Linear-Real Codes and Coders*

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In linear-real coding, the transmitted signals are (possibly redundant) linear combinations of the data signals. The linear combination of data signals can have a block pattern, resulting in linear-real block coders, or a stationary pattern, resulting in linear-real stationary (shift-register) coders. Stationary coding is shown to be a limiting case of block coding. Both methods appear to be practical for the control of burst and impulse noise. However, stationary coding appears to have some advantages and is the only one we study here. We propose shift register implementations which promise the required precision and dispersion at less cost than tuned RLC circuits.

Error properties of both block and stationary coders are similar, but it is easier to learn concepts by analyzing the block coders. When the receiver is able, by using some of the techniques we discuss, to estimate the noise covariance matrix for each codeblock, the resulting noise power is less than that for receivers not using the statistics for each codeblock.

Nonlinear memoryless filters, such as clippers, are especially effective when used with linear-real coders. We propose a memoryless filter which attenuates the input signal more severely when a second input to the filter indicates the channel is having a noise burst. If the memoryless filter is designed for the worst case noise, then performance will not degrade with decreased noise when the nonlinearity is odd and monotonic.

I. INTRODUCTION

Many communications channels, including telephone channels, contain noise which comes in short bursts, such as noise from impulses. Such noise is particularly deleterious when the channel is used for the transmission of digital data.

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At least as early as 1958 it was discovered that it is sometimes possible to reduce digital errors in such channels without reducing the noise power by using a scheme such as Fig. 1 shows. In some formulations[1–7] the transformation A consisted of a continuous all-pass filter whose Fourier transform magnitude was unity at all frequencies but whose phase characteristic varied with frequency; the inverse linear transformation was the continuous all-pass filter with the conjugate phase characteristic. The linear filter was called the smear operation, and its inverse the desmear operation. Later papers considered linear transformations to be real-number matrices operating upon the data in blocks.[8–10]

In all schemes to which Fig. 1 applies, a single impulse of noise into the inverse linear filter will be transformed into an output noise which is dispersed in time. With proper design, this dispersed noise will be small enough at all times to not produce errors at the output of the quantizer.

Our purpose is to investigate coding schemes which fall in the general pattern of Figure 1 to gain conceptual insight and learn practical design. Such study is useful because the practicality of the matrix version has never been studied, and the continuous all-pass filter was limited by cost and filter imprecision. The shift registers we might propose avoid the problems which hindered the application of continuous all-pass filters.

We show that the real-number linearity of the transformations of Fig. 1 will permit the receiver to use any available information about noise correlation or position. All of the proposed means for using this information are simple in concept, and some are simple to implement.

II. DESCRIPTION

Linear-real block coding is a form of coding in which A, an n by k matrix of real numbers, is used to produce an output vector $\mathbf{b}$ from an input vector $\mathbf{r}$ according to the equation

$$\mathbf{b} = A\mathbf{r}. \tag{1}$$

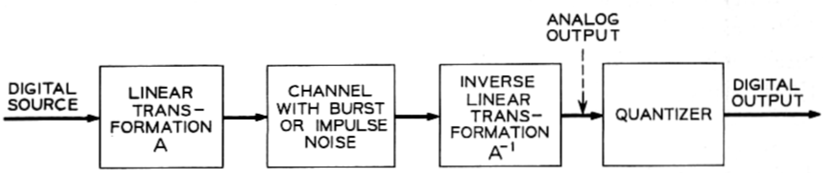

Fig. 1 — A general arrangement for placing linear filters A and A^{-1} to reduce digital errors.

If $n > k$, then $\mathbf{b}$ will be redundant in the sense that not all of its components are independent. The word "real" is used in order to emphasize the fact that the arithmetic in equation 1, and all other equations in this paper, is real number arithmetic. The use of real number arithmetic distinguishes this work from generalized parity-check coders which are linear in finite-field arithmetic.

Stationary (shift register) linear-real coding is a limiting case of linear-real block coding, but is best described as being the convolution summation given by

$$b_i = \sum_{j=-\infty}^{\infty} h_{i-j} r_j \tag{2}$$

where b_i is the i^{th} signal transmitted, r_i is the i^{th} data number, and where h_q can naturally be called the unit pulse response of the encoding filter at time-step q.

The conclusions to be reached on practical applications are that moderate cost encoders and decoders of considerable use for burst and impulse noise channels can be built as soon as low-cost tapped digital delay lines are available. Magnetic domain-wall digital delay lines,[11] for example, might well make these coders practical.

There are two general ways in which noise is controlled by means of linear-real coding. We give the complete details and mathematics later. Briefly, the qualitative aspects are:

The total noise power in the decoded signal is made less than that without coding. We discuss three distinct ways of doing this:

(*i*) When linear-real block coding is used, and when the noise covariance matrix is known (or can be adaptively deduced by the receiver) then this knowledge can be used to reduce the noise power. It can be correlation type knowledge, as accounts for the effectiveness of Wiener filtering. If the noise process is *a posteriori* nonstationary, then a receiver which estimates the noise correlation matrix for each code block may effectively use the available information on the position of burst noises within the block. This is particularly effective in burst noise channels having block coders using rectangular A matrices.

(*ii*) A stationary memoryless nonlinear filter (such as a clipper) can be used to reduce the noise power before the inverse linear transformation is applied. Such a filter would of course reduce noise power in the absence of an inverse filter when it immediately precedes the quantizer, but it would not then reduce errors. When placed before the inverse transformation, the stationary memoryless nonlinear filter reduces both errors and noise power. We refer to equations for analyzing design and performance of the memoryless nonlinear filter. A simulation example in Section VI shows these devices to be surprisingly effective.

(*iii*) A memoryless nonlinear filter can be used which has both the noisy signal and an estimate of the instantaneous noise power for inputs. The output is an optimized estimate of the signal given the estimated instanteous noise power. This filter always reduces noise power, as does the filter in method *ii*, and only reduces errors if there is a filter such as the inverse linear transformation between it and the quantizer. We describe several methods for estimating the instantaneous noise power in Section V. One of these, which appears in Fig. 6, uses the fact that practical pam signals have more bandwidth then the Nyquist bandwidth for their pulse interval.

The remaining noise power is distributed more evenly among all decoded signal components and (in the limit of infinite smearing) made Gaussian. This type of noise control is especially effective in quantized-signal burst and impulse noise channels which have a thermal noise which is small compared with the separation between quantization levels. In this case a burst noise with power which is small compared with the thermal noise would be unable to produce many errors if it were evenly dispersed, although it could when bunched up. Dispersal of the burst noise power is sometimes unfavorable, but if the noise power is reduced enough and the noise dispersed enough, then the effect is very favorable. The decoding operation also tends to make the decoded signal have a Gaussian first-order probability distribution, which reduces the probability of a large peak and thereby reduces errors for quantized signals.

The design equations for the nonlinear memoryless filter (clipper) to which we refer assume a known probability distribution on the noise, as does the simulation reported. In practice, the actual noise can be less noisy than that used for design purposes, and the resulting mean square error will not be larger than that with the design noise, provided the noise probability density is even and the nonlinearity has certain properties. We give precise details in Appendix D.

III. BLOCK CODES AND THEIR NOISE COVARIANCE MATRIX

In general, assuming **r** and **c** are independent zero-mean column vector random variables, which represent the signal to be encoded and the channel noise, respectively, and assuming **r** and **c** have nonsingular covariance matrixes Q and N, respectively, and assuming that $\mathbf{f} = \mathbf{b} + \mathbf{c}$ is decoded by some linear operator T, where **b** is given in equation 1, then a straightforward evaluation of the covariance matrix of $\mathbf{u} = \mathbf{r} - T\mathbf{f}$ will show that

$$\begin{aligned} M &= E[\mathbf{u}\mathbf{u}^t] \\ &= (I_{(k\times k)} - TA)Q(I_{(k\times k)} - TA)^t + TNT^t \end{aligned} \tag{3}$$

where $(\ )^t$ denotes the transpose of a matrix or column vector. This formula can be used to compare the performance of encoder-decoder pairs with good and bad choices for matrix A, and good and bad choices of matrix T.

Table I shows three possible T matrices. The first was shown to be the least mean square linear estimator in (9), and for Gaussian signal and noise gives the conditional mean of the transmitted vector given the received vector. The second is the first evaluated for infinite signal power in all degrees of freedom (which implies $Q^{-1} = 0$) and produces a decoded error uncorrelated with the signal. The third does not require the use of the N matrix. All assume the columns of A to be linearly independent.

Table II gives further insights into the behavior of the decoded error by presenting a number of special cases of equation (3). The justification of the equations of Table II is given in Appendix A. In one of the special cases in Table II, namely when equation (7) applies, the decoded noise energy is proportional to the arithmetic mean of the received noise energy. In other cases, such as that of equation (12), the eigenvalues of $A^tN^{-1}A$ play a crucial role in formulas for the mean square decoded noise.

