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Electron Beam Heating of a Thin Film on a Highly Conducting Substrate

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An analysis is made of the steady-state temperature distribution in a poorly conducting plane film on a highly conducting semi-infinite substrate, owing to a time-independent heat input in a cylindrical region of the film and substrate. The problem is of interest in connection with the localized hardening of anodic oxide films on silicon by electron beam bombardment in order to produce oxide diffusion masks for the manufacture of integrated circuits. A formal solution is obtained for arbitrary dependence of the heat input on radius and depth, and a detailed study is made of a particular case in which the heat input is independent of radius across the beam, and varies in a realistic manner with depth in the film. Approximate formulas are given for the temperature in the film when the radius of the beam is large compared to the thickness of the film, and also when the conductivity of the film is small compared to the conductivity of the substrate. The approximate formulas are compared with the results of calculations based on the exact solution. Finally, a crude estimate is made of the time required to reach the steady state.

I. INTRODUCTION AND SUMMARY

Recently considerable interest has developed in the application of electron beam technology to microelectronics.[1] A number of papers have been concerned with the heat-flow problems encountered when a high-power electron beam interacts with a target. Previous investigations have considered electron heating of a uniform semi-infinite target,[2] of a target consisting of a highly conductive metal film on a less conductive substrate,[3] and of a thin film not supported by a substrate.[4] The heating of a poorly conductive film on a highly conductive substrate does not appear to have been treated before, and forms the topic of the present investigation. It corresponds to the case of an electron beam incident upon an oxidized silicon substrate.

This analysis may find an application in the fabrication of oxide diffusion masks for integrated circuits by electron beam bombardment.[5] The etch rate of an anodic oxide film on silicon in hydrofluoric acid has been shown to decrease strongly under electron bombardment, and a proposal for producing patterns is to harden some areas of the film and then to remove the surrounding oxide with dilute HF.

It should be noted that electron beam bombardment produces radiation damage as well as thermal effects. The radiation damage alone would increase the etch rate of the film in HF, but in conjunction with high temperature it also facilitates ionic rearrangement in the SiO_2 film during irradiation, which leads to a decrease in etch rate. The latter effect predominates by far in the case of anodic SiO_2 films, so that the net result is a strong decrease in etch rate.

While the rise in temperature during irradiation is thus not the only factor contributing to the "hardening" of the oxide film, it is still the major factor, and a knowledge of the temperature distribution during irradiation is highly desirable. On the one hand one is interested in working at a high temperature in order to increase the rate of oxide "hardening"; on the other hand one must stay below the melting point of silicon (1415°C), or perhaps even lower in order not to generate excessive thermal stresses in the silicon. The edge definition of the hardened region in the oxide film is also of paramount interest for mask fabrication. Effects due to radiation damage will not be considered here, but it may be noted that radiation damage will be generated exclusively in the oxide and not in the silicon at the accelerating voltages of interest (less than 10 kv).

In this paper we consider the mathematical problem of calculating the steady-state temperature distribution due to an axially symmetric, time-independent heat input throughout a cylindrical volume of the film and substrate. The thermal properties of both materials are assumed independent of temperature, and radiation from the outer surface of the film is neglected. A formal solution of the problem is given in Section II for an arbitrary dependence of the heat input on radius and depth; but in the subsequent analysis we assume that at any given depth the heat input is independent of radius across the beam and zero outside the beam. We also confine our attention to the temperature distribution in the film itself. The only thing we really need to know about the temperature in the substrate is that its maximum, which occurs on the axis at the film-substrate interface, is not high enough to melt the substrate.

With a fixed distribution of input heat, the normalized temperature distribution in the film depends on two dimensionless parameters, namely the ratio of beam radius to film thickness and the ratio of film conductivity to substrate conductivity. In the physical problem, the beam radius may be several times the film thickness, and the conductivity of the oxide film is between a tenth and a hundredth of the conductivity of the silicon substrate. In Section III an asymptotic approximation is given for the temperature distribution when the normalized beam radius is large. Section IV contains the solution for a perfectly conducting substrate, as well as an estimate of the first-order effect of finite but large substrate conductivity.

In order to calculate the temperature distribution numerically, it is necessary to assume a definite dependence of heat input on distance into the film ("depth-dose function"). In Section V we assume a depth-dose function which approximates the form determined empirically by Grün[6] and also employed by Wells.[7] The parameters are adjusted so that the power input is maximum at a depth equal to 40 per cent of the film thickness and zero at the bottom of the film, since, in general, one wishes to avoid direct heating of the substrate by the electron beam. Contour plots of normalized temperature have been calculated from the formulas of Section II for selected beam diameters and conductivity ratios. In addition, the exact temperature distributions along the axis and at the top and bottom of the oxide film are compared with the approximate formulas of Section III.

As a typical numerical result, we find that for an SiO_2 film of thickness 0.5 micron, bombarded by a 5 kv electron beam of diameter 20 microns with a current of 628 μa and a uniform power density of 10^6 watts/cm^2, the steady-state temperature rise on the axis is about 1800°C at the surface of the film, and about 800°C at the surface of the silicon substrate.

In Section VI a crude estimate is made of the time required to reach the steady-state temperature after the electron beam is instantaneously switched on. It appears that in an example such as the preceding, the transient time would be of the order of a few tenths of a microsecond.

II. STEADY-STATE TEMPERATURE DISTRIBUTION

The geometry of the problem to be considered is shown in Fig. 1. A plane film of thermal conductivity K_1 fills the region $0 \leqq z \leqq c$, and overlies a semi-infinite substrate of thermal conductivity K_2 which fills the region $z < 0$. We wish to find the steady-state temperature rise $T(r,z)$ under the influence of an axially symmetric, distributed heat source of strength $Q(r,z)$.

The temperature rise satisfies Poisson's equation,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \begin{cases} -Q/K_1, & 0 < z < c, \\ -Q/K_2, & z < 0, \end{cases} \tag{1}$$

and the boundary conditions are

$$\begin{aligned} T_z(r,c) &= 0, \\ T(r,0^+) &= T(r,0^-), \\ K_1 T_z(r,0^+) &= K_2 T_z(r,0^-), \\ T(r,z) \to 0 \quad &\text{as} \quad r^2 + z^2 \to \infty, \end{aligned} \tag{2}$$

where T_z denotes $\partial T/\partial z$. The first of the boundary conditions asserts that there is no heat flow across the upper boundary of the film. The method of solution which we are going to use would also allow for a linearized radiation condition at the surface, i.e., a linear relation between $T(r,c)$ and $T_z(r,c)$, if one knew the appropriate coefficients. The second and third conditions insure the continuity of temperature and heat flow across the interface between film and substrate, and the fourth condition says that the temperature rise tends to zero at great distances from the source.

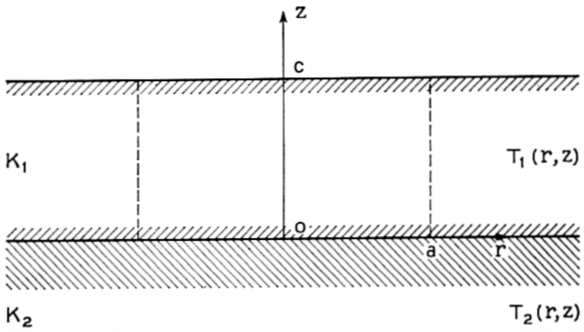

Fig. 1 — Cross section of plane film on semi-infinite substrate.

It will be convenient henceforth to work in terms of normalized, dimensionless quantities. In particular, we shall take the film thickness as the unit of length, and denote the ratio of film conductivity to substrate conductivity by ε. We also introduce a representative heat source strength Q_0 having the dimensions of power per unit volume. Thus, let

$$\begin{aligned} \xi &= r/c = \text{normalized radius} \\ \zeta &= z/c = \text{normalized depth} \\ \varepsilon &= K_1/K_2 = \text{conductivity ratio} \\ q &= Q/Q_0 = \text{normalized heat input} \\ U &= (K_1/c^2Q_0)T = \text{normalized temperature rise.} \end{aligned}$$

In terms of these normalized quantities, (1) takes the form

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial U}{\partial \xi} + \frac{\partial^2 U}{\partial \zeta^2} = \begin{cases} -q & 0 < \zeta < 1, \\ -\varepsilon q, & \zeta < 0, \end{cases} \tag{3}$$

and the boundary conditions (2) become

$$\begin{aligned} U_\zeta(\xi,1) &= 0, \\ U(\xi,0^+) &= U(\xi,0^-), \\ \varepsilon U_\zeta(\xi,0^+) &= U_\zeta(\xi,0^-), \\ U(\xi,\zeta) \to 0 \quad &\text{as} \quad \xi^2 + \zeta^2 \to \infty. \end{aligned} \tag{4}$$

In what follows, we shall treat separately the cases of heat input to the film and heat input to the substrate. The general case follows by superposition.

