improvements in the system; and A. R. Lingenfelter, who was responsible for recording and reducing the data.

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A Note on a Type of Optimization Problem that Arises in Communication Theory

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A problem that has arisen[1] in connection with the use of transversal filters to reduce the effect of intersymbol interference in digital communication systems is to determine a real N-vector $\boldsymbol{c} \triangleq (c_1, c_2, \cdots, c_N)$ such that, with $n_0 \,\varepsilon\, \mathfrak{F} \triangleq \{1,2, \cdots, N\}$,

$$\sum_{\substack{n=-\infty \\ n\neq n_0}}^{\infty} \left| \sum_{j \varepsilon \mathfrak{F}} c_j x_{n-j} \right| \tag{1}$$

is minimized subject to the constraint

$$1 = \sum_{j\varepsilon\mathfrak{F}} c_j x_{n_0-j} \,. \tag{2}$$

Here $\{x_n\}_{-\infty}^{\infty}$ denotes a set of real constants such that $| x_0 | > \sum_{n\neq 0} | x_n |$. Lucky[1] has proved the interesting theorem that the optimal choice of $\boldsymbol{c}$ coincides with the unique solution[2] of the equations

$$\begin{aligned} 1 &= \sum_{j\varepsilon\mathfrak{F}} c_j x_{n_0-j} \\ 0 &= \sum_{j\varepsilon\mathfrak{F}} c_j x_{n-j} \,, \qquad n \,\varepsilon\, \mathfrak{F} - \{n_0\}. \end{aligned} \tag{3}$$

The proof of Ref. 1 consists of establishing a contradiction to the assertion that (1), with c_{n_0} eliminated with the aid of (2), is minimized for some c for which (3) is not satisfied. The reader is referred to Ref. 1 for the details.

The purpose of this note is to show that Lucky's result, and far more general results of similar type, can be directly deduced from the following proposition.

Proposition: Let f^* and $\mathcal{R}(f^*)$, respectively, denote an abstract element and a set such that $f^* \not\in \mathcal{R}(f^*)$. Let $\mathcal{S} \triangleq \{f^*\} \cup \mathcal{R}(f^*)$, and let Q denote a mapping of $\mathcal{S}$ into the set of nonnegative numbers. Let $\mathcal{S}_0$ denote a normed linear space with norm $\| \cdot \|$, and let R denote a mapping of $\mathcal{S}$ into $\mathcal{S}_0$. Let

$$\sigma(f) \triangleq Qf + \| Rf \|$$

for all $f \in \mathcal{S}$. Suppose that

(*i*) $Qf^* = 0$

(*ii*) for all $g \in \mathcal{R}(f^*)$,

$$Qg \geqq \| Rg - Rf^* \| . \tag{4}$$

Then for all $f \in \mathcal{S}$,

$$\sigma(f) \geqq \sigma(f^*) \tag{5}$$

and, if (4) holds with strict inequality for all $g \in \mathcal{R}(f^*)$, then (5) holds with strict inequality for all $f \in \mathcal{S}$ except $f = f^*$.

Proof: Let $f \in \mathcal{R}(f^*)$. Then

$$\begin{aligned} \sigma(f) - \sigma(f^*) &= Qf + \| Rf \| - Qf^* - \| Rf^* \| \\ &= Qf + \| Rf \| - \| Rf^* \| \\ &\geqq Qf - \| Rf - Rf^* \| , \end{aligned}$$

from which the validity of the proposition is evident.

An Application of the Proposition

For each $j \in \mathfrak{F} \triangleq \{1,2, \cdots , N\}$, let $\{x_{nj}\}_{-\infty}^{\infty}$ denote a set of real numbers such that $| x_{jj} | > \sum_{n \neq j} | x_{nj} |$. Let $\mathfrak{F}'$ denote a proper subset of $\mathfrak{F}$ containing at least one element, and let $\{a_n \mid n \in \mathfrak{F}'\}$ be a set of real numbers. Consider the problem of determining a real N-vector $c \triangleq (c_1, c_2, \cdots, c_N)$ such that

$$\delta(c) \triangleq \sum_{\substack{n=-\infty \\ n \notin \mathfrak{F}'}}^{\infty} \left| \sum_{j \in \mathfrak{F}} c_j x_{nj} \right|$$

is minimized subject to the constraints

$$a_n = \sum_{j \in \mathfrak{F}} c_j x_{nj} , \qquad n \in \mathfrak{F}'. \tag{6}$$

Our assumption that $| x_{jj} | > \sum_{n \neq j} | x_{nj} |$ for $j \in \mathfrak{F}$ implies[2] that there exists a unique solution c^* to the set of equations

$$a_n = \sum_{j \varepsilon \mathfrak{F}} c_j x_{nj}, \qquad n \varepsilon \mathfrak{F}'$$

$$0 = \sum_{j \varepsilon \mathfrak{F}} c_j x_{nj}, \; n \varepsilon (\mathfrak{F} - \mathfrak{F}').$$

We shall prove that if $c \neq c^*$ and c satisfies the constraints of (6), then $\delta(c) > \delta(c^*)$. For the special case in which $\mathfrak{F}'$ contains a single element, this result can be proved[3] with a modification of Lucky's technique.