Equation (13) of Table II shows that the average of the eigenvalues of $A^tN^{-1}A$ appears in a formula for a lower bound for the mean

TABLE I—THREE DIFFERENT LINEAR OPERATORS FOR DECODING f INTO r.

Name	Formula
Mean estimator (Gives least mean square error)	$T = (Q^{-1} + A^tN^{-1}A)^{-1}A^tN^{-1}$
Unattenuated estimator	$T = (A^tN^{-1}A)^{-1}A^tN^{-1}$
Unadaptive estimator (The generalized inverse of A)	$T = (A^tA)^{-1}A^t$

Table II — Some Special Cases of the Error Covariance Matrix of Equation (3) and the Resulting Mean Square Error

Unadaptive Estimator

$$M = (A^tA)^{-1}A^tNA[(A^tA)^{-1}]^t. \tag{4}$$

$$\text{m.s. error} = 1/k \operatorname{tr} (A^tA)^{-1}A^tNA[(A^tA)^{-1}]^t. \tag{5}$$

When the columns of A are orthogonal and each of length $(n/k)^{\frac{1}{2}}$:

$$M = (k/n)^2A^tNA. \tag{6}$$

When in addition $N = \operatorname{diag}(n_1, n_2, \cdots, n_n)$, and AM is the arithmetic mean of these n_i's, and A is $1/(k)^{\frac{1}{2}}$ times the first k columns of a Hadamard matrix (see Appendix A for a definition):

$$\text{m.s. error} = M_{ii} = (k/n)AM. \tag{7}$$

Unattenuated Estimator

$$M = (A^tN^{-1}A)^{-1}. \tag{8}$$

$$\text{m.s. error} = 1/k \operatorname{tr} (A^tN^{-1}A)^{-1}. \tag{9}$$

Mean Estimator

$$M = (Q^{-1} + A^tN^{-1}A)^{-1}. \tag{10}$$

$$\text{m.s. error} = 1/k \operatorname{tr} (Q^{-1} + A^tN^{-1}A)^{-1}. \tag{11}$$

Mean Estimator ($\Omega = 1$) *or Unattenuated Estimator* ($\Omega = 0$)

$$\text{m.s. error} = 1/k \sum_{i=1}^{k} \frac{1}{\lambda_i(\Omega Q^{-1} + A^tN^{-1}A)} \tag{12}$$

where $\lambda_i(Z)$ denotes the i^{th} unordered eigenvalue of Z. Special case of above when $Q = sI$, s scalar:

$$\text{m.s. error} = 1/k \sum_{i=1}^{k} \frac{1}{\Omega s^{-1} + \lambda_i(A^tN^{-1}A)} \geq \frac{1}{\Omega s^{-1} + 1/k \sum_{i=1}^{k} \lambda_i(A^tN^{-1}A)}. \tag{13}$$

Special case of equation (12) when $Q = sI$, and A is square, orthogonal, and each column has length $(n/k)^{\frac{1}{2}}$:

$$\text{m.s. error} = 1/k \sum_{i=1}^{k} \frac{1}{s^{-1} + \dfrac{1}{\lambda_i(N)}}. \tag{14}$$

The following assumptions are referred to as equation (15):

$\Omega = 1$: T is the mean estimator.
$\Omega = 0$: T is the unattenuated estimator, and $A^tN^{-1}A$ is positive definite.
$Q = sI$, s scalar.
A is $1/(k)^{\frac{1}{2}}$ times the first k columns of an $n \times n$ Hadamard matrix.
$N = \operatorname{diag}(n_1, n_2, \cdots n_n)$.

The n_i variables are independent, identically distributed random variables such that $E(1/n_i)$ exists, has finite variance σ^2, and the harmonic mean of the n_i variables

$$HM = \left[1/n \sum_{i=1}^{n} 1/n_i\right]^{-1} \tag{15}$$

is finite.
k is large enough for the weak law of large numbers to apply.

Assuming equation (15):

$$\frac{1}{\Omega s^{-1} + \frac{n}{k\,HM}} \leq \text{m.s. error} \tag{16}$$

$$\text{m.s. error} \leq \frac{0.5}{s^{-1} + \frac{n}{k\,HM} - \sqrt{\frac{\sigma 2_n}{\tau}}} + \frac{0.5}{s^{-1} + \frac{n}{k\,HM} + \sqrt{\frac{\sigma 2_n}{\tau}}} \tag{17}$$

provided that the first denominator is positive, where τ is given by equation 35 of Appendix I.

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square error; furthermore, that the mean square error equals this lower bound only when all the eigenvalues are the same. Thus the deviations of the eigenvalues of $A^tN^{-1}A$ detemine the closeness of the lower bound of equation (13), which Appendix A shows is sometimes related to the harmonic mean of the eigenvalues of N, which appears in equations (16) and (17).

A geometric illustration of the eigenvalues of $A^tN^{-1}A$ for rectangular A with orthonormal columns begins with the observation that the eigenvectors of N^{-1} form the semiaxes of an n-dimensional ellipsoid. The projection of this ellipsoid by the transformation A^t forms another ellipsoid, which will be called the k-dimensional shadow of the original n-dimensional ellipsoid.*

The semiaxes of the shadow ellipsoid have the lengths of the eigenvalues of $A^tN^{-1}A$. In order for the equation (13) bound to be close to the actual value, the semiaxes of the shadow ellipsoid have to be generally near their mean length; in other words, the shadow has to be round. A sufficient condition for the shadow to be round is that the ellipsoid is the shadow of a round ellipsoid, but this is not necessary. For some of the possible spacial orientations, for example, a football's shadow is rounder than the football.

IV. THE LIMITING CASE OF STATIONARY (SHIFT REGISTER) CODING

The purpose of this section is to show that—in the limit— all linear-real coding and decoding operations can become time stationary, so that they can be implemented by shift registers with time-invariant impulse responses. The limit is taken in the sense that the transmitted digits are obtained as a single block code whose output is a column

* An ordinary planar shadow of a three-dimensional object will be an orthogonal projection only when the light rays are parallel, and are normal to the plane of the shadow.

vector with components from $-n$ to n, where n approaches infinity.

There are two reasons why a study taking linear-real coding to the limit of being time stationary can be advantageous or useful:

(*i*) Stationary encoders and decoders appear to be more economical to implement than the block type of encoders and decoders.

(*ii*) The mathematical investigations to be made in the passage to the limit will add insights to linear-real coding by showing that a special case of it is Wiener filtering, and will add insights to Wiener filtering by showing that a Wiener filter is related to the least mean square estimator of matrix-encoded noise data vectors.

Toeplitz matrices, defined later, and Z-transforms (Ragazzini and Franklin),[12] are our main mathematical techniques to reach these ends.

4.1 *Stationary Coders*

The transmitted signal b_i is assumed to be obtained from the data stream r_j by the convolution summation of equation (2), which can be put in matrix form by means of the doubly infinite vectors

$$\mathbf{b} = \begin{bmatrix} \vdots \\ b_{-1} \\ b_0 \\ b_1 \\ \vdots \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} \vdots \\ r_{-1} \\ r_0 \\ r_1 \\ \vdots \end{bmatrix}, \qquad \text{etc.,}$$

and the Toeplitz matrix (defined in section 4.2)

$$A_{ij} = a_{i-j} = h_{i-j}$$

so that equation (2) can be expressed in matrix form by

$$\mathbf{b} = A\mathbf{r}.$$

The problem of how to perform the infinite matrix multiplications, either analytically or with hardware, will be shown to be solvable by the use of Z-transforms.

4.2 *Infinite Toeplitz Matrices*

An infinite matrix A, with elements A_{ij}, $i, j = 0, \pm 1, \pm 2, \cdots$, will be called Toeplitz* if some sequence $\ldots, a_{-1}, a_0, a_1, \ldots$ exists

* Hermitian matrices of the type of Equation (14) are called Toeplitz forms, and are described by Grenander and Szego.[13] The Hermitian property is not assumed in this paper's definition, since it is not needed for some of the results.

such that

$$A_{ij} = a_{(i-j)} \tag{18}$$

for all i, j. Associated with this Toeplitz matrix will be the two-sided Z-transform

$$a(z) = \sum_{q=-\infty}^{\infty} a_q z^{-q}. \tag{19}$$

The convergence properties of Toeplitz matrices could prove troublesome in some cases, but in this paper most difficulties will be avoided by using only those matrices whose associated Z-transform, according to equations (18) and (19), has all its poles some finite distance from the circle $|z| = 1$, and which is absolutely convergent on $|z| = 1$. (If the matrix is to be inverted, it also must have its zeros some finite distance from $|z| = 1$.)