2.1 *Heat Input to Film*

Assume that $q(\xi,\zeta)$ differs from zero only in the film. We denote the normalized temperature rise in the film by $U_1(\xi,\zeta)$, and seek a solution of Poisson's equation in the form

$$U_1(\xi,\zeta) = \int_0^\infty f(w,\zeta)J_0(w\xi)dw, \qquad 0 \leqq \zeta \leqq 1, \tag{5}$$

where $f(w,\zeta)$ is a function to be determined. Similarly, for the normalized temperature rise in the substrate, $U_2(\xi,\zeta)$, we seek a solution of Laplace's equation in the form

$$U_2(\xi,\zeta) = \int_0^\infty g(w)e^{w\zeta}J_0(w\xi)dw, \qquad \zeta \leqq 0, \tag{6}$$

which vanishes as $\zeta \to -\infty$.

We shall consider the case in which the normalized heat input can be written as

$$q(\xi,\zeta) = \psi(\xi)\varphi(\zeta), \tag{7}$$

that is, as the product of a function of radius times a function of depth. This will probably be justifiable if the increase in beam width with depth due to electron scattering is negligible. Furthermore, by taking $\psi(\xi)$ and $\varphi(\zeta)$ equal to δ-functions it is possible to derive the Green's function, in terms of which one can express the solution for an arbitrary axially symmetric heat input.

Substituting (5) and (7) into (3) and making use of Bessel's equation, we obtain

$$\int_0^\infty [f_{\zeta\zeta}(w,\zeta) - w^2 f(w,\zeta)]J_0(w\xi)dw = -\psi(\xi)\varphi(\zeta), \quad \text{for} \quad 0 < \zeta < 1. \tag{8}$$

From the Hankel inversion formula,[8] it follows that

$$f_{\zeta\zeta}(w,\zeta) - w^2 f(w,\zeta) = -w\bar{\psi}(w)\varphi(\zeta), \tag{9}$$

where $\bar{\psi}(w)$ is the Hankel transform of $\psi(\xi)$, defined by

$$\bar{\psi}(w) = \int_0^\infty \xi\psi(\xi)J_0(w\xi)d\xi. \tag{10}$$

Extensive tables[9] of Hankel transforms are available; we note in particular the following pairs.

(*i*) Uniform beam of normalized radius α:

$$\psi(\xi) = \begin{cases} 1, & 0 \leqq \xi < \alpha, \\ 0, & \xi > \alpha, \end{cases} \qquad \bar{\psi}(w) = (\alpha/w)J_1(\alpha w). \tag{11a}$$

(*ii*) Gaussian beam:

$$\psi(\xi) = \exp(-\xi^2/\alpha^2), \qquad \bar{\psi}(w) = (\alpha^2/2)\exp(-\alpha^2 w^2/4). \tag{11b}$$

(*iii*) Infinitesimally thin, hollow beam of radius ξ_0:

$$\psi(\xi) = \delta(\xi - \xi_0), \qquad \bar{\psi}(w) = \xi_0 J_0(w\xi_0). \tag{11c}$$

To satisfy the boundary conditions (4), we must have

$$\begin{aligned} f_\zeta(w,1) &= 0, \\ f(w,0) &= g(w), \\ \varepsilon f_\zeta(w,0) &= wg(w), \end{aligned} \tag{12}$$

from which, eliminating $g(w)$,

$$f_\zeta(w,1) = 0, \qquad \varepsilon f_\zeta(w,0) = wf(w,0). \tag{13}$$

The solution of the two-point boundary value problem for $f(w,\zeta)$ by standard methods leads to

$$\begin{aligned} f(w,\zeta) = \bar{\psi}(w) \Bigg[& \frac{\sinh w\zeta + \varepsilon \cosh w\zeta}{\cosh w + \varepsilon \sinh w} \int_0^1 \varphi(\eta) \cosh w(1 - \eta) d\eta \\ & - \int_0^\zeta \varphi(\eta) \sinh w(\zeta - \eta) d\eta \Bigg]. \end{aligned} \tag{14}$$

From the second of (12),

$$g(w) = \frac{\varepsilon \bar{\psi}(w)}{\cosh w + \varepsilon \sinh w} \int_0^1 \varphi(\eta) \cosh w(1 - \eta) d\eta. \tag{15}$$

In principle, the normalized temperature rise is completely given by (5), (6), (10), (14), and (15), provided that the heat input to the film can be represented as a product $\psi(\xi)\varphi(\zeta)$, and that there is no heat input to the substrate. A complete numerical solution for arbitrary ψ and φ would, however, involve the evaluation of five integrals, each of which depends on one or more parameters. In practice, one would try to approximate the source function in such a way that at least some of the integrations could be done analytically. An example is discussed in the following sections.

2.2 *Heat Input to Substrate*

Again we take the heat input in the product form (7), but now assume that $\varphi(\zeta)$ differs from zero only in the substrate, $\zeta < 0$. For the normalized temperature rise we assume

$$U_1(\xi,\zeta) = \int_0^\infty l(w) \cosh w(1 - \zeta) J_0(w\xi) dw, \qquad 0 \leqq \zeta \leqq 1, \tag{16}$$

$$U_2(\xi,\zeta) = \int_0^\infty m(w,\zeta) J_0(w\xi) dw, \qquad \zeta \leqq 0, \tag{17}$$

in the film and substrate, respectively, where $l(w)$ and $m(w,\zeta)$ are functions to be determined. The expression for $U_1(\xi,\zeta)$ already satisfies Laplace's equation and the first of the boundary conditions (4).

As before, it is easy to show that $m(w,\zeta)$ must satisfy the differential equation

$$m_{\zeta\zeta}(w,\zeta) - w^2 m(w,\zeta) = -\varepsilon w \bar{\psi}(w)\varphi(\zeta), \tag{18}$$

and the boundary conditions

$$\begin{aligned} l(w) \cosh w &= m(w,0), \\ -\varepsilon l(w) \sinh w &= m_\zeta(w,0), \\ m(w,\zeta) \rightarrow 0 \quad &\text{as} \quad \zeta \rightarrow -\infty. \end{aligned} \tag{19}$$

By standard methods we find

$$m(w,\zeta) = D(w)e^{w\zeta} + \tfrac{1}{2}\varepsilon\bar{\psi}(w) \int_{-\infty}^{0} \exp\,(-w\,|\,\eta - \zeta\,|)\,\varphi(\eta)d\eta, \quad \text{for} \quad \zeta \leqq 0. \tag{20}$$

It is clear that $m(w,\zeta)$ satisfies the last of the boundary conditions (19) if $w > 0$ and $\varphi(\zeta)$ vanishes for all sufficiently large negative ζ. In practical cases, it will certainly be justifiable to set the heat input identically equal to zero below some finite depth. We shall not take space to investigate the mathematical question of how slowly $\varphi(\zeta)$ could approach zero, and still have $m(w,\zeta)$ also approach zero, as $\zeta \rightarrow -\infty$.

From the first two boundary conditions (19), it is straightforward to calculate

$$l(w) = \frac{\varepsilon\bar{\psi}(w)}{\cosh w + \varepsilon \sinh w} \int_{-\infty}^{0} \exp\,(w\eta)\varphi(\eta)d\eta, \tag{21}$$

$$D(w) = \tfrac{1}{2}\varepsilon\bar{\psi}(w) \left[\frac{\cosh w - \varepsilon \sinh w}{\cosh w + \varepsilon \sinh w}\right] \int_{-\infty}^{0} \exp\,(w\eta)\varphi(\eta)d\eta. \tag{22}$$

The first of these, substituted into (16), gives the normalized temperature rise in the film; and the second, together with (17) and (20), gives the temperature rise in the substrate.

