real matrix of order $(2p + q)$, and suppose that $M^{-1}H \in P_0$ for all $M \in \mathfrak{J}(\alpha)$. Then

$$(T + D_1)^{-1}(H + D_2) \in P_0$$

for all diagonal matrices $D_1 \geqq 0$ and $D_2 \geqq 0$.

2.14 *Proof of Theorem 5*

Suppose that for some $D_1 \geqq 0$ and $D_2 \geqq 0$

$$(T + D_1)^{-1}(H + D_2) \notin P_0 .$$

Then there exists[2] a diagonal matrix $D > 0$ such that

$$(T + D_1)^{-1}(H + D_2) + D$$

is singular. It follows that

$$H + \Delta + TD$$

is singular, in which $\Delta = D_2 + D_1 D$. Since

$$\Delta + TD = M(\Delta + D)$$

in which $M \in \mathfrak{J}(\alpha)$, it follows that

$$H + M(\Delta + D)$$

is singular, and therefore that

* As suggested, if $q = 0$, then $M = M_1 \oplus M_2 \oplus \cdots \oplus M_p$, while if $p = 0$, then $M = I_q$.

$$M^{-1}H + (\Delta + D)$$

is singular, which is a contradiction since[1] $M^{-1} H \in P_0$ and $(\Delta + D)$ is a diagonal matrix with positive diagonal elements. □

2.15 *Theorem 6*

Let $M^{-1}G \in P_0$ for all $M \in \mathfrak{I}$, and let det $G \neq 0$. Let R be as defined in Section 1.4. Then for any $T \in \mathfrak{I}$

$$(T + D_1)^{-1}[(I + GR)^{-1}G + D_2] \in P_0$$

for all diagonal matrices $D_1 \geqq 0$ and $D_2 \geqq 0$.

2.16 *Proof of Theorem 6*

Since det $G \neq 0$ and $M^{-1}G \in P_0$ for all $M \in \mathfrak{I}$, it follows (see the proof of Theorem 7 of Ref. 2) that

$$M^{-1}(I + GR)^{-1}G \in P_0$$

for all $M \in \mathfrak{I}$.

Suppose that for some $T \in \mathfrak{I}$ and some $D_1 \geqq 0$ and $D_2 \geqq 0$

$$(T + D_1)^{-1}[(I + GR)^{-1}G + D_2] \notin P_0 .$$

Then, following the proof of Theorem 5, we would have

$$\det \{M^{-1}(I + GR)^{-1}G + (\Delta + D)\} = 0$$

for some $M \in \mathfrak{I}$ and some diagonal matrix $(\Delta + D)$ with positive diagonal elements, which is a contradiction. □

III. ACKNOWLEDGMENT

The writer is indebted to A. N. Willson, Jr. for carefully reading the draft.

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