Any poles outside $|z| = 1$ arise from a_q sequences which are nonzero for $q < 0$. This should not cause alarm, as noncausality of unit pulse reponses for decoders is not a serious practical obstacle, since actual noncausal unit pulse responses can be arbitrarily well approximated by accepting a decoding delay. These restrictions on the poles of the associated Z-transforms require that a_q be bounded by a geometrically decreasing sequence as $q \to \pm\infty$.

Section B.1 of Appendix B presents theorems which are useful in relating Toeplitz matrix operations to Z-transforms, and shows how least mean square matrix operators of the Toeplitz type can be related to Weiner-filter types of sampled data estimators.

4.3 *Error Analysis*

When A, Q, and N are Toeplitz and nonsingular, the expressions for the mean square error equivalent to the equations of Table 2 are

$$T_{\text{MEAN}} = (Q^{-1} + A^t N^{-1} A)^{-1} A^t N^{-1}$$

or

$$t_{\text{MEAN}}(z) = \frac{q(z)a\left(\frac{1}{z}\right)}{n(z) + a\left(\frac{1}{z}\right)q(z)a(z)}$$

gives

$$\begin{matrix}\text{m.s.}\\ \text{error}\end{matrix} = \begin{bmatrix}\text{on diagonal component}\\ \text{of } M_{\text{MEAN}} = (Q^{-1} + A^t N^{-1} A)^{-1}\end{bmatrix} \tag{20}$$

or

$$\begin{matrix}\text{m.s.}\\ \text{error}\end{matrix} = Z^{-1}\left\{\frac{q(z)n(z)}{n(z) + a\left(\frac{1}{z}\right)q(z)a(z)}\right\}\Bigg|_{k=0} . \tag{21}$$

where Z^{-1} is the inverse Z-transform integral operator. When A is either finite or Toeplitz but nonsingular, $T_{\text{UNATTENUATED}}$ and $T_{\text{UNADAPTIVE}}$ give the same decoding matrix, namely A^{-1}, which will be called T_{INVERSE}.

$$T_{\text{INVERSE}} = A^{-1}$$

or for the Toeplitz case

$$t_{\text{INVERSE}}(z) = \frac{1}{a(z)}$$

gives

$$\begin{matrix}\text{m.s.}\\ \text{error}\end{matrix} = \begin{bmatrix}\text{on diagonal component}\\ \text{of } A^{-1}N(A^{-1})^{t}\end{bmatrix} \tag{22}$$

$$\begin{matrix}\text{m.s.}\\ \text{error}\end{matrix} = Z^{-1}\left\{\frac{n(z)}{a(z)a\left(\frac{1}{z}\right)}\right\}\Bigg|_{k=0} . \tag{23}$$

The above error can be evaluated by these three methods:

(*i*) Truncate A and N and then compute an on-diagonal component of $(A^{t}N^{-1}A)^{-1}$ near the center of the matrix.

(*ii*) Use Z-transforms to find $t_{\text{UNATTENUATED}}(z)$. Invert the Z-transform by either

(*a*) Using the inversion integral for Z-transforms, or

(*b*) Using pole-zero expansions and a small table of Z-transforms.

Method (*ii-a*) is the Z-transform analog of using Parseval's theorem to find mean square errors of stationary nonsampled systems.

Lemma 1: *When A is Toeplitz with columns orthogonal and of length* 1, *then*

$$(a) \qquad A^{t}A = I$$

$$(b) \qquad a\left(\frac{1}{z}\right)a(z) = 1.$$

The proof is trivial. Also notice that (*a*) ⇔ (*b*).

Corollary 1: *When* $T = A^{-1}$ *and* A *is orthogonal,*

$$\begin{matrix} m.s. \\ error \end{matrix} = m.s.\ noise.$$

For design purposes it is desirable to make the following definitions; both assume $T = A^{-1}$ which is assumed to exist.

For Toeplitz A and N:

$$\begin{matrix}\text{noise}\\ \text{power}\\ \text{amplification}\end{matrix} = \frac{\begin{bmatrix}\text{on diagonal component} \\ \text{of } A^{-1}N(A^{-1})^t\end{bmatrix}\begin{bmatrix}\text{on diagonal component} \\ \text{of } A^tA\end{bmatrix}}{[\text{on diagonal component of } N]} \tag{24}$$

For Block Coders:

$$\begin{matrix}\text{noise}\\ \text{power}\\ \text{amplification}\end{matrix} = \frac{\left[\frac{1}{k}\,\mathrm{tr}\ A^{-1}N(A^{-1})^t\right]\left[\frac{1}{k}\,\mathrm{tr}\ A^tA\right]}{\left[\frac{1}{k}\,\mathrm{tr}\ N\right]}. \tag{25}$$

Physically, this corresponds to the actual amplification of noise in a channel which encodes with a matrix proportional to the A matrix, where the proportionality constant is selected to make the encoder give unity power amplification to a white signal, and where the decoder is T_{INVERSE}. For the stationary coder and channel, the Z-transform version is:

$$\begin{matrix}\text{noise}\\ \text{power}\\ \text{amplification}\end{matrix} = \frac{Z^{-1}\left\{\dfrac{n(z)}{a(z)a\left(\frac{1}{z}\right)}\right\}\Bigg|_{k=0} Z^{-1}\left\{a(z)a\left(\frac{1}{z}\right)\right\}\Bigg|_{k=0}}{n_0}. \tag{26}$$

The block code version of the trace formula can also be used to show that if the impulse response of the stationary encoder is $\ldots a_{-1}, a_0, a_1, \ldots$, and its inverse is $\ldots b_{-1}, b_0, b_1, \ldots$, so that $a_q * b_q = \delta_{q,0}$, then for $N \propto I_\infty$ the noise power amplification can be evaluated from the impulse reponses by:

$$\begin{matrix}\text{noise}\\ \text{power}\\ \text{amplification}\\ \text{(for white noise)}\end{matrix} = \left[\sum_{q=-\infty}^{\infty} a_q^2\right]\left[\sum_{q=-\infty}^{\infty} b_q^2\right]. \tag{27}$$

The Z-transform version for $N \propto I_\infty$ is:

$$\begin{matrix}\text{noise}\\ \text{power}\\ \text{amplification}\\ \text{(for white noise)}\end{matrix} = Z^{-1}\left\{\frac{1}{a(z)a\left(\frac{1}{z}\right)}\right\}\Bigg|_{k=0} \quad Z^{-1}\left\{a(z)a\left(\frac{1}{z}\right)\right\}\Bigg|_{k=0}. \tag{28}$$

It can be readily seen from equation (26) that:

Lemma 2: *When A is Toeplitz, the noise power amplification will be unity whenever*

$$a(z)a\left(\frac{1}{z}\right) = \text{constant}$$

whether or not the noise is white, so long as it is Toeplitz.

An equivalent statement is that when A and N are Toeplitz, a sufficient condition for the noise power amplification to be unity is that $A^t A = I_\infty$, which is equivalent to $a(z)a(1/z) = \text{constant}$.

The above lemma will be seen to be especially significant after it is proved that unity noise power amplification is the least which can ever be obtained, and when it is shown that simple $a(z)$ functions, namely all-pass functions, obey the conditions of the lemma. Notice that the noise power amplification definition was based upon a receiver which performed the inverse of the encoding operation, and not upon a receiver which made a least square estimate of the signal given the *a posteriori* noise statistics. Consequently, statements about least possible noise power amplification are not applicable to adapative types of receivers such as those employing T_{MEAN}.

The following theorem is for block codes with $n = k$.

Theorem 1: *When square block coding is used and N is proportional to the identity, then the noise power amplification is always greater than or equal to one, and it is one only when A is proportional to an orthogonal matrix.*

Proof: What is required is a demonstration that:

$$(i) \qquad \frac{1}{k^2}[\operatorname{tr} A^{-1}(A^{-1})^t][\operatorname{tr} AA^t] \geqq 1 \tag{29}$$

and

(*ii*) Equality occurs if and only if A is proportional to an orthogonal matrix. (30)

These are established in Section 2 of Appendix B.

The following corollary is the Toeplitz matrix limit version of the above.

Corollary 2: When A and N are Toeplitz, and N is proportional to I_∞*, a necessary and sufficient condition for unity noise power amplification is that* $A^tA = I_\infty$*, which is equivalent to* $a(1/z)\,a(z) =$ *constant. Otherwise the noise power amplification is greater than one.*

When stationary (shift-register) linear-real decoding is used, then the decoding filter passes the noise through a Z-transform transfer function. When the noise is statistically stationary, the expected value of the mean square of the output noise is stationary, and depends only upon the amplitude of the transfer function averaged over the values of z. However, for burst noise the variance of the mean square of the decoded noise does depend upon the phase of the transfer function. For burst-noise or impulse-noise channels, this variance is minimized if the impulse response from the noise to the analog output of the decoder consists of many small terms instead of a few big ones.