III. APPROXIMATIONS FOR A UNIFORM BEAM OF LARGE RADIUS

We shall consider henceforth only the case in which the heat input is radially uniform out to the normalized radius $\xi = \alpha$, and zero for $\xi > \alpha$. Also we shall be interested only in the temperature distribution in the film itself. The dependence of heat input on depth will, however, still be taken as arbitrary.

In the case where heat is applied to the film by means of a radially uniform beam which does not penetrate the substrate, the normalized temperature rise in the film is given by (5), (11a), and (14). In practice, the SiO_2 film may be only half a micron thick while the beam radius is several microns. We accordingly seek an asymptotic expansion of the temperature distribution for large α. In the analysis we assume that the conductivity ratio ε is fixed with $\varepsilon \lesssim 1$. Our results will also include the physically interesting case $\varepsilon \ll 1$.

Combining (5), (11a), and (14), we may write the temperature distribution in the film in the form

$$U_1(\xi,\zeta;\,\varepsilon) = \int_0^\infty \left[\frac{\varepsilon\alpha}{w}\int_0^1 \varphi(\eta)d\eta + h(w,\zeta;\,\varepsilon)\right] J_1(\alpha w)J_0(\xi w)dw, \qquad 0 \leqq \zeta \leqq 1, \tag{23}$$

where

$$h(w,\zeta;\,\varepsilon) = \frac{\alpha}{w}\left[\frac{\sinh w\zeta + \varepsilon\cosh w\zeta}{\cosh w + \varepsilon \sinh w}\int_0^1 \varphi(\eta)\cosh w(1-\eta)d\eta - \varepsilon\int_0^1 \varphi(\eta)d\eta - \int_0^\zeta \varphi(\eta)\sinh w(\zeta-\eta)d\eta\right]. \tag{24}$$

It is clear that the function $h(w,\zeta;\,\varepsilon)$ may be expanded in a power series around $w = 0$; that is,

$$h(w,\zeta;\,\varepsilon) = \sum_{m=0}^{\infty} h^{(m)}(0,\zeta;\,\varepsilon)w^m/m!, \tag{25}$$

where the superscripts denote derivatives with respect to w. In particular,

$$h(0,\zeta;\,\varepsilon) = \alpha\left[(\zeta - \varepsilon^2)\int_0^1 \varphi(\eta)d\eta - \int_0^\zeta (\zeta-\eta)\varphi(\eta)d\eta\right]. \tag{26}$$

Let

$$\rho = \xi/\alpha, \qquad \xi = \alpha\rho, \tag{27}$$

so that the boundary of the heat input region is $\rho = 1$. We have[10]

$$
\begin{aligned}
P(\rho) &\equiv \int_0^\infty J_1(\alpha w) J_0(\rho\alpha w) \frac{dw}{w} \\
&\equiv \int_0^\infty J_1(x) J_0(\rho x) \frac{dx}{x} \\
&= \begin{cases} \dfrac{2}{\pi} E(\rho), & 0 \leqq \rho < 1, \\[2ex] \dfrac{2\rho}{\pi}\left[E\left(\dfrac{1}{\rho}\right) - \left(1 - \dfrac{1}{\rho^2}\right) K\left(\dfrac{1}{\rho}\right)\right], & \rho > 1, \end{cases}
\end{aligned} \tag{28}
$$

where E and K are complete elliptic integrals.

Furthermore, assuming that ρ, ζ, and ε are fixed, the following asymptotic expansions for large α are derived in the Appendix, in terms of the derivatives of $h(w,\zeta;\varepsilon)$ at $w = 0$:

$$
\int_0^\infty h(w,\zeta;\varepsilon) J_1(\alpha w) J_0(\rho\alpha w) dw
$$

$$
\sim \begin{cases} \dfrac{h(0,\zeta;\varepsilon)}{\alpha} + \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \\ \qquad \cdot F(n+\frac{3}{2}, n+\frac{1}{2}; 1; \rho^2) \dfrac{h^{(2n+1)}(0,\zeta;\varepsilon)}{\alpha^{2n+2}}, & 0 \leqq \rho < 1, \\[2ex] \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)\rho^{2n+3}} \\ \qquad \cdot F\left(n+\frac{3}{2}, n+\frac{3}{2}; 2; \dfrac{1}{\rho^2}\right) \dfrac{h^{(2n+1)}(0,\zeta;\varepsilon)}{\alpha^{2n+2}}, & \rho > 1, \end{cases} \tag{29}
$$

where $F(a,b;c;z)$ is the hypergeometric function. Since when ρ is near unity, we have[11]

$$
\begin{aligned}
&F(n+\tfrac{3}{2}, n+\tfrac{1}{2}; 1; \rho^2) = O[(1-\rho)^{-(2n+1)}], \\
&\qquad\qquad\qquad\qquad 0 < (1-\rho) \ll 1, \\
&F\left(n+\tfrac{3}{2}, n+\tfrac{3}{2}; 2; \frac{1}{\rho^2}\right) = O[(\rho-1)^{-(2n+1)}], \\
&\qquad\qquad\qquad\qquad 0 < (\rho-1) \ll 1,
\end{aligned} \tag{30}
$$

it follows that the asymptotic expansions (29) are useful only for $\alpha\,|1-\rho| \gg 1$, but not in the neighborhood of $\rho = 1$.

Combining (23), (26), (28), and (29), we obtain finally,

$$U_1(\rho\alpha,\zeta;\varepsilon) \sim \varepsilon\alpha P(\rho)\int_0^1 \varphi(\eta)d\eta + (\zeta - \varepsilon^2)\int_0^1 \varphi(\eta)d\eta - \int_0^\zeta (\zeta - \eta)\varphi(\eta)d\eta + \mathrm{O}[\varepsilon/\alpha(1-\rho)], \quad 0 \leqq \rho < 1;$$

$$U_1(\rho\alpha,\zeta;\varepsilon) \sim \varepsilon\alpha P(\rho)\int_0^1 \varphi(\eta)d\eta + \mathrm{O}[\varepsilon/\alpha(\rho-1)\rho^2], \quad \rho > 1, \tag{31}$$

where $P(\rho)$ is defined by (28). That the remainder terms are $\mathrm{O}(\varepsilon)$ when $\varepsilon \ll 1$ may be seen from the relations

$$h(w,\zeta;\varepsilon) = h(-w,\zeta;-\varepsilon), \qquad h^{(2n+1)}(0,\zeta;\varepsilon) = -h^{(2n+1)}(0,\zeta;-\varepsilon); \tag{32}$$

i.e., the odd derivatives of h with respect to w at $w = 0$ are odd functions of ε. Note, however, that setting $\varepsilon = 0$ in the asymptotic solution (31) does not give the exact solution for a perfectly conducting substrate, inasmuch as there are exponentially small terms in α which never appear in the asymptotic solution. The exact solution for $\varepsilon = 0$ is given in Section IV.

When the product $\varepsilon\alpha$ is sufficiently large, the leading terms in the asymptotic solution (31) are proportional to $P(\rho)$; that is, they are functions of ρ $(= \xi/\alpha)$ only, and are independent of the depth ζ in the film. The function $P(\rho)$ is plotted in Fig. 2. It is continuous, with a logarithmically infinite slope, at $\rho = 1$. Numerical comparisons between the exact solution (23) and the approximate solution (31) are made in Section V.

