Phase Principle for Detecting Narrow-Band Gaussian Signals

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This paper describes a phase principle for detecting a weak, narrow-band Gaussian signal in the presence of receiver noise. The phase principle leads to a phase detector which performs 2.5 db worse than the time-honored square-law detector when gain fluctuations are negligible. However, when gain fluctuations are significant the phase detector can perform better than the square-law detector. The phase principle can be implemented by using radio interferometer type receivers or monopulse radar-type receivers.

I. INTRODUCTION

In many branches of science and technology one is often confronted with the problem of detecting a weak, narrow-band Gaussian signal in the presence of receiver noise. For example, this problem occurs in the fields of radio and radar astronomy, radar detection, and radio communication. The classical solution to this problem utilizes the time-honored square-law detector. The output of the square-law detector is proportional to the total power applied at its input. One detects the presence of a Gaussian signal in the presence of receiver noise by monitoring the total power. When the total power is relatively high the signal is supposed to be present. Present-day radiometers use this principle.

In this paper we shall describe a phase principle for detecting a narrow-band Gaussian signal in the presence of receiver noise. We shall show that the phase principle leads to a phase detector whose performance is comparable to the square-law detector when gain fluctuations are negligible. Furthermore, the phase detector is relatively insensitive to system gain fluctuations; whereas the square-law detector is highly sensitive to system gain fluctuations. That is, the phase detector enjoys this important property shared by the polarity-coincidence correlator,[1,2] the phase detector analyzed by Huggins and Middleton,[3] and the zero-crossing detector.[4] The latter two detectors are suitable for detecting an extremely narrow-band signal immersed in narrow-band noise, and they are insensitive to a "white" Gaussian signal.

II. IMPLEMENTATION AND RELATIVE PERFORMANCE OF THE PHASE DETECTOR

Fig. 1 illustrates a simplified implementation of the phase detector for detecting a narrow-band Gaussian signal. We assume that two receivers are available for detection purposes. $S(t)$, and $N_1(t)$ and $N_2(t)$ represent zero mean, independent, narrow-band Gaussian processes. $N_1(t)$ and $N_2(t)$ are considered to be receiver noises of equal variances. $S(t)$ represents the narrow-band Gaussian signal to be detected. η_i represents the ith independent sample of the phase difference between $S(t) + N_1(t)$ and $S(t) + N_2(t)$. η_i is taken to be in the primary interval $(-\pi, \pi)$. After n such samples the output $\eta^\dagger$ of the phase detector is given by

$$\eta^\dagger = \frac{1}{n} \sum_{i=1}^{n} \cos \eta_i . \tag{1}$$

One compares $\eta^\dagger$ with some constant threshold value Λ_1 and decides that signal is present if $\eta^\dagger > \Lambda_1$. Otherwise the signal is supposed to be absent.

We shall compare the performance of the phase detector with the performance of the square-law detector on the basis of the "deflection criterion."[6] In general, if f_{S+N} and f_N represent the output of a detector with and without signal, respectively, then the "deflection criterion" bases the performance of the detector on the detection index

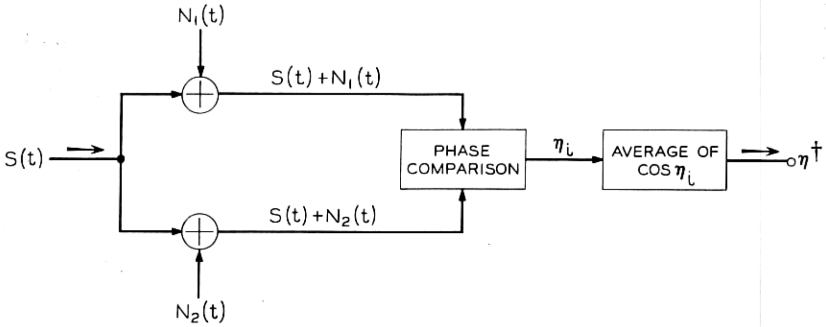

Fig. 1 — Simplified implementation of the phase detector for detecting a narrow-band Gaussian signal. ($S(t)$, $N_1(t)$, and $N_2(t)$ represent independent, narrow-band Gaussian processes.)

k where

$$k \equiv \frac{| Ef_{S+N} - Ef_N |}{[\text{Var } f_{S+N} + \text{Var } f_N]^{\frac{1}{2}}} \tag{2}$$

$$E = \text{Expectation}$$

$$\text{Var} = \text{Variance.}$$

This form of the detection index was also used in Ref. 3. Incidentally, comparing the detectors on the basis of the detection index when large sample sizes and weak signals are involved is equivalent to comparing the detectors on the basis of probability of error and probability of false alarm. This equivalence is demonstrated in Ref. 4.