For quantized signals it is important to minimize the variance of noise power because fluctuations above the mean of the variance increase the error rate far more than fluctuations below the mean of the variance decrease it. In order to make the variance of the noise power small, the impulse response from noise to analog output must be near its peak for many times longer than the periods of fluctuation in the noise process.

Because trace and expected value operators commute, the expected value of the output mean square error can be found by substituting $E(N)$ where N appears, provided the noise process is stationary. This cannot be done for error probabilities after the quantizer, however.

4.4 *All-Pass Z-Transforms*

A Z-transform $a(z)$ is defined to be *all-pass* if $|a(z)| =$ *constant for* $|z| = 1$. These are the Z-transform version of two-sided Laplace (or Fourier) transformed all-pass functions. Figure 2 shows some important properties of all-pass Z-transforms, including the fact that $a(z)a(1/z) =$ *constant* is an alternative definition of an all-pass Z-transform. The proofs of relationships in the figure not proved previously are straightforward. The practical implications of these relationships are that all stationary (shift-register) linear-real coders should have Z-transforms which are all-pass, in order not to increase the noise power amplification.

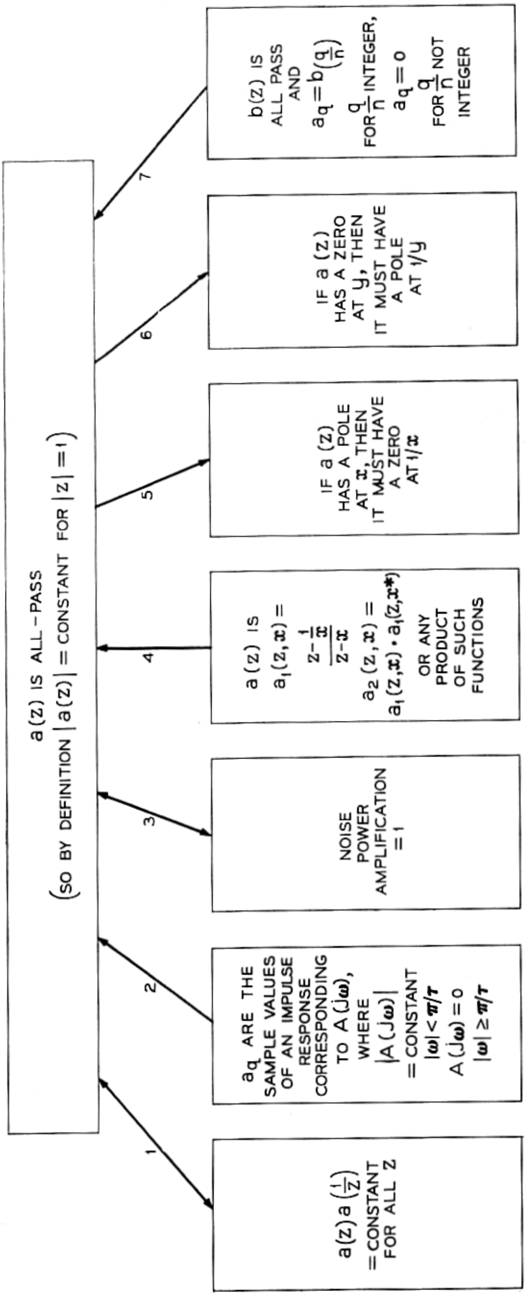

Fig. 2 — Some important relationships for all-pass Z networks.

V. A. Kisel' has made an excellent short study of all-pass Z-transforms, with a view toward using them as phase-correcting networks.[14] He has shown that networks whose Z-transform transfer function are of the form

$$a(z) = \frac{1 + \beta_1 z + \beta_2 z^2 + \beta_3 z^3}{\beta_3 + \beta_2 z + \beta_1 z^2 + z^3}$$

are all-pass, and that Fig. 3 synthesizes such functions. Additional modifications are added to this basic structure and implementations are proposed in the next section.

V. IMPLEMENTATION STUDIES

The decoder for block coding with adaptive mean decoding appears to require a large modern digital computer, and even then it could probably only operate "on line" with a slow channel and a block size not much over one hundred. Further research may lead to A matrices for which $(Q^{-1} + A^t N^{-1} A)$ can be easily inverted for realistic Q and N, or further research may lead to quicker inversion procedures, but with the present techniques, block coding with adaptive mean decoding appears to be decidely less practical than other methods of error control.

The decoder for unadaptive block decoding appears to be generally feasible if certain simplifying techniques are used. The most impor-

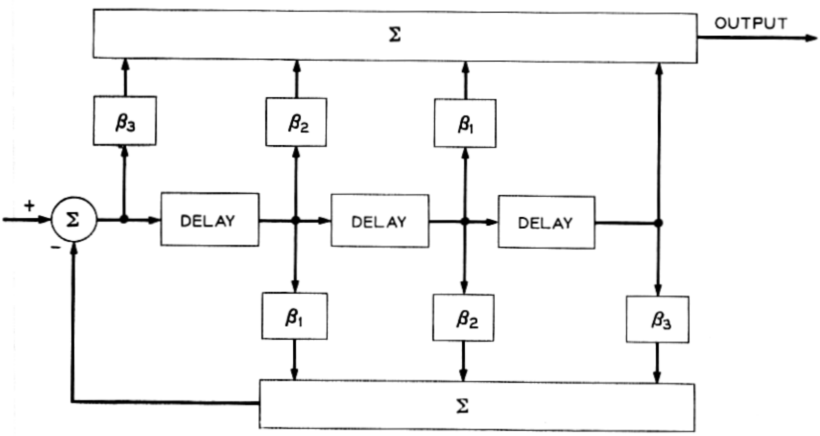

Fig. 3 — A shift register (real-number arithmetic) whose Z-transform transfer function is all-pass. (After V. A. Kisel', with modifications and a correction.)

tant of these is the use of an A matrix which is a permutation matrix* times diag$(A_0, A_0, \ldots, A_0)$, where A_0 is itself a matrix. A_0 must be large enough to give adaquate smear, whereas A must be large enough to make error burst lengths considerably shorter than the length of a code word. The hardware simplification achieved is that the inverse of the small A_0 can be repeatedly applied in time by the same hardware so as to invert the larger A. The practicality of block coding appears to be slightly overshadowed by stationary (shift register) coding, which offers somewhat simpler circuits and freedom from the problem of block synchronization.

Stationary (shift register) coding appears to be the most practical form of linear-real coding. In effect, such coding is a smear-desmear type of signal processing whenever the encoding and decoding filters are inverses of each other and of the all-pass type. The fundamental reason for the practicality of shift register all-pass filters is that accurately tuned shift registers can be relatively inexpensively synthesized, even when the dispersion times are several seconds. This is partly so because the "absolute" tuning of a shift register is determined by the clock pulses and not the precision of the components used in making the register, and partly because the "relative" tuning in a shift register is controlled by gains which in practice can be resistor values. As will be seen, analog shift registers can be implemented digitally, in which case complexity grows only as the logarithm of accuracy. In RLC filter synthesis, in contrast, cost grows rapidly with accuracy.

Figure 4 is a block diagram for coding of the basic stationary (shift register) type. The decoder, because it must handle the analog signals from the channel instead of the digital input signals, is selected to have the impulse response simplest to implement, namely an all-pass causal $1/a(z)$ obtained by a shift register made from a tapped delay line with a relatively moderate number of taps. The encoder is consequently left with approximating the noncausal $a(z)$, which it does with a delay by means of a tapped delay line.

The decoding shift register of Fig. 4 can be implemented by the arrangement of Fig. 5, which is a particular synthesis of the all-pass shift register shown in Fig. 3. In Fig. 5 all the digital-to-analog conversion is done by resistor summing networks. This is relatively inexpensive, although it does require that the flip-flop registers be designed for relatively precise voltage levels on the "on" and "off" states.

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* A permutation matrix is a matrix with a single one in each column and each row; it is always nonsingular.

