Some Stability Results Related to Those of V. M. Popov*

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In a recent paper by this writer, some new techniques are described for obtaining sufficient conditions for the $\mathcal{L}_2