We now look briefly at the case of heat input to the substrate by a radially uniform beam of normalized radius α and depth dependence $\varphi(\zeta)$, for $\zeta \leqq 0$. The normalized temperature rise in the film is, from (11a), (16), (21), and (27),

$$U_1(\rho\alpha,\zeta;\varepsilon) = \varepsilon\alpha\int_0^\infty \frac{\cosh w(1-\zeta)}{w(\cosh w + \varepsilon\sinh w)}\left[\int_{-\infty}^0 \exp(w\eta)\,\varphi(\eta)d\eta\right] \cdot J_1(\alpha w)J_0(\rho\alpha w)dw, \qquad 0 \leqq \zeta \leqq 1. \tag{33}$$

When ρ, ζ, and ε are fixed, and both $\alpha \gg 1$ and $\alpha\,|\,1-\rho\,| \gg 1$, an analysis entirely similar to the preceding gives

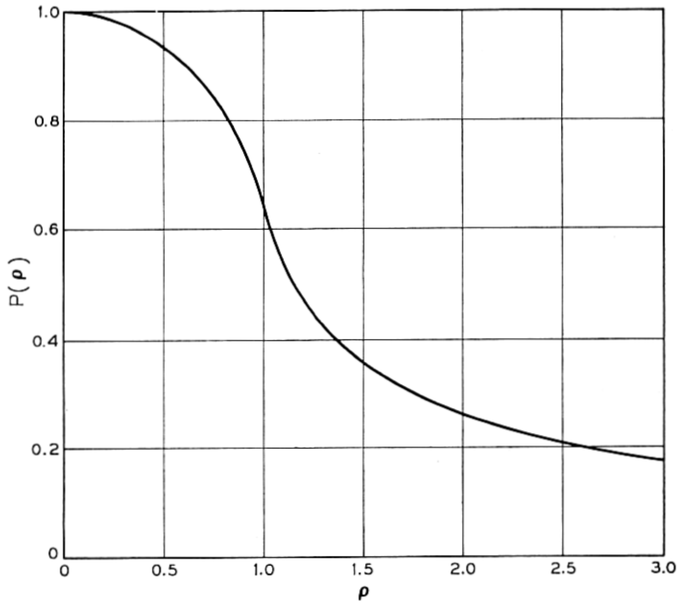

Fig. 2 — The function $P(\rho) = \int_0^\infty \frac{J_1(x)J_0(\rho x)}{x}\,dx$.

$$
\begin{aligned}
U_1(\rho\alpha,\zeta;\,\varepsilon) &\sim \varepsilon\alpha P(\rho)\int_{-\infty}^{0}\varphi(\eta)d\eta + \varepsilon\int_{-\infty}^{0}(\eta-\varepsilon)\varphi(\eta)d\eta \\
&\qquad + \mathrm{O}[\varepsilon/\alpha(1-\rho)], \qquad 0 \leqq \rho < 1; \\
U_1(\rho\alpha,\zeta;\,\varepsilon) &\sim \varepsilon\alpha P(\rho)\int_{-\infty}^{0}\varphi(\eta)d\eta + \mathrm{O}[\varepsilon/\alpha(\rho-1)\rho^2], \\
&\qquad \rho > 1.
\end{aligned} \tag{34}
$$

IV. APPROXIMATIONS FOR A UNIFORM BEAM WITH LARGE SUBSTRATE CONDUCTIVITY

We now assume that $\varepsilon \ll 1$; that is, the conductivity of the substrate is large compared to the conductivity of the film. (For an SiO_2 film on silicon, ε is between 0.1 and 0.01.) Again we consider a radially uniform beam, with an arbitrary depth-dose function $\varphi(\zeta)$. No restrictions are placed on the normalized beam radius α.

For heat input to the film only, the normalized temperature rise in the film is given by (5), (11a), and (14). Referring to (14), we expand the function $f(w,\zeta)$ in powers of ε. After a little algebra, we find that the normalized temperature rise in the film can be written as

$$U_1(\xi,\zeta;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n U_1^{(n)}(\xi,\zeta), \tag{35}$$

where

$$U_1^{(0)}(\xi,\zeta) = \alpha \int_0^\infty F(w,\zeta) J_1(\alpha w) J_0(\xi w) dw, \tag{36}$$

with

$$F(w,\zeta) = \frac{1}{w \cosh w}\left[\sinh w\zeta \int_\zeta^1 \varphi(\eta) \cosh w(1-\eta) d\eta + \cosh w(1-\zeta) \int_0^\zeta \varphi(\eta) \sinh w\eta \, d\eta\right], \tag{37}$$

and for $n \geqq 1$,

$$U_1^{(n)}(\xi,\zeta) = (-1)^{n-1}\alpha \int_0^\infty \left[\int_0^1 \varphi(\eta) \cosh w(1-\eta) d\eta\right] \times \frac{\cosh w(1-\zeta) \tanh^{n-1} w}{\cosh^2 w} J_1(\alpha w) J_0(\xi w) \frac{dw}{w}. \tag{38}$$

The quantity $U_1^{(0)}(\xi,\eta)$ is the normalized temperature rise in the film when the substrate is perfectly conducting. In this case, the temperature rise at the bottom of the film is zero, and in fact it is obvious from (37) that $F(w,0) \equiv 0$. The quantity $U_1^{(n)}(\xi,\zeta)$ represents the nth order correction if ε is small but not zero.

We may evaluate $U_1^{(0)}(\xi,\zeta)$ by contour integration. Let S_n denote the semicircle of radius $n\pi$ in the upper half-plane ($n = 1, 2, \cdots$), with diameter along the real axis indented at the origin. From (37) it follows that $F(w,\zeta)$ is uniformly bounded on S_n, and has simple poles within S_n at $w = (m - \frac{1}{2})\pi i$ ($m = 1, \cdots, n$). The choice of integrand depends on whether $0 \leqq \xi \leqq \alpha$, or $\xi \geqq \alpha$. For $0 \leqq \xi \leqq \alpha$ we consider

$$\int_{S_n} F(w,\zeta) J_0(\xi w) H_1^{(1)}(\alpha w) dw, \tag{39}$$

and for $\xi \geqq \alpha$ we consider

$$\int_{S_n} F(w,\zeta) J_1(\alpha w) H_0^{(1)}(\xi w) dw, \tag{40}$$

where $H_0^{(1)}$ and $H_1^{(1)}$ are Hankel functions. The integrals are evaluated by the calculus of residues and then the limit $n \to \infty$ is taken. Since the procedure is a standard one, we omit the details and merely state the results.

We obtain, for $0 \leqq \xi \leqq \alpha$,

$$\left\{U_1^{(0)}(\xi,\zeta) - \left[\zeta \int_\zeta^1 \varphi(\eta)d\eta + \int_0^\zeta \eta\varphi(\eta)d\eta\right]\right\}$$
$$= -\frac{2\alpha}{\pi}\sum_{m=1}^{\infty} \frac{\sin[(m-\frac{1}{2})\pi\zeta]}{(m-\frac{1}{2})} I_0[(m-\tfrac{1}{2})\pi\xi]K_1[(m-\tfrac{1}{2})\pi\alpha] \tag{41}$$
$$\cdot \int_0^1 \varphi(\eta)\sin[(m-\tfrac{1}{2})\pi\eta]d\eta,$$

and, for $\xi \geqq \alpha$,

$$U_1^{(0)}(\xi,\zeta) = \frac{2\alpha}{\pi}\sum_{m=1}^{\infty}\frac{\sin[(m-\frac{1}{2})\pi\zeta]}{(m-\frac{1}{2})}$$
$$\cdot I_1[(m-\tfrac{1}{2})\pi\alpha]K_0[(m-\tfrac{1}{2})\pi\xi]\int_0^1 \varphi(\eta)\sin[(m-\tfrac{1}{2})\pi\eta]d\eta. \tag{42}$$

Here I_0, I_1, K_0, and K_1 are modified Bessel functions. The continuity of $\partial U_1^{(0)}/\partial\xi$ at $\xi = \alpha$ is readily verified, while that of $U_1^{(0)}$ at $\xi = \alpha$ follows from the identities

$$I_0(x)K_1(x) + I_1(x)K_0(x) = 1/x, \tag{43}$$

and

$$\sum_{m=1}^{\infty}\frac{\cos[(m-\frac{1}{2})\pi\theta]}{(m-\frac{1}{2})^2} = \frac{\pi^2}{2}(1-\theta), \qquad 0 \leqq \theta \leqq 2. \tag{44}$$

We remark that (41) and (42) could have been derived by separation of variables in (3). For $|\alpha - \xi| \gg 1$ the right-hand sides of (41) and (42) are exponentially small. It follows that (41) and (42) are consistent with the asymptotic expansions (31) and (32) for $\alpha \gg 1$, when $\varepsilon = 0$ and asymptotically small terms are neglected.

It does not appear possible to evaluate the first-order correction $U_1^{(1)}(\xi,\zeta)$, as given by (38), using contour integration, because the integrand has the wrong parity in w. We can, however, obtain a bound on the value of $U_1^{(1)}(0,0)$, at the "hot spot" of the film-substrate interface, where the zero-order solution $U_1^{(0)}(0,0)$ vanishes.

