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Realizability Conditions for the Impedance Function of the Lossless Tapered Transmission Line

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In the study of tapered transmission lines or accoustical horns, an unsolved problem of great practical interest is the determination of the taper function (inductance or capacitance per unit length as a function of distance; it is assumed that the product of these quantities is unity) for the structure which will possess a prescribed driving point impedance function. For the case where the structure may be modeled by a cascade of sections of *uniform* transmission line segments, physical realizability conditions and a synthesis procedure have been given by B. K. Kinariwala.[1] For the case of continuous taper no results of a general nature are known.

In this note, we shall give an almost complete characterization of driving point impedances for structures possessing once continuously differentiable taper functions. Although the proof of the realizability theorem will not be given here, the author wants to point out that the sufficiency is, in fact, proved by a construction of the taper function. However, this construction is too unwieldy to be of practical use.

The mathematical formulation of the problem is as follows.

Suppose that for all complex s $y(x,s)$, $0 \leqq x \leqq l$ is the solution of the Horn equation

$$(c(x)y'(x,s))' = s^2 c(x)y(x,s) \tag{1}$$

satisfying the boundary condition

$$y(0,s) = -i, \qquad y'(0,s) = \frac{is}{c(0)}.^{*}$$

If by driving point impedance we mean the function

$$Z(s) = -\frac{s}{c(l)} \frac{y(l,s)}{y'(l,s)},$$

then the following theorem is true.

Necessity: If $c(x)$ is a positive real function continuously differentiable on $0 \leqq x \leqq l$ then

* Unit terminating resistance at zero is assumed.

(*i*) $Z(s)$ is a positive real function.

(*ii*) There exist entire functions of the exponential type* l, $N_i(s)$, $D_i(s)$, $i = 1, 2$, such that

$$(a)\ Z(s) = \frac{N_1(s) + N_2(s)}{D_1(s) + D_2(s)},$$

$$(b)\ N_1D_1 - N_2D_2 \equiv e^{2ls},$$

$$(c)\ N_1(s) = N_1(-s),\ D_1(-s) = D_1(s)$$
$$N_2(s) = -N_2(-s),\ D_2(s) = -D_2(-s).$$

(*iii*) If for real ω

$$f(\omega) = \operatorname{Re} e^{-2il\omega} Z(i\omega) \quad \text{or} \quad \operatorname{Re} e^{-2il\omega} \frac{1}{Z(i\omega)}$$

then the function $f(\omega)$ has an asymptotic expansion† at $\pm\infty$ of the following kind

$$f(\omega) \approx 1 + \frac{a}{\omega^2} + \frac{b}{\omega^4} + \frac{c}{\omega^6} + 0\left(\frac{1}{\omega^7}\right)$$

(The constants of course may be different).

Sufficiency: In order that a complex function $Z(s)$ be the driving point impedance of the differential equation (1) for some continuously differentiable positive taper function $C(x)$ it is sufficient that

(*i'*) $Z(s)$ be positive real,

(*ii'*) there exist complex functions $N_i(s)$, $D_i(s)$, $i = 1, 2$ of the exponential type satisfying (2) (*a*), (*b*), (*c*), and

(*iii'*) the function $f(\omega)$ defined in (3) have asymptotic expansion at infinity

$$f(\omega) \approx 1 + \frac{a}{\omega^2} + \frac{b}{\omega^4} + \frac{c}{\omega^6} + \frac{d}{\omega^8} + 0\left(\frac{1}{\omega^9}\right).$$

Remarks: (*i*) It is conjectured that the existence of the two asymptotic expansions are not independent, that is (*i'*), (*ii'*), and one expansion may be sufficient.

(*ii*) A similar result is probably valid for infinite transmission lines.

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* The function $h(s)$ is called exponential type l if $e^{-lr}M(r)$ remains bounded for all $r > 0$ but for any $l' < l$ $e^{-l'r}M(r)$ grows to infinity. Here $M(r) = \max_{|s| \leqq r} |h(s)|$.

† This means that $\lim_{\omega=\pm\infty} \omega^2 (f(\omega) - 1) = a$,

$$\lim_{\omega=\pm\infty} \omega^4 \left(f(\omega) - 1 - \frac{a}{\omega^2}\right) = b, \text{ etc.}$$

Substituting the words "functions of order unity" [2] for "functions of the exponential type l" should yield the correct theorem.

(*iii*) As a last conjecture we offer the following. Let $f(\omega)$ be a positive even function possessing the properties

(*a*) $f(\omega)$ has an asymptotic expansion at infinity as in (*iii'*).

(*b*) The Fourier transform of $1/f(\omega) - 1$ vanishes outside the interval $(-2l, 2l)$.

Then there exists a unique taper function such that if $Z(s)$ is the impedance function of the associated differential equation (1) then

$$\text{Re } Z(i\omega)e^{-2i\omega l} = \frac{1}{f(\omega)}.$$

REFERENCES

1. Kinariwala, B. K., Theory of Cascaded Structures: Lossless Transmission Lines, B.S.T.J., *45*, April, 1966, pp. 631–650.
2. Titchmarsh, *Theory of complex functions*, Oxford Univ. Press, 1937.