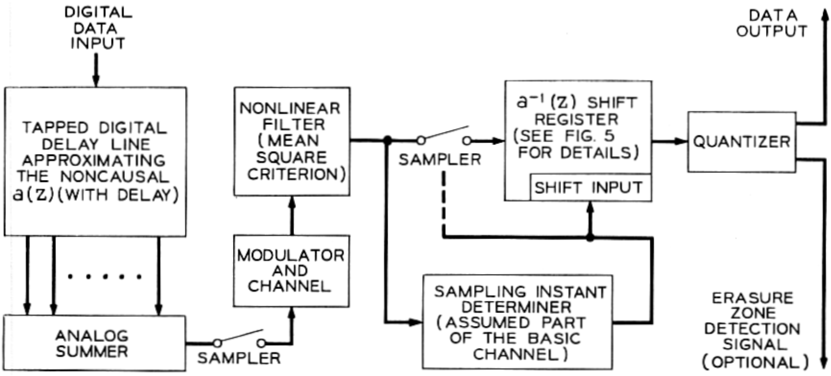

Fig. 4 — One possible general arrangement for unadaptive stationary (shift-register) linear-real encoders and decoders. For multilevel signals, a Gray encoder can be used before the analog summer, and the quantizer would incorporate a Gray decoder.

Notice that in Fig. 5 there is only one analog-to-digital converter, because the analog feedback signal is added to the input signal before the conversion which is necessary in order to place the signals in the digital delay line.

The cost of the encoding and decoding shift registers will be roughly proportional to the amount of smear that they introduce. The amount of smear necessary for given performance depends upon the noise power. It follows that a considerable economic saving can be obtained at given performance if circuits, inexpensive compared to the decoder, can be found to reduce the noise during bursts.

A new circuit with this purpose for PAM systems is as shown in Fig. 6. The operation of the circuit requires that the interval between signal pulses be longer than the Nyquist interval for the bandwidth of the pulse shape. A way to find part of the noise component is to sample at the sampling instants, reconstruct the waveform which would be transmitted if these sample values were the data-signal values, and then subtract this signal from the actual received signal. (For proof of this statement, see appendix C.) An estimate of the instantaneous noise power can be made directly from those noise components which can be found. These components, for example, can be used to deduce the presence or absence of a noise burst. The circuit in Fig. 6 can obtain some noise components,* provided that the taps

* Specifically, Fig. 6 obtains the sample values of $\Delta(t)$ of Appendix C at $t = nT/2$, n integer. Notice that by construction, $\Delta(nT/2) = 0$ for n even. By the sampling theorem, just the samples of $\Delta(t)$ will be sufficient to reconstruct $\Delta(t)$ provided that $C(\omega)$ is zero for $|\omega| \geq 2\,\pi/T$.

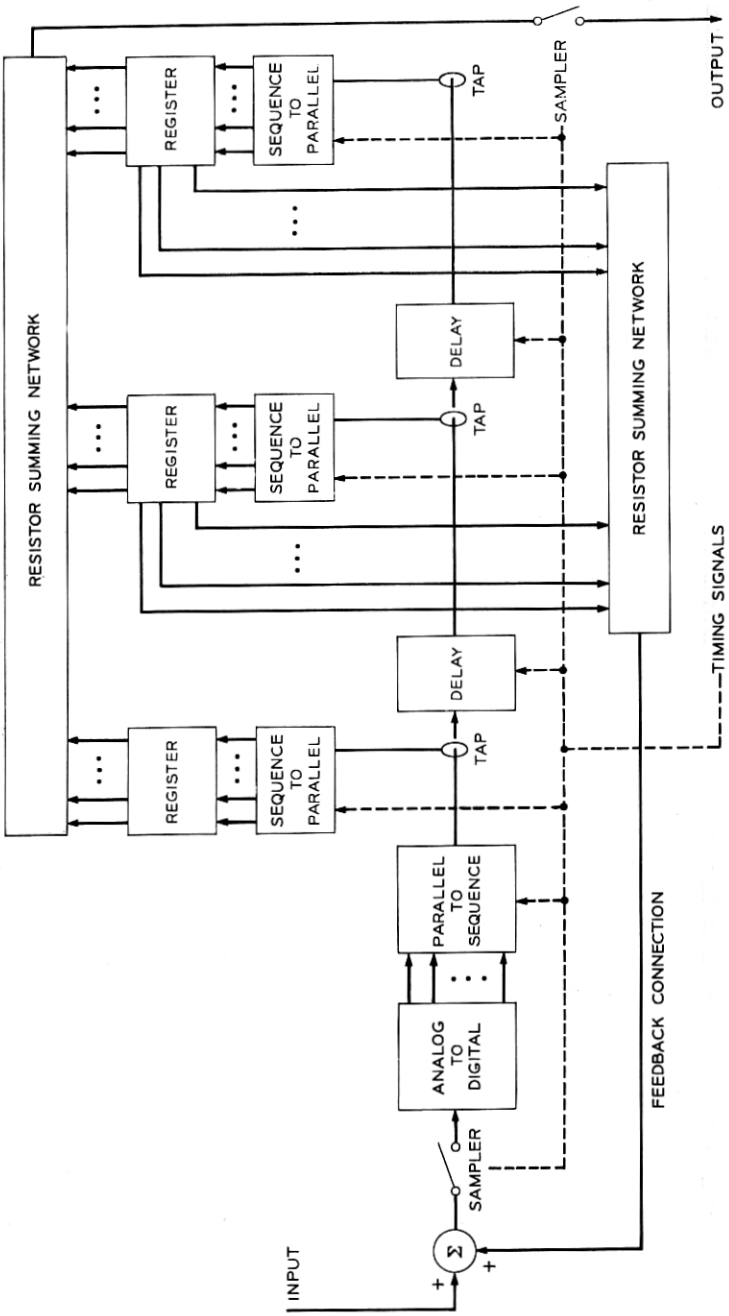

Fig. 5 — A possible arrangement for implementing the decoding shift register.

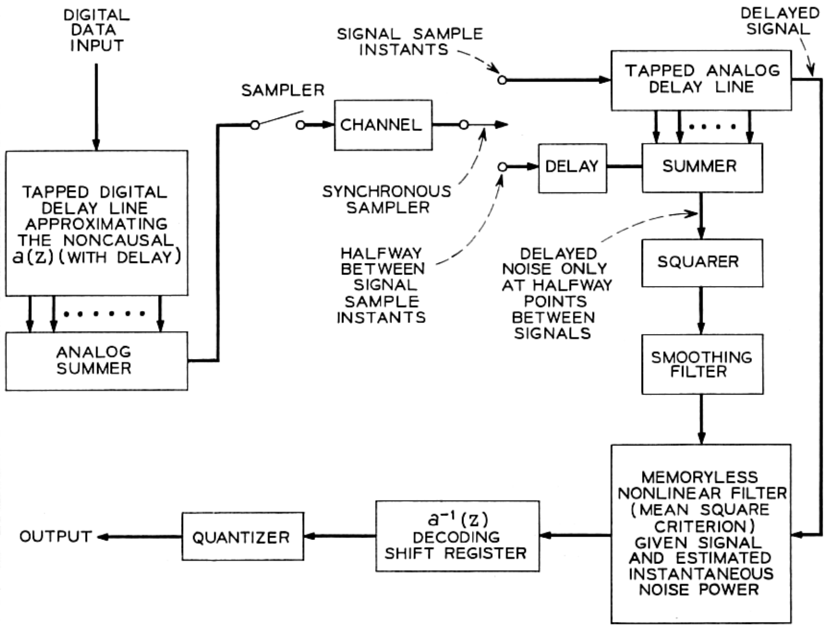

Fig. 6 — A stationary (shift register) coder with an adaptive decoder for PAM channels with white burst noise and pulse rates less than the Nyquist rate.

on the delay line represent the PAM pulse value at $t = nT/2$, n odd.

The output noise estimate (specifically $\Delta(nT/2, n$ odd, in the language of Appendix C and the previous footnote) is then squared to produce the sample variance of the noise; then the sample variance function is put through a smoothing filter, as shown in Fig. 6. The optimization of this filter is complicated by the absence of an appropriate error criterion, but Wiener filtering principles could be used to optimize a mean square criterion. The problem formulation would specify that the sample variance is the true ensemble variance contaminated by small sample-size noise, and that the cross-correlation between the halfway sample process and the sample process could be found from the autocorrelation function of the channel noise.

Finally, a two-input nonlinear memoryless filter is used, also shown in Fig. 6. It is reasonable to optimize this filter using a mean square criterion because in the limit of infinite smearing only the power of the noise will be significant because of the smearing and Gaussianizing effects of the decoding shift register. Some improvement may be possible by using other criteria, but the details appear to be very difficult and are unsolved.

The general scheme of Fig. 6 appears to be the most economical form of linear-real coding when the channel is used for PAM at less than the Nyquist rate. Telephone lines are used at less than the Nyquist rate because they are used with signals with nonsharp-cutoff frequency characteristics. Radio links can obtain information on nontuned burst noise, such as static, by listening on adjacent frequencies, and could therefore provide the smoothed estimate of instantaneous noise power, needed as an input to the two-input memoryless filter, by other means. Instantaneous carrier-to-noise ratios could be used for carrier systems, for example.