From (38), setting $n = 1$ and changing the order of integration, which is justified since the double integral is absolutely convergent,

$$U_1^{(1)}(0,0) = \alpha\int_0^1 \varphi(\eta)\left[\int_0^\infty \frac{\cosh w(1-\eta)}{\cosh w}\frac{J_1(\alpha w)}{w}\,dw\right]d\eta. \tag{45}$$

When $\eta = 0$, the inner integral is equal[12] to 1. When $\eta > 0$, we transform the inner integral by substituting the integral representation[13]

$$J_1(\alpha w) = (2\alpha w/\pi) \int_0^{\pi/2} \cos(\alpha w \cos\theta) \sin^2\theta \, d\theta, \tag{46}$$

and again invoking the absolute convergence of the double integral to change the order of integration. This leads to

$$\begin{aligned} &\int_0^\infty \frac{\cosh w(1-\eta)}{\cosh w} \frac{J_1(\alpha w)}{w} dw \\ &\quad = \frac{2\alpha}{\pi} \int_0^{\pi/2} \left[\int_0^\infty \frac{\cosh w(1-\eta)}{\cosh w} \cos(\alpha w \cos\theta) dw \right] \sin^2\theta \, d\theta \\ &\quad = \alpha \sin\frac{\pi\eta}{2} \int_0^{\pi/2} \frac{\cosh(\frac{1}{2}\pi\alpha\cos\theta)\sin^2\theta\, d\theta}{\sinh^2(\frac{1}{2}\pi\alpha\cos\theta) + \sin^2(\frac{1}{2}\pi\eta)} \\ &\quad < \alpha \sin\frac{\pi\eta}{2} \int_0^{\pi/2} \frac{\cosh(\frac{1}{2}\pi\alpha\cos\theta)\sin\theta\, d\theta}{\sinh^2(\frac{1}{2}\pi\alpha\cos\theta) + \sin^2(\frac{1}{2}\pi\eta)} \\ &\quad = \frac{2}{\pi}\tan^{-1}\left[\frac{\sinh(\frac{1}{2}\pi\alpha)}{\sin(\frac{1}{2}\pi\eta)}\right] < 1, \end{aligned} \tag{47}$$

where the third line follows from a table of Fourier transforms.[14] Hence, finally, from (45) and (47),

$$\begin{aligned} &U_1^{(1)}(0,0) \\ &\quad = \alpha^2 \int_0^1 \varphi(\eta) \sin\frac{\pi\eta}{2} \left[\int_0^{\pi/2} \frac{\cosh(\frac{1}{2}\pi\alpha\cos\theta)\sin^2\theta\, d\theta}{\sinh^2(\frac{1}{2}\pi\alpha\cos\theta) + \sin^2(\frac{1}{2}\pi\eta)} \right] d\eta \\ &\quad < \alpha \int_0^1 \varphi(\eta) d\eta. \end{aligned} \tag{48}$$

We see from (31) that asymptotically, for $\alpha \gg 1$, the upper bound in (48) is attained, since $P(1) = 1$.

Now suppose that heat is put only into the substrate, so that the normalized temperature rise in the film is given by (33). Expanding in powers of ε leads to

$$U_1(\xi,\zeta;\varepsilon) = \sum_{n=1}^\infty \varepsilon^n U_1^{(n)}(\xi,\zeta), \tag{49}$$

where

$$\begin{aligned} U_1^{(n)}(\xi,\zeta) = \alpha \int_0^\infty &\frac{(-1)^{n-1}\cosh(1-\zeta)w \tanh^{n-1} w}{w \cosh w} \\ &\times \left[\int_{-\infty}^0 \exp(w\eta)\varphi(\eta) d\eta \right] J_1(\alpha w) J_0(\xi w) dw. \end{aligned} \tag{50}$$

In particular we have, on changing the order of integration,

$$\begin{aligned} U_1^{(1)}(0,0) &= \alpha \int_{-\infty}^{0} \varphi(\eta) \left[\int_0^{\infty} \frac{\exp (w\eta) J_1(\alpha w)}{w} \, dw \right] d\eta \\ &= \int_{-\infty}^{0} \varphi(\eta)[(\eta^2 + \alpha^2)^{\frac{1}{2}} + \eta] d\eta, \end{aligned} \tag{51}$$

after substituting the known value[15] of the inner integral.

V. NUMERICAL RESULTS

In this section we give the results of some numerical computations using the exact formulas of Section II, and some comparisons with the approximations of Sections III and IV. We asume that the electron beam voltage is such that the electrons penetrate to the bottom of the oxide film (about 5 kv for an 0.5 μ film[7,16]), but do not enter the substrate. For the depth-dose function we take

$$\varphi(\zeta) = \sin \beta\zeta, \qquad \beta = 5\pi/6, \qquad 0 \leqq \zeta \leqq 1, \tag{52}$$

which is plotted in Fig. 3. The assumed depth-dose function vanishes at

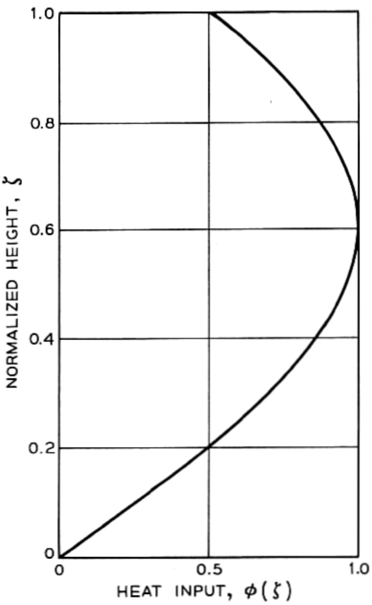

Fig. 3 — Heat input function $\varphi(\zeta) = \sin 5\pi\zeta/6$.

the bottom of the film, has a maximum at a depth equal to 40 per cent of the film thickness, and in general corresponds very closely to the empirical function used by Grün[6] and Wells.[7] The position of the maximum could be varied, of course, by changing the parameter β.

Substituting $\varphi(\zeta)$ into (5), (11a), and (14), we find after some algebra that the normalized temperature rise in the film is given by

$$U_1(\xi,\zeta) = \alpha V(\xi) \sin \beta\zeta + \alpha\beta W(\xi,\zeta), \tag{53}$$

where

$$V(\xi) = \int_0^\infty \frac{J_1(\alpha w)J_0(\xi w)}{w^2 + \beta^2}\, dw \tag{54}$$

and

$$W(\xi,\zeta) = \int_0^\infty \frac{J_1(\alpha w)J_0(\xi w)}{w(w^2 + \beta^2)} \cdot \left[\frac{\varepsilon \cosh(1 - \zeta)w - (\sinh \zeta w + \varepsilon \cosh \zeta w) \cos \beta}{\cosh w + \varepsilon \sinh w}\right] dw. \tag{55}$$

The integral on the right side of (54) can be expressed in terms of modified Bessel functions. We have[17]

$$\int_0^\infty \frac{wJ_1(\alpha w)J_1(tw)}{(w^2 + \beta^2)}\, dw = \begin{cases} I_1(\beta t)K_1(\beta\alpha), & t \leqq \alpha, \\ I_1(\beta\alpha)K_1(\beta t), & t \geqq \alpha. \end{cases} \tag{56}$$

Integrating both sides with respect to t from α to ξ and using the relationship[18]

$$\int_0^\infty \frac{J_1(\alpha w)J_0(\alpha w)}{w^2 + \beta^2}\, dw = \frac{I_1(\alpha\beta)K_0(\alpha\beta)}{\beta} \tag{57}$$

we obtain

$$V(\xi) = \begin{cases} \dfrac{1}{\beta}\left[\dfrac{1}{\beta\alpha} - K_1(\beta\alpha)I_0(\beta\xi)\right], & 0 \leqq \xi \leqq \alpha, \\ \dfrac{I_1(\beta\alpha)K_0(\beta\xi)}{\beta}, & \xi \geqq \alpha. \end{cases} \tag{58}$$

The integral for $W(\xi,\zeta)$, on the other hand, has to be evaluated numerically. The integrand is oscillatory, and falls off exponentially for large w if $0 < \zeta < 1$. If $\zeta = 0$ or $\zeta = 1$, it falls off like $1/w^4$ if $\xi \neq 0$, and like $1/w^{7/2}$ if $\xi = 0$. The numerical integration was done by Simpson's rule on an IBM 7094 computer. Combined analytic and empirical investigations of the accuracy were made in order to guarantee that the relative error in any value of U is less (in most cases, much less) than one per cent.

