A Stability Criterion for Nonuniformly Sampled and Distributed Parameter Systems

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This paper presents a stability criterion for distributed parameter and uniformly or nonuniformly sampled systems. Specifically, a finite algorithm is presented which tests whether all the zeros of a function of the form

$$F(s) = \sum_{n=0}^{N} c_n e^{su_n} ,$$

lie within the interior of the left half s-plane. Thus, the algorithm tests the stability of those systems whose system functions are ratios of finite sums of exponentials. Included in such systems are all distributed systems whose components are uniform, lossless transmission lines and all sampled systems with a periodically varying sampling rate.

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$$F(s) = \sum_{n=0}^{N} c_n e^{su_n} , \qquad \begin{pmatrix} c_n \text{ and } u_n \text{ real} \\ u_0 = 0 \\ u_{n+1} > u_n \end{pmatrix} \tag{1}$$

lie within the interior of the left half s-plane. Thus, the algorithm tests the stability of those systems whose system functions are ratios of finite sums of exponentials. Included in such systems are all distributed systems whose components are uniform, lossless transmission lines and all sampled systems with a periodically varying sampling rate.

Criterion: $F(s)$ is of type L_I (i.e., it has all its zeros in the interior of the left half s-plane) if and only if

$$\Psi_o(s) = \frac{F(s) - F(-s)e^{su_N}}{F(s) + F(-s)e^{su_N}} = \frac{\sum_{n=0}^{m} a_n e^{sx_n}}{\sum_{n=0}^{m} b_n e^{sx_n}}, \qquad \begin{pmatrix} x_0 = 0 \\ x_{n+1} > x_n \end{pmatrix} \tag{2}$$

reduces to zero/K, (K a constant), under the repeated application of the following

Algorithm:

Given

$$\Psi_{k-1}(s) = \frac{\sum_n a_n e^{sx_n}}{\sum_n b_n e^{sx_n}}, \qquad \begin{pmatrix} x_0 = 0 \\ x_{n+1} > x_n \end{pmatrix}.$$

(*i*) Terminate the algorithm unless $0 < R_k < \infty$ where

$$R_k = -\frac{a_0}{b_0}.$$

(*ii*) Using

$$a_{nu} = R_k b_{nu} = a_n + R_k b_n$$

$$a_{nv} = -R_k b_{nv} = a_n - R_k b_n$$

decompose

$$\Psi_{k-1}(s) = \frac{\sum_n a_{nu} e^{sx_n} + \sum_n a_{nv} e^{sx_n}}{\sum_n b_{nu} e^{sx_n} + \sum_n b_{nv} e^{sx_n}}.$$

(*iii*) Identify the lowest x_n with nonzero a_{nu} as τ_k.
(*iv*) Obtain

$$\Psi_k(s) = \frac{\sum_n a_{nu} e^{s(x_n - \tau_k)} + \sum_n a_{nv} e^{sx_n}}{\sum_n b_{nu} e^{s(x_n - \tau_k)} + \sum_n b_{nv} e^{sx_n}}.$$

The algorithm is repeated for $k = 1, 2, \cdots, M$ until it terminates. The number of steps M is always finite.

Proof: The following three observations based upon the results of Ref. 1 lead to the above criterion.

(*i*) $F(s)$ is of type L_I if and only if $\Psi_o(s)$ is a positive-real function of s.

The Hadamard factorization theorem[2] shows that $e^{su_N}F(-s)/F(s)$ is analytic in the right half s-plane and it is bounded by one on the imaginary axis. Hence, $\Psi_o(s)$ is a positive-real function of s. Alternatively, for real problems (i.e., u_n rational) a mapping $\lambda = \tanh s\tau$ can be used to reduce Ψ_o to a rational function which is positive-real in λ and the conclusion follows.[1]

(*ii*) An odd function of the form (2) is positive-real if and only if it is the impedance function of a SCULL, i.e., a short-circuited cascade of uniform, lossless transmission lines.[1]

This is a special case of the realizability theorem in Ref. 1.

(*iii*) A function of the form (2) is the impedance function of a SCULL if and only if it reduces to (zero/a const.) under a repeated application of the above specified algorithm.[1]

This is actually a synthesis algorithm for the short-circuited cascade structure.

Q.E.D.

Example:

$$F(s) = 10e^{14.3s} - 2e^{8.2s} - e^{6.1s} + 5$$

$$\Psi_o(s) = \frac{5e^{14.3s} - e^{8.2s} + e^{6.1s} - 5}{15e^{14.3s} - 3e^{8.2s} - 3e^{6.1s} + 15}$$

$$\Psi_1(s) = \frac{6e^{6.1s} - 6}{12e^{6.1s} + 12}$$

$$\Psi_2(s) = \frac{\text{zero}}{24}.$$

Hence, $F(s)$ is of type L_I and the associated system is stable.

REFERENCES

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2. Titchmarsh, E. C., *The Theory of Functions*, Oxford University Press, London, 1939, Chapter VIII, p. 250.