It is also possible to use a different principle of instantaneous noise power estimation which does not require a PAM channel used below the Nyquist rate. The other principle uses the quantized structure of the data stream. It is implemented by a decoder with a "pilot" decoder which decodes, followed by an operator which squares the difference between the signal and the nearest quantization level, which is then smoothed and put into a two-input memoryless filter like that of Fig. 6, following which is the regular decoding shift register and quantizer. This scheme is probably less practical than Figs. 4 and 6, but it does give conceptual insights into some of the signal properties which can be used in decoding, especially for burst channels.

VI. COMMENTS AND SIMULATION RESULTS

Any sample of the decoded noise is a weighted sum of the random channel noises at many other sample instants. When the number of terms in this sum approaches infinity and the relative size of the largest term in the sum approaches zero, the central limit theorem applies. It will probably be true that practical designs will not have the conditions of the central limit theorem fulfilled to the extent that very small digital error probabilities can be computed by using integrals of the tails of the gaussian distribution.

Nevertheless, the fact that the decoded noise at any instant is a sum of the random channel noises at many instants will tend to make the decoded noise have some of the characteristics of a gaussian distribution. One characteristic that the decoded noise will have is the small probability that the decoded noise is larger than three or four standard deviations. This effect of the decoding filter (or matrix) will be called the gaussianizing property.

The use of nonlinear filters in conjunction with linear-real coders is extremely effective, since such filters can considerably reduce both the noise power and the probability that the noise has a large peak. By reducing the probability that the noise has a large peak, the desirable gaussian distribution of the decoded noise occurs with smaller matrices, smaller shift-registers, or simpler all-pass filters. In the limit when the decoded noise is actually gaussian, the noise power is the only significant statistic; the higher-order moments of the noise become insignificant due to the gaussian-distributing property of the decoder. It is therefore quite appropriate to design the nonlinear filter using a mean square error criterion, as is done in Section VIII of Reference 9.

Linear-real coding has features which could greatly improve error detection in channels with burst noise. When erasure zones are used to detect errors, the gaussian-distributing property of the decoder greatly increases the ratio of the probability in the erasure zone to the probability beyond the erasure zone. In addition, the noise spreading gives more opportunities for a signal to land in an erasure zone in the presence of impulses or bursts, because of randomness of the decoded noise, and, with suitable designs, because of deterministic reasons.

If the communications channel is, in order, digital processor to analog transmitter to analog receiver to digital processor, then linear-real block coding permits the energy per transmitted data digit to be altered by reprogramming the digital processors, instead of physically retuning bandwidths of analog equipment. Although this option does not in itself affect error control, it perhaps could greatly simplify the implementation of adaptive communications systems in which the signal energy per digit is adjusted to be appropriate for the transmission conditions, message importance, or message load.

A digital computer simulation was run of an additive-noise channel with a linear-real block-code encoder at the input, and several types of decoders at the output. Table III shows the results of the simulation. The listed results are averages. The A matrix is the Hadamard matrix which is generated recursively according to the procedure described by Golomb and his colleagues (p. 55, first paragraph in proof of Theorem 4.5).[15] The N matrix had zeros in all off-diagonal components, and independent random variables on the diagonals, which were 0.3 with probability 0.7 and 8.3 with probability 0.3. In accordance with Theorem 4 in Appendix D, these can be worst-case values which then give the worst-case decoded mean square error.

Once the N matrix was generated, the channel noises were gen-

TABLE III — SIMULATED PERFORMANCE OF LINEAR-REAL CODERS

	MS ERROR IN DECODED COMPONENTS		COMMENTS
	When receiver uses perfect N matrix	When receiver uses $N = \text{diag}(n_1', \cdots, n_n')$ where $n_i' = \max(0.3, f_i^2 - 1)$	
Mean estimator	0.516	0.711	The lower bound of equation (13) is somewhat loose; it gives 0.297.
Unattenuated estimator	2.821	2.821	Equation (12) has correctly predicted that the error would be the same as that of the unadaptive estimator because A is square.
Unadaptive estimator	2.821		Equation (7) averaged over the possible N matrices gives m.s. error of 2.70. The randomness of the N matrix accounts for difference.
Clip estimator parameters (1.2, 0.9, 4.0)	1.805		
Clip estimator parameters (1.0, 0.75, 3.0)	1.152		
Clip estimator parameters (0.8, 0.6, 2.0)	0.771		
Clip estimator parameters (0.6, 0.6, 1.5)	0.677		
Clip estimator parameters (0.5, 0.5, 1.3)	0.645		
Clip estimator parameters (0.4, 0.5, 1.0)	0.649		

Channel: Additive noise channel sending +1 and −1 binary numbers and block encoding with an A which is $k^{-\frac{1}{2}}$ times the first k columns of an n by n Hadamard matrix.

$n = 16.$

$k = 16.$

Number of words in simulation: 10. Noise type: Zero-mean white Gaussian noise has variance 0.3 with probability 0.7 and variance 8.3 with probability 0.3.

erated randomly from a Gaussian distribution having the given N for a covariance matrix. The clip estimator used a decoder which first put each received component through a memoryless nonlinearity, and then decoded the resulting components with the unadaptive estimator. The parameters (x, y, z) indicate that the nonlinearity is a continuous odd function having slope 1 for inputs of magnitude less than x, and slope y for inputs of magnitude between x and z, and slope 0 for inputs of magnitude exceeding z. These parameters can be chosen to approximate the least mean square memoryless nonlinear filter referred to earlier, or they can be found by a trial-and-error procedure with either analysis or simulations to evaluate the resulting error.

The following two conclusions can be drawn from the simulation, but it would not be appropriate to generalize them to cases of nonsquare A matrices:

(*i*) For intermittent additive impulse noise of the type simulated, the simple clip estimator scheme, for appropriate parameters, is almost as good as the mean estimator, even though it is unadaptive and therefore requires only a simple receiver.

(*ii*) The use of rather crude algorithms for generating an estimate of N appeared to be inferior to clip estimator decoding with appropriate parameters.

ACKNOWLEDGMENT

I am grateful to Dr. Sidney Berkowitz for discussions, and the substitution appearing in equation (31). Many astute and helpful suggestions from the referees have been incorporated into the paper.

APPENDIX A

Justification of Table 2 Equations

A.1 *Unadaptive Estimator*

In the case of the unadaptive estimator $TA = I_{(k \times k)}$, so equation (3) reduces to equation (4) shown in Table II. Now in general, when M is the covariance matrix of the decoded noise, the mean square error will be the average of the on-diagonal terms of M, or in other words, $(1/k) tr\ M$. In this way (5) follows from (4). Equation (6) follows from (4) because $A^t A = (n/k) I_{(k \times k)}$ in this case.

A Hadamard matrix is a square matrix with $+1$ or -1 elements and orthogonal columns. (Golomb and his associates fully describe Hadamard matrices and their application to binary block codes.[15])

In deriving equation (7), a straightforward evaluation of (6) under the assumption of diagonal N gives the result that

$$M_{ij} = \left(\frac{k}{n}\right)^2 \sum_{l=1}^{n} a_{li} a_{lj} n_l .$$

assuming:

T is the unadaptive estimator

$$N = \text{diag}\,(n_1, n_2, \cdots, n_n).$$

The on-diagonal terms of the above can be evaluated by using the Hadamard assumption, which causes $(a_{li})^2$ to equal $1/k$ for all l and i. This gives

$$M_{ii} = \left(\frac{k}{n}\right)\left[\frac{1}{n} \sum_{l=1}^{n} n_l\right]$$

assuming:

T is the unadaptive estimator

$N = \text{diag}\,(n_1, n_2, \cdots, n_n)$

A is $1/(k)^{\frac{1}{2}}$ times the first k columns of any Hadamard matrix.

Notice that the term in brackets is AM, the arithmetic mean of the set $(n_1, n_2, \ldots, n_n)$.

A.2 *Unattenuated Estimator*

Equation (8) comes from (3) by direct substitution for the T matrix.

A.3 *Mean Estimator*

In the case of the mean estimator,

$$\begin{aligned} I_{(k\times k)} - TA &= I_{(k\times k)} - (Q^{-1} + A^t N^{-1} A)^{-1} A^t N^{-1} A \\ &= I_{(k\times k)} - (Q^{-1} + A^t N^{-1} A)^{-1} (A^t N^{-1} A + Q^{-1} - Q^{-1}) \\ &= (Q^{-1} + A^t N^{-1} A)^{-1} Q^{-1}. \end{aligned} \tag{31}$$

Substituting $(Q^{-1} + A^t N^{-1} A)^{-1} Q^{-1}$ for $(I_{(k\times k)} - TA)$ in equation (3) readily shows that

$$M = (Q^{-1} + A^t N^{-1} A)^{-1}$$

assuming T is the mean estimator.