Four different sets of parameters were chosen; namely, $\alpha = 2$, 10, and 20 with $\varepsilon = 1/40$, and $\alpha = 2$ with $\varepsilon = 1/4$. The normalized temperature rises at the surface of the film and at the film-substrate interface are plotted against normalized radius in Fig. 4. Note the differences in scale; in each case the edge of the beam, $\xi = \alpha$, is at the center of the plot. The temperature distribution along the vertical axis is shown for the same four cases in Fig. 5.

It is seen that the temperature distribution at the surface of the film becomes more flat-topped, and the fall-off at the edge of the beam becomes relatively (although not absolutely) mote abrupt as α increases in the first three cases of Fig. 4. Also note that the temperature levels are somewhat higher and the temperature variation through the film is less in Fig. 4(d) than in Fig. 4(a), since for the same value of α the relative conductivity of the substrate is only $\frac{1}{10}$ as large in Fig. 4(d) as in Fig. 4(a).

The dashed curves in Figs. 4 and 5 correspond to the approximate formulas (31) for large α. If $\varphi(\zeta)$ is given by (52), these approximations read:

$$U_1(\rho\alpha,\zeta;\,\varepsilon) \sim \varepsilon\alpha P(\rho)\,\frac{1-\cos\beta}{\beta} + \frac{1}{\beta}\left[\frac{\sin\beta\zeta}{\beta} - \zeta\cos\beta - \varepsilon^2(1-\cos\beta)\right], \qquad 0 \leqq \rho < 1, \tag{59}$$

$$U_1(\rho\alpha,\zeta;\,\varepsilon) \sim \varepsilon\alpha P(\rho)\,\frac{1-\cos\bar{\beta}}{\beta}, \qquad \rho > 1,$$

where $\rho = \xi/\alpha$. As expected, the approximations are discontinuous at the edge of the beam, $\rho = 1$; and they are not much good when $\alpha = 2$ (worse for the larger value of ε). They are remarkably good, however, for $\alpha = 10$ and $\alpha = 20$; the dashed curves essentially coincide with the solid ones except on the surface of the film in the immediate neighborhood of the beam edge.

Contour plots for the temperature distribution in the film are given in Fig. 6 for $\alpha = 2$ and $\alpha = 10$ with $\varepsilon = 1/40$, and for $\alpha = 2$ with $\varepsilon = 1/4$. Contour plots were not made for $\alpha = 20$, because the numerical

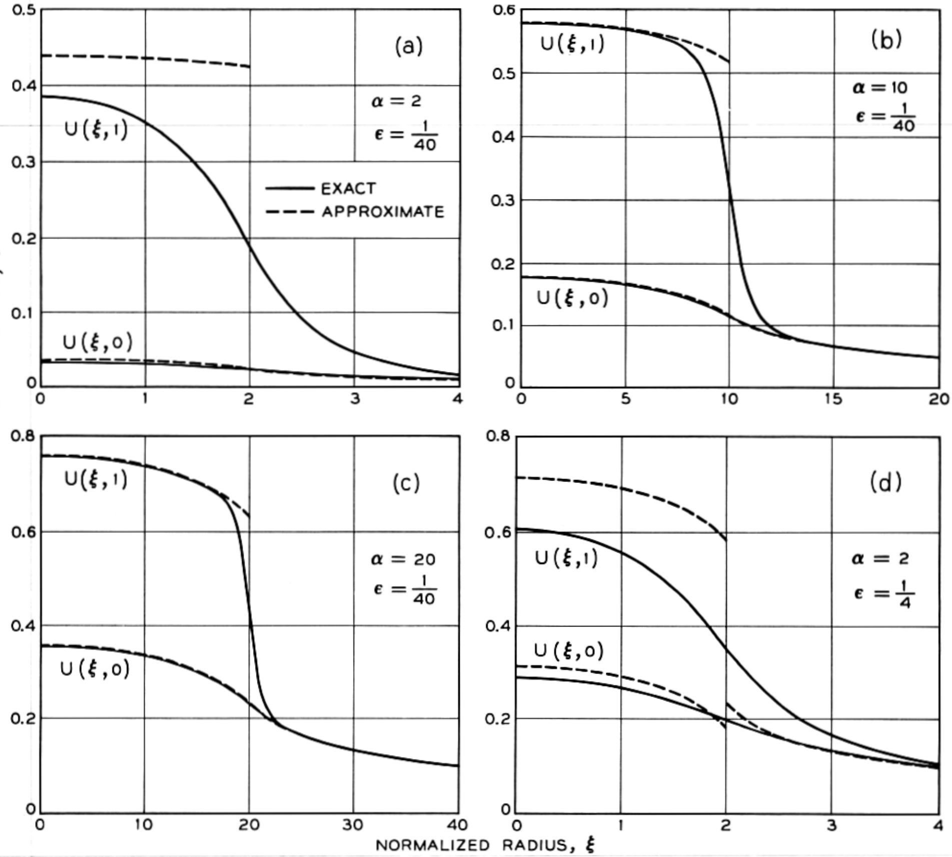

Fig. 4 — Normalized temperature rise at upper and lower surfaces of film: (a) $\alpha = 2, \varepsilon = 1/40$; (b) $\alpha = 10, \varepsilon = 1/40$; (c) $\alpha = 20$, $\varepsilon = 1/40$; (d) $\alpha = 2$, $\varepsilon = 1/4$.

integration is slow for large α (the integrands oscillate more rapidly); but it is clear that for large α the approximate formulas (59) would yield accurate contours, except very close to the beam edge.

We may also compare the bound on the first-order correction term for small ε, as given in Section IV, with the exact results. At the center of the film-substrate interface, (48) and (52) give

$$\varepsilon U_1^{(1)}(0,0) < \varepsilon\alpha(1 - \cos\beta)/\beta = 0.713\varepsilon\alpha, \tag{60}$$

which leads to the following comparison with the exact solution $U_1(0,0)$.

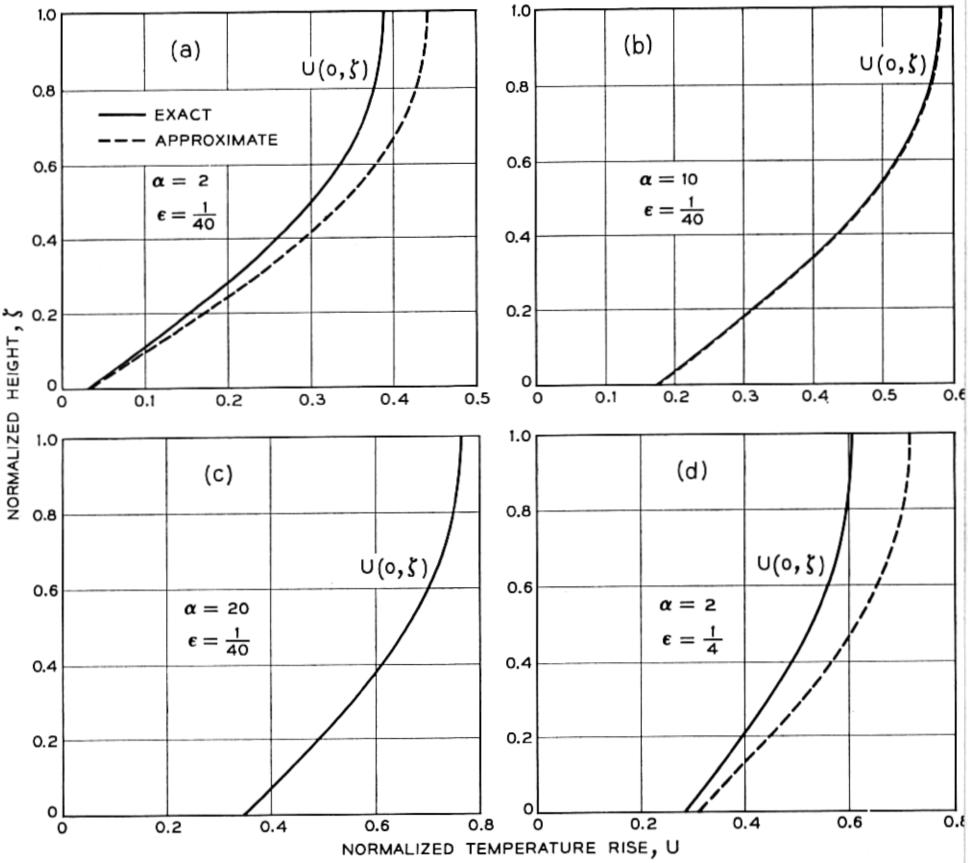

Fig. 5 — Normalized temperature rise along axis: (a) $\alpha = 2$, $\varepsilon = 1/40$; (b) $\alpha = 10$, $\varepsilon = 1/40$; (c) $\alpha = 20$, $\varepsilon = 1/40$; (d) $\alpha = 2$, $\varepsilon = 1/4$.