General statistical properties of η_i were derived in Ref. 5. Using these results we find that

$$E \cos \eta_i = \frac{\pi l}{4} \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; l^2\right) \tag{3}$$

$$E \cos^2 \eta_i = \frac{1}{2} + \frac{l^2}{4} \, {}_2F_1(1, 1; 3; l^2) \tag{4}$$

where ${}_2F_1$ is the Gaussian hypergeometric function

$${}_2F_1(\alpha, \beta; \gamma; x) \equiv 1 + \frac{\alpha\beta}{\gamma} x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)} \frac{x^2}{2!} + \cdots$$

and

$$l = \frac{a}{1 + a}$$

$$a = \frac{\text{Var } S(t)}{\text{Var } N_1(t)} = \frac{\text{Var } S(t)}{\text{Var } N_2(t)} .$$

Thus,

$$E\eta^\dagger = E \cos \eta_i \tag{5}$$

$$\text{Var } \eta^\dagger = \frac{\text{Var} \cos \eta_i}{n} = \frac{E[\cos^2 \eta_i] - E^2[\cos \eta_i]}{n} . \tag{6}$$

Thus, for small values of "a", the only case of interest in this paper, the detection index k for the phase detector is given by

$$k = \frac{E\eta^\dagger}{[2 \text{ Var } \eta^\dagger]^{\frac{1}{2}}} = \frac{\pi a \sqrt{n}}{4} . \tag{7}$$

The output, I_n, of a square-law detector operating on $2n$ independent samples of the Gaussian sum signal $N_1(t) + N_2(t) + 2S(t)$ has the following mean and variance[7]:

$$EI_n = 2\,[\text{Var } N_1(t) + 2 \text{ Var } S(t)] \tag{8}$$

$$\text{Var } I_n = \frac{4\,[\text{Var } N_1(t) + 2 \text{ Var } S(t)]^2}{n}. \tag{9}$$

Thus, for small values of "a" the detection index k_1 for the square-law detector is given by

$$k_1 = \frac{4 \text{ Var } S(t)}{\left[\dfrac{8 \text{ Var}^2 N_1(t)}{n}\right]^{\frac{1}{2}}} = a\sqrt{2n}\,. \tag{10}$$

Incidentally, the square-law detector operating on the Gaussian sum signal is equivalent to the Neyman-Pearson detector of reference 2.

For equal detector performances the detection indices must be equal. Thus, from (7) and (10) we see that the performance of the square-law detector, in terms of signal-to-noise power ratio "a," is approximately 2.5 *db* better than the performance of the phase detector when gain fluctuations are negligible. Clearly, when gain fluctuations are significant the phase detector can perform better than the square-law detector since the phase difference η_i is relatively insensitive to gain fluctuations. The phase detector can be implemented by using radio interferometer-type receivers or monopulse radar-type receivers.[5]

It will now be shown that the test statistic $\eta^\dagger$ defined by (1) is the optimum test statistic for processing n independent samples of the phase difference η_i. Equation (34) of Ref. 5 gives the probability density $p_2(\eta)$ of each independent sample η_i as

$$p_2(\eta) = \frac{1 - l^2}{2\pi}(1 - \beta_2^{\,2})^{-3/2}\left[\beta_2 \sin^{-1}\beta_2 + \frac{\pi\beta_2}{2} + \sqrt{1 - \beta_2^{\,2}}\right] \tag{11}$$

where

$$\beta_2 = l \cos \eta$$

$$l = \frac{a}{1 + a}.$$

By using (11) for small values of "a" and applying the likelihood-ratio test[8,9] associated with n independent samples of η_i one decides that a signal is present only if

$$\log \frac{\prod_{i=1}^{n} \frac{1}{2\pi} \left(1 + \frac{\pi}{2} a \cos \eta_i\right)}{\left(\frac{1}{2\pi}\right)^n} > \Lambda_0 , \tag{12}$$

or

$$\sum_{i=1}^{n} \log \left(1 + \frac{\pi a}{2} \cos \eta_i\right) \doteq \frac{\pi}{2} a \sum_{i=1}^{n} \cos \eta_i > \Lambda_0 , \tag{13}$$

or

$$\frac{1}{n} \sum_{i=1}^{n} \cos \eta_i = \eta^{\dagger} > \Lambda_1 \tag{14}$$

where Λ_0 and Λ_1 are constant threshold values. This establishes the optimum property of $\eta^{\dagger}$.

III. CHARACTERISTICS OF THE PHASE DETECTOR

Let us summarize the important characteristics of the phase detector:

(*i*) The phase detector is relatively insensitive to system gain fluctuations.

(*ii*) For detecting weak signals, the performance of the phase detector is comparable to the performance of the square-law detector even with no gain fluctuations. With significant gain fluctuations, the phase detector can perform better than the square-law detector. Furthermore, unlimited post-detection integration is permitted with the phase detector.

(*iii*) The narrow-band signal applied to the phase detector may be a "white" Gaussian signal, a sinusoidal signal, or an arbitrary narrow-band Gaussian signal.

(*iv*) The phase detector can be implemented by using radio interferometer-type receivers or monopulse radar-type receivers.

(*v*) The phase detector utilizes two receivers.

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