A.4 *Joint Mean and Unattenuated Estimator*

The unattenuated estimator is the special case of the mean estimator when $Q^{-1} \to 0$. It is convenient to handle the two cases together by using the variable $\Omega = 1$ when the mean estimator is used, and $\Omega = 0$ when the unattenuated estimator is used.

The next two equations use an approach from Berkowitz.[10] Equation (9) or (11) can be simplified by using the fact that, for any nonsingular Z,

$$\operatorname{tr} Z^{-1} = \sum_i \frac{1}{\lambda_i(Z)}$$

where $\lambda_i(Z)$ denotes the i^{th} unordered eigenvalue of Z. The result is equation (12). When the signal is white, the relation $\lambda_i(\tau I + Z) = \tau + \lambda_i(Z)$ can be used, giving the equality in equation (13). When $\Omega = 1$ the positive semidefiniteness of $A^t N^{-1} A$ causes its eigenvalues to be real and nonnegative; when $\Omega = 0$ the positive definiteness of $A^t N^{-1} A$ will now need to be assumed. Because $1/(\Omega s^{-1} + \lambda)$ is a convex upward function of λ in the region of possible λ, the inequality part of (13) follows by convexity. This inequality will prove useful later when—under additional assumptions—the term in brackets will be found in closed form.

For square orthonormal A, it follows that $A^{-1} = A^t$, so

$$\lambda_i(A^t N^{-1} A) = \lambda_i(A^{-1} N^{-1} A) = \lambda_i(N^{-1}) = \frac{1}{\lambda_i(N)}.$$

Equation (14) results when the above is substituted into (13). Notice that when Ω is zero and N is diagonal, this will reduce to AM. On the other hand, when Ω is one, this will be less than AM.

When A is rectangular, the next analysis leads to a closed form solution for the average of the eigenvalues of $A^t N^{-1} A$, under the assumptions of equation (15), and it also leads to upper bounds upon the m.s. error. The exact values of the components of A may enter into the formulas for some statistics of the error. However, in the first and second moment statistics to be investigated under the particular assumptions made, it turns out that the only important property of the A matrix is the inner product between the i^{th} and j^{th} columns. This will always be $(n/k)\ \delta_{ij}$, independent of the particular Hadamard matrix upon which A is based. However, since higher-order moments are significant, especially in quantized channels, it is likely that some Hadamard matrices might be more useful for practical purposes than others.

Under the assumptions of equation (15), straightforward calculations will show the following. HM is the harmonic mean of the diagonal components of N; equation (15) includes its formula.

$$(i) \qquad A^tN^{-1}A = \frac{n}{k\,HM} I + Y_k$$

where, for large k

$$\left.\begin{aligned} E[(Y_k)_{ij}] &= 0 \\ E[(Y_k)_{ij}^2] &= \frac{\sigma^2 n}{k^2} \end{aligned}\right\} \quad \text{all } i, j \tag{32}$$

$$(ii) \qquad E\left[\frac{1}{k} \operatorname{tr} A^tN^{-1}A\right] = \frac{n}{k\,HM}. \tag{33}$$

$$(iii) \qquad \operatorname{Var}\left[\frac{1}{k} \operatorname{tr} A^tN^{-1}A\right] = \frac{n}{k^2}\sigma^2. \tag{34}$$

The above equations are especially useful because they show that

$$\textit{the average of the eigenvalues of } A^tN^{-1}A = \frac{n}{k\,HM}.$$

This can be substituted into equation (13) to prove equation (16). Equation (16) becomes an equality when all of the eigenvalues of $A^tN^{-1}A$ are equal; otherwise the mean square error is greater.

Because the m.s. error evaluated according to equation (12) requires the computation of eigenvalues of typically a rather large matrix, or the trace formula of (9) or (11) yields little insight, and because the bound of equation (16) is a simple closed-form equation, the question arises of whether the bound given by (16) is really close enough to be used for design and analysis purposes as an equality. The analysis which follows will derive an upper bound for the m.s. error, which could be used to develop some sufficient conditions for near equality of equation (16)

Let equation (32) be used to define Y_k, let $\lambda'(Y_k)$ denote

$$\max_i |\lambda_i(Y_k)| ,$$

and let τ be any number such that

$$\tau \leqq \frac{\sum_{i=1}^{k} |\lambda_i(Y_k)|^2}{[\lambda'(Y_k)]^2}. \tag{35}$$

Notice that τ can always be as large as 1 and never exceeds k. The second of the following inequalities is Schur's inequality, which is valid for any square Y_k.[16] The first comes from (35).

$$\tau[\lambda'(Y_k)]^2 \leqq \sum_{i=1}^{k} |\lambda_i(Y_k)|^2 \leqq \sum_{i=1}^{k} \sum_{j=1}^{k} |(Y_k)_{ij}|^2. \tag{36}$$

Assuming that k is large enough for the weak law of large numbers to hold permits (32) to be used to evaluate the above double sum, so that with a few manipulations (36) reduces to

$$\lambda'(Y_k) \leqq \sqrt{\frac{\sigma^2 n}{\tau}}. \tag{37}$$

By using equations (13), (32), (33), and (37), and a relatively obvious property of convex functions,* equation (17) is established.

APPENDIX B

Relating Teoplitz Matrix Operations with Z-Transforms

Theorem 2: If

$$A_{ij} = a_{i-j}$$

and if

$$a(z) = \sum_{q=-\infty}^{\infty} a_q z^{-q}$$

converges on $|z| = 1$ *and has no poles or zeros for a finite distance from* $|z| = 1$*, then* A^{-1} *exists and*

$$a^{-1}(z) = \frac{1}{a(z)}.$$

Proof: Let

$$b(z) = \frac{1}{a(z)} \quad \text{for} \quad |z| = 1.$$

* The property is that if $f(x)$ is convex downward, and

$$\sum_{i=1}^{k} x_i = 0, \qquad \max_i |x_i| \leqq R,$$

then

$$\sum_{i=1}^{k} \frac{1}{k} f(u + x_i) \leqq \tfrac{1}{2} f(u - R) + \tfrac{1}{2} f(u + R).$$

The assumptions on $a(z)$ cause a_q and b_q to have geometrical decay, and therefore the following converge absolutely:

$$(BA)_{ij} = \sum_{q=-\infty}^{\infty} B_{iq}A_{qj} = \sum_{q=-\infty}^{\infty} b_{i-q}a_{q-j} .$$

Also reducing to the above is $(AB)_{ij}$. Letting $q' = q - j$ gives

$$(BA)_{ij} = (AB)_{ij} = \sum_{q'=-\infty}^{\infty} b_{(i-j)-q'}a_{q'} = b_q * a_q \,|_{i-j}$$

where the * denotes the convolution sum in the line above. Because $b(z)a(z) = 1$, it follows that

$$bq * a_q \,|_{i-j} = \delta_{i,j} .$$

So $BA = AB = I_\infty$, thus proving that B is the inverse of A, which completes the proof.

The following have proofs similar to that of the theorem.

Lemma 3: If A and B are Toeplitz, then $C = AB$ is Toeplitz with

$$c(z) = a(z)b(z).$$

Lemma 4: The half-power of a Toeplitz matrix N can be defined by

$$n^{\frac{1}{2}}(z) = \sqrt{n(z)}.$$

The following has a straightforward proof:

Lemma 5: If A is Toeplitz, then A^t is Toeplitz and $a^t(z) = a(1/z)$.

The following relates linear-real coding for Toeplitz matrices with Wiener filtering.

Theorem 3: When A, Q, and N are infinite Toeplitz, then the least mean square estimator

$$T = (Q^{-1} + A^tN^{-1}A)^{-1}A^tN^{-1}$$

is the infinite Toeplitz, and the noncausal Wiener filter, given by

$$t(z) = \frac{q(z)a\left(\frac{1}{z}\right)}{n(z) + a\left(\frac{1}{z}\right)q(z)a(z)}.$$

Proof: By Theorem 2 and Lemmas 3 and 5,

$$t(z) = \frac{1}{\dfrac{1}{q(z)} + \dfrac{a\left(\frac{1}{z}\right)a(z)}{n(z)}} \cdot a\left(\frac{1}{z}\right) \cdot \frac{1}{n(z)}.$$

This equals the stated result, which completes the proof.

Corollary 3: When $A = I_\infty$

$$T = (Q^{-1} + N^{-1})^{-1}N^{-1}$$

and

$$t(z) = \frac{q(z)}{q(z) + n(z)}$$

is the noncausal Wiener filter.