ε	α	$U_1(0,0)$	$0.713\varepsilon\alpha$
1/40	2	0.0314	0.0356
1/40	10	0.1770	0.1782
1/40	20	0.3556	0.3564
1/4	2	0.2874	0.3564

In order to relate the dimensionless temperature rise U_1 to the physical temperature rise T_1, it is convenient to introduce the power density (i.e., per unit area) in the incident beam. For a uniform beam of radius αc with depth-dose function $\varphi(z/c)$, the dimensional factor Q_0, which normalizes the heat input per unit volume (Section II), is related to the incident power density P_0 by

$$\pi(\alpha c)^2 P_0 = 2\pi Q_0 \int_0^c \int_0^{\alpha c} \varphi(z/c) r \, dr \, dz, \tag{61}$$

$$Q_0 = P_0 \bigg/ c\int_0^1 \varphi(\zeta) d\zeta \, . \tag{62}$$

Hence, the actual temperature rise at the point (r,z) of the film is given by

$$\begin{aligned} T_1(r,z) &= (c^2 Q_0/K_1) U_1(r/c, z/c) \\ &= (1.403 c P_0/K_1) U_1(r/c, z/c), \end{aligned} \tag{63}$$

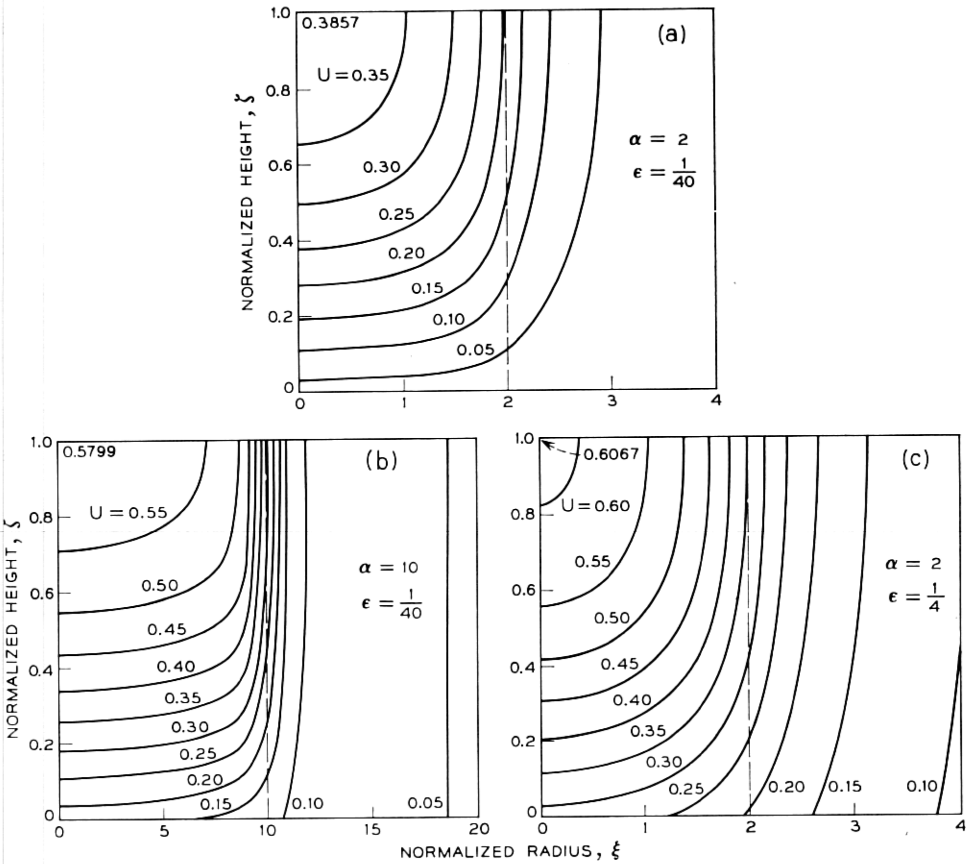

Fig. 6 — Isothermal contours. (a) $\alpha = 2$, $\varepsilon = 1/40$; (b) $\alpha = 10$, $\varepsilon = 1/40$; (c) $\alpha = 2$, $\varepsilon = 1/4$.

where the numerical coefficient corresponds to the depth-dose function (52). Consistent units for (63) are:

$$
\begin{aligned}
T_1 &= \text{temperature rise in °C} \\
r,z &= \text{coordinates in cm} \\
c &= \text{film thickness in cm} \\
P_0 &= \text{incident power density in watts/cm}^2 \\
K_1 &= \text{film conductivity in watt/cm °K} \\
0.239\, K_1 &= \text{film conductivity in cal/sec cm °K.}
\end{aligned}
$$

We shall now look at a numerical example. It turns out that the constant-conductivities model which has been analyzed in the present paper is not a very good approximation to the real problem of a silicon dioxide film on a silicon substrate, because the thermal conductivities of both materials depend strongly on temperature. In fact, the conductivity[19] of SiO_2 *increases* from about 0.015 watt/cm °K at room temperature (various values are reported — not for thin films — which differ among themselves by as much as 2:1 depending on the crystalline orientation of the sample) to about 0.03 watt/cm °K at 900°C. For Si, on the other hand, the conductivity[20] *decreases* from about 1 watt/cm °K at room temperature to about 0.03 watt/cm °K at 900°C. For purposes of calculation we shall more or less arbitrarily assume the values

$$
\begin{aligned}
K_1 &= 0.03 \text{ watt/cm °K} \\
K_2 &= 1.2 \text{ watt/cm °K} \\
\varepsilon &= 1/40 \\
c &= 0.5\,\mu = 5 \times 10^{-5} \text{ cm} \\
P_0 &= 10^6 \text{ watts/cm}^2.
\end{aligned}
\tag{64}
$$

Since these conductivities may be somewhat larger than the actual conductivities, the temperatures which we shall compute may be somewhat lower than the actual temperatures. For a 5-kv beam, the assumed power density corresponds to a current density of 200 amps/cm^2.

Table I gives the total beam current and the maximum temperature rise (i.e., on the axis) at the top and bottom of the film, for beams of diameter 2 μ, 10 μ, and 20 μ, corresponding to the previous computations with $\alpha = 2$, 10, and 20, and a conductivity ratio of 1/40. It appears, therefore, that in each case at least a part of the irradiated spot would be raised to the temperature at which the oxide hardens (900°C), but in no case would the substrate melt (1415°C).

It is probably worth repeating that the physical problem of interest is nonlinear, because of the dependence of conductivity on temperature. Bounds on the solution may be obtained from linear models, by using

TABLE I

Diameter	Current	$U_1(0,1)$	$U_1(0,0)$	$T_1(0,c)$	$T_1(0,0)$
2μ	$6.28\mu a$	0.3857	0.0314	902°	73°
10μ	$157\mu a$	0.5799	0.1771	1356°	414°
20μ	$628\mu a$	0.7589	0.3556	1775°	832°

the theorem that with a fixed heat input the steady-state temperature is not increased anywhere (usually, it is decreased everywhere) if the conductivity is increased anywhere, and vice versa. However, only a full-dress numerical treatment of the nonlinear partial differential equation, assuming that one knew the temperature dependence of the conductivity, would be likely to yield really accurate results.

VI. TRANSIENT EFFECTS

It is of interest to know how long it will take to reach the steady state if the electron beam is suddenly switched onto the film, since this gives an idea of how rapidly the beam may be scanned in laying out a mask. There have been some published analyses[7,21] of transient heating effects in electron beam machining, but we shall content ourselves with a crude estimate of the time scale in the present problem.