Let $\mathfrak{R}(c^*)$ denote the set of all real N-vectors g, except the vector c^*, such that

$$a_n = \sum_{j \varepsilon \mathfrak{F}} g_j x_{nj}, \qquad n \varepsilon \mathfrak{F}'.$$

Let Q be the mapping of $\mathcal{S} \triangleq \{c^*\} \cup \mathfrak{R}(c^*)$ into the set of nonnegative numbers defined by

$$Qv = \sum_{n \varepsilon (\mathfrak{F} - \mathfrak{F}')} \left| \sum_{j \varepsilon \mathfrak{F}} v_j x_{nj} \right|$$

for all $v \varepsilon \mathcal{S}$.

Let $\mathcal{S}_0$ denote the linear space of vectors $u = (\cdots, u_{-1}, u_0, u_{N+1}, u_{N+2}, \cdots)$ with norm

$$\| u \| = \sum_{j \varepsilon \mathfrak{F}} | u_j |,$$

and let R denote the mapping of $\mathcal{S}$ into $\mathcal{S}_0$ defined by

$$(Rv)_n = \sum_{j \varepsilon \mathfrak{F}} v_j x_{nj}, \qquad n \not\varepsilon \mathfrak{F}$$

for all $v \varepsilon \mathcal{S}$. Then we have

$$\delta(c) = Qc + \| Rc \|$$

for all $c \varepsilon \mathcal{S}$. Since $Qc^* = 0$, if

$$Qg > \| Rg - Rc^* \|$$

for all $g \varepsilon \mathfrak{R}(c^*)$, that is, if

$$\rho \triangleq \sum_{n \not\varepsilon (\mathfrak{F} - \mathfrak{F}')} \left| \sum_{j \not\varepsilon \mathfrak{F}} g_j x_{nj} \right| - \sum_{n \not\varepsilon \mathfrak{F}} \left| \sum_{j \not\varepsilon \mathfrak{F}} (g_j - c_j^*) x_{nj} \right| > 0 \qquad (7)$$

for all $g \varepsilon \mathfrak{R}(c^*)$, then, by the proposition, $\delta(c) > \delta(c^*)$. To show that (7) is satisfied, observe that

$$\sum_{n \varepsilon (\mathfrak{F}-\mathfrak{F}')} \left| \sum_{j \varepsilon \mathfrak{F}} g_j x_{nj} \right| = \sum_{n \varepsilon (\mathfrak{F}-\mathfrak{F}')} \left| \sum_{j \varepsilon \mathfrak{F}} (g_j - c_j^*) x_{nj} \right|$$
$$= \sum_{n \varepsilon \mathfrak{F}} \left| \sum_{j \varepsilon \mathfrak{F}} (g_j - c_j^*) x_{nj} \right|$$

for $g \varepsilon \mathcal{R}(c^*)$, and that, with $w_j \triangleq (g_j - c_j^*)$,

$$\sum_{n \varepsilon \mathfrak{F}} \left| \sum_{j \varepsilon \mathfrak{F}} w_j x_{nj} \right| \geqq 2 \sum_{n \varepsilon \mathfrak{F}} | w_n x_{nn} | - \sum_{n \varepsilon \mathfrak{F}} \sum_{j \varepsilon \mathfrak{F}} | w_j | \cdot | x_{nj} |$$

and

$$\sum_{n \notin \mathfrak{F}} \left| \sum_{j \varepsilon \mathfrak{F}} w_j x_{nj} \right| \leqq \sum_{n \notin \mathfrak{F}} \sum_{j \varepsilon \mathfrak{F}} | w_j | \cdot | x_{nj} | .$$

Therefore,

$$\begin{aligned} \rho &\geqq 2 \sum_{n \varepsilon \mathfrak{F}} | w_n x_{nn} | - \sum_{k=-\infty}^{\infty} \sum_{n \varepsilon \mathfrak{F}} | w_n | \cdot | x_{kn} | \\ &\geqq \sum_{n \varepsilon \mathfrak{F}} | w_n | \left(| x_{nn} | - \sum_{k \neq n} | x_{kn} | \right), \end{aligned} \tag{8}$$

which completes our proof, since the right side of (8) is positive for all $g \varepsilon \mathcal{R}(c^*)$

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