The following proof of equation (29) and statement (30) follows the ideas of J. E. Mazo. For square A,

$$\text{tr}\,[A^{-1}(A^{-1})^t] = \text{tr}\,[(A^{-1})^t A^{-1}],$$

since in general $tr\ HC = tr\ CH$ for square H and C. Now let $B = AA^t$. Notice that $(A^{-1})^t A^{-1}$ is B^{-1}. Equation (29) is then:

$$\frac{1}{k^2}\,\text{tr}\,B^{-1}\,\text{tr}\,B \geqq 1.$$

But

$$\text{tr}\,B = \sum_{i=1}^{k} \lambda_i(B)$$

$$\text{tr}\,B^{-1} = \sum_{i=1}^{k} \frac{1}{\lambda_i(B)}$$

so

$$\frac{1}{k^2}\,\text{tr}\,B^{-1}\,\text{tr}\,B = \frac{\dfrac{1}{k}\displaystyle\sum_{i=1}^{k} \lambda_i(B)}{\left[\dfrac{1}{\dfrac{1}{k}\displaystyle\sum_{i=1}^{k}\dfrac{1}{\lambda_i(B)}}\right]}.$$

The numerator and denominator are respectively the arithmetic and harmonic means of the eigenvalues of the B matrix. Hardy, Littlewood, and Polya (p. 26, special case of 2.9.1) show that this ratio always exceeds one, except when the eigenvalues are all the same, in which case it is one.[17] This proves (29).

At equality B has equal eigenvalues, and since it is symmetric the eigenvectors span the space and B is proportional to an orthogonal matrix:

$$
\begin{aligned}
B = \lambda U &= P^{-1}(\lambda I)P \\
&= P^{t}(\lambda I)P \\
&\qquad \text{because } B \text{ is symmetric} \\
&= \lambda I.
\end{aligned}
$$

Therefore $AA^t = \lambda I$ and $A^{-1} = \lambda A^t$, so A is proportional to an orthogonal matrix at equality, thereby establishing (30) and completing the proof of (29) and (30), thereby completing the proof of Theorem 1.

APPENDIX C

Finding Noise Component

In the text we discuss the circuit shown in Fig. 6 and state that a way to find part of the noise component is to sample at the sampling instants, reconstruct the waveform which would be transmitted if these sample values were data-signal values, and then subtract this signal from the actual received signal.

The proof of this statement requires the use of the valid converse of the sampling theorem, which states that an arbitrary function with frequency components out to $|\omega| = \pi/T_1$ cannot be reconstructed from samples every T seconds if $T > T_1$. If it is assumed that

(*i*) $h(O) = 1$
(*ii*) $h(nT) = 0$
(*iii*) $H(\omega)$ is nonzero for $|\omega| < \pi/T_1$
(*iv*) $T_1 < T$
(*v*) The additive noise $c(t)$ has components at all frequencies for which $H(\omega)$ has components,

then it follows that

$$\text{actual sample at } t = c(t) + \sum_n r_n h(t - nT)$$

predicted sample at t based upon

samples at nT, $n = 0, \pm 1, \pm 2, \cdots = \sum_n [r_n + c(nT)]h(t - nT)$

$\Delta(t)$ = difference of the above $= c(t) - \sum_n c(nT)h(t - nT)$.

By using a well-known result in sampling theory[12] the Fourier transform of $\Delta(t)$ can be written as either of the following.

$$\boldsymbol{\Delta}(\omega) = \mathfrak{F}[\Delta(t)] = C(\omega) - H(\omega) \sum_n c(nT)e^{-i\omega nT}$$

$$= C(\omega) - H(\omega) \sum_n C\left(\omega - \frac{n2\pi}{T}\right).$$

By the converse to the sampling theorem, no $H(\omega)$ will make $\boldsymbol{\Delta}(\omega)$ zero for all ω. Consequently, $\boldsymbol{\Delta}(\omega)$ contains some components of the additive noise. If $T_1 \geqq T/2$, then the direct sampling theorem shows that samples every $T/2$ are sufficient to reconstruct $\Delta(t)$.

APPENDIX D

The purpose of this appendix is to state and prove the following theorem.

Theorem 4: Assuming

(*i*) *Channel I has additive noise c independent of the signal b*
(*ii*) *Channel II has additive noise g independent of the signal b*
(*iii*) *c and g are zero mean, and each is even about its mean*
(*iv*) $F(\alpha) = p(|c| \leqq \alpha)$, *(α is defined to be nonnegative)*
(*v*) $K(\alpha) = p(|g| \leqq \alpha)$
(*vi*) *In both channels signal plus noise are passed through the memoryless nonlinearity nl( ) at the receiver*
(*vii*) *nl(x) is odd*
(*viii*) *nl(x) has a slope bounded between* 0 *and* 1 *for all x, and this slope is monotonically decreasing in* $|x|$
(*ix*) *The mean square errors of channels I and II are* MSE_I *and* MSE_{II}, *respectively.*
(*x*) *Channel I is noisier than channel II in the sense that* $F(\alpha) \leqq K(\alpha)$ *for all α, which means that for every bit of probability density c has at* $\pm\beta$, *g has an equal amount at a distance which is at least* $\pm\beta$,

then

$$MSE_I \geqq MSE_{II}.$$

(Thus, worst-case noise gives worst-case results with these nonlinearities.)

The next definition and lemma are used in the proof of Theorem 4.

Let $MS(\alpha)$ denote the special case of MSE_I of Theorem 4 when

$$p_2(c) = \tfrac{1}{2}\delta(c + \alpha) + \tfrac{1}{2}\delta(c - \alpha) \tag{38}$$

where α is a positive constant, and $\delta(\ )$ denotes the Dirac impulse function.

Lemma 6: Under the conditions of Theorem 4, $(\partial MS(\alpha)/\partial\alpha) \geq 0$.

Proof:

$$MS(\alpha) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [nl(b + c) - b]^2 p_1(b)p_2(c)\, dc\, db. \tag{39}$$

Substituting equation (38) for $p_2(c)$, integrating with respect to c, and then taking partial derivations with respect to α gives

$$\frac{\partial MS(\alpha)}{\partial\alpha} = \int_{-\infty}^{\infty}\Biggl\{\underbrace{[nl(b + \alpha) - b]}_{A}\underbrace{\frac{dnl(x)}{dx}\bigg|_{b+\alpha}}_{B} - \underbrace{[nl(b - \alpha) - b]}_{C}\underbrace{\frac{dnl(x)}{dx}\bigg|_{b-\alpha}}_{D}\Biggr\}p_1(b)\, db. \tag{40}$$

Now

$$[\text{assumptions 7, 8}] \Rightarrow [C \leqq 0 \text{ for } b \geqq 0,\ A \leqq 0 \text{ for } b \leqq 0] \tag{41}$$

$$[\text{assumption 8}] \Rightarrow [D \geqq 0,\ B \geqq 0] \tag{42}$$

$$[\text{assumption 8}] \Rightarrow [B \leqq D \text{ when } b \geqq 0,\ B \geqq D \text{ when } b \leqq 0]. \tag{43}$$

Therefore

$$-CD \geqq -CB \quad \text{when} \quad b \geqq 0 \tag{44}$$

$$AB \geqq AD \quad \text{when} \quad b \leqq 0. \tag{45}$$

Consequently

$$\frac{\partial MS(\alpha)}{\partial\alpha} \geqq \int_{-\infty}^{0-} [(A - C)D]p_1(b)\, db + \int_{0}^{\infty} [(A - C)B]p_1(b)\, db. \tag{46}$$

Now

$$A - C = \int_{b-\alpha}^{b+\alpha} \frac{dnl(x)}{dx}\, dx \tag{47}$$

and by assumption 8 the integrand is nonnegative, so both sides of (47) are nonnegative. This fact and (42), and the nonnegativeness of $p_1(b)$, make the right side of (46) nonnegative, which proves the lemma.

Proof of Theorem 4: The assumed evenness of the noises, and the linearity of the expectation operator, permit the $MS(\alpha)$ function to be used to evaluate the mean square error, as follows

$$MSE_{\text{I}} - MSE_{\text{II}} = \int_0^\infty MS(\alpha)\, dF(\alpha) - \int_0^\infty MS(\alpha)\, dK(\alpha). \qquad (48)$$

The above right side can be combined into one integral, such that integrating by parts gives zero for the end conditions plus the resulting integral.

$$MSE_{\text{I}} - MSE_{\text{II}} = \int_0^\infty [K(\alpha) - F(\alpha)] \left\{ \frac{dMS(\alpha)}{d\alpha} \right\} d\alpha. \qquad (49)$$

Assumption 10 makes the bracketed term nonnegative, whereas Lemma 6 makes the braced term nonnegative, so the right side of (49) when integrated is nonnegative, which proves the theorem.

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