Consider the case of a film on a perfectly conducting substrate, with the film initially at zero temperature, and with a time-independent heat input starting at $t = 0$. Then the instantaneous temperature distribution satisfies the heat flow equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = -\frac{Q}{K_1} + \frac{C\delta}{K_1}\frac{\partial T}{\partial t}, \tag{65}$$

where K_1 is the thermal conductivity, C the heat capacity, and δ the density. The total temperature $T(r,z,t)$ may be written as the sum of a steady-state part and a transient part,

$$T(r,z,t) = T_1(r,z) + \Theta(r,z,t), \tag{66}$$

where $T_1(r,z)$ satisfies Poisson's equation (cf. Section II) and $\Theta(r,z,t)$ satisfies the homogeneous equation

$$\frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r}\frac{\partial \Theta}{\partial r} + \frac{\partial^2 \Theta}{\partial z^2} - \frac{C\delta}{K_1}\frac{\partial \Theta}{\partial t} = 0, \tag{67}$$

and vanishes as $t \to \infty$.

A sufficiently general solution of (67) may be written in the form

$$\Theta(r,z,t) = \sum_{m=0}^{\infty} \int_0^{\infty} A_m(w) \exp \{-(K_1/C\delta)[w^2 + (m + \tfrac{1}{2})^2\pi^2/c^2]t\} \times J_0(wr) \sin \frac{(m + \frac{1}{2})\pi z}{c} \, dw. \tag{68}$$

The initial condition requires that the total temperature vanish at $t = 0$; that is,

$$T_1(r,z) + \Theta(r,z,0) = 0. \tag{69}$$

Hence, from a knowledge of the steady-state temperature one can in principle use the properties of Fourier series and Fourier-Bessel integrals to determine the functions $A_m(w)$ and the transient solution $\Theta(r,z,t)$.

We would like to know how fast $\Theta(r,z,t)$ approaches zero with increasing time. It is clear that (68) cannot be characterized by any single exponential decay; but we observe that the most slowly decaying exponential is

$$\exp [-(K_1\pi^2/4c^2C\delta)t],$$

and it is therefore reasonable to define a crude "transient time" as

$$\tau = 4c^2C\delta/\pi^2K_1 . \tag{70}$$

We assume the following numerical values for the SiO_2 film:

$$\begin{aligned} c &= 5 \times 10^{-5} \text{ cm} \\ C &= 1 \text{ watt sec/gm °K} \\ \delta &= 2.2 \text{ gm/cm}^3 \\ K_1 &= 0.03 \text{ watt/cm °K}. \end{aligned} \tag{71}$$

Then

$$\tau = 3.0 \times 10^{-7} \text{ sec}, \tag{72}$$

so the transient time is a fraction of a microsecond.

VII. ACKNOWLEDGMENTS

The authors are grateful to P. F. Schmidt for posing this problem and for providing several references. They wish to thank J. A. Lewis and J. McKenna for various discussions, and Miss E. Lauh for the extensive IBM computations and plots.

APPENDIX

Asymptotic Expansion of a Class of Integrals

In this appendix, the asymptotic expansion of

$$J(\rho,\alpha) \equiv \int_0^\infty h(w)J_1(\alpha w)J_0(\rho\alpha w)\,dw \tag{73}$$

is derived, where

$$h(w) = \sum_{m=0}^{\infty} h^{(m)}(0)w^m/m!, \tag{74}$$

for $\alpha \gg 1$ and $\alpha\,|\,1 - \rho\,| \gg 1$, that is, for ρ not in the neighborhood of 1. It is clear that the asymptotic expansions will break down in the neighborhood of $\rho = 1$, since then the integrand will contain a term which is not rapidly oscillating.

We start from a result given by Tranter.[22] Namely, if we have the expansion

$$\int_0^\infty \exp(-\gamma w)\,\mathfrak{F}(\alpha,w)dw = \sum_{n=0}^{\infty} A_n(\alpha)\gamma^n, \tag{75}$$

then, formally,

$$\int_0^\infty h(w)\mathfrak{F}(\alpha,w)dw \sim \sum_{n=0}^{\infty} (-1)^n A_n(\alpha)h^{(n)}(0). \tag{76}$$

In the case at hand,

$$\mathfrak{F}(\alpha,w) = J_1(\alpha w)J_0(\rho\alpha w). \tag{77}$$

Assume first that ρ is fixed and $0 \leqq \rho < 1$. Then if $(\alpha^2 + \gamma^2)^{\frac{1}{2}} > \rho\alpha + \gamma$, we have[23]

$$\begin{aligned}\int_0^\infty \exp(\gamma w)\,J_1(\alpha w)J_0(\rho\alpha w)dw &= \frac{1}{\alpha}\sum_{m=0}^{\infty} \frac{(-1)^m\rho^{2m}\Gamma(2m+2)}{2^{2m+1}[\Gamma(m+1)]^2} \\ &\cdot\left(\frac{\alpha^2}{\alpha^2+\gamma^2}\right)^{m+1} F\left(m+1,\, -m+\tfrac{1}{2};\, 2;\, \frac{\alpha^2}{\alpha^2+\gamma^2}\right).\end{aligned} \tag{78}$$

Using standard transformations,[24]

$$\int_0^\infty \exp(-\gamma w)\, J_1(\alpha w) J_0(\rho\alpha w)\, dw$$

$$= \frac{1}{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m \rho^{2m} \Gamma(2m+2)}{2^{2m+1}[\Gamma(m+1)]^2} \left(\frac{\alpha^2}{\gamma^2}\right)^{m+1} \times F\left(m+1, m+\tfrac{3}{2}; 2; -\frac{\alpha^2}{\gamma^2}\right)$$

$$= \frac{1}{\alpha} + \frac{\gamma}{\alpha^2} \sum_{m=0}^{\infty} \frac{(-1)^m \rho^{2m} \Gamma(2m+2)\Gamma(-\frac{1}{2})}{2^{2m+1}[\Gamma(m+1)]^3 \Gamma(\frac{1}{2}-m)} \times F\left(m+\tfrac{3}{2}, m+\tfrac{1}{2}; \tfrac{3}{2}; -\frac{\gamma^2}{\alpha^2}\right) \tag{79}$$

$$= \frac{1}{\alpha} - \frac{2\gamma}{\pi\alpha^2} \sum_{m=0}^{\infty} \frac{\rho^{2m} \Gamma(m+\frac{1}{2})\Gamma(m+\frac{3}{2})}{[\Gamma(m+1)]^2} \times F\left(m+\tfrac{3}{2}, m+\tfrac{1}{2}; \tfrac{3}{2}; -\frac{\gamma^2}{\alpha^2}\right)$$

$$= \frac{1}{\alpha} + \frac{\gamma}{\sqrt{\pi}\alpha^2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \gamma^{2n} \Gamma(n+\frac{1}{2})}{\alpha^{2n}\Gamma(n+1)} \times F(n+\tfrac{3}{2}, n+\tfrac{1}{2}; 1; \rho^2),$$

where in the last step we have expanded the hypergeometric function in a power series and interchanged the order of summation. Comparing (79) with (75), we see that

$$A_{2n}(\alpha) = \begin{cases} 1/\alpha & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{80}$$

$$A_{2n+1}(\alpha) = \frac{(-1)^{n+1}\Gamma(n+\frac{1}{2})}{\alpha^{2n+2}\Gamma(\frac{1}{2})\Gamma(n+1)} F(n+\tfrac{3}{2}, n+\tfrac{1}{2}; 1; \rho^2).$$

It follows from (76) that

$$\int_0^\infty h(w) J_1(\alpha w) J_0(\rho\alpha w)\, dw$$

$$\sim \frac{h(0)}{\alpha} + \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \times F(n+\tfrac{3}{2}, n+\tfrac{1}{2}; 1; \rho^2) \frac{h^{(2n+1)}(0)}{\alpha^{2n+2}} \tag{81}$$

for $0 < \rho < 1$.

An entirely similar derivation, the details of which will be omitted, leads to the expansion

$$\int_0^\infty h(w)J_1(\alpha w)J_0(\rho\alpha w)\,dw \sim \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)\rho^{2n+3}} \times F\left(n+\tfrac{3}{2}, n+\tfrac{3}{2}; 2; \frac{1}{\rho^2}\right)\frac{h^{(2n+1)}(0)}{\alpha^{2n+2}}, \quad (82)$$

for $\rho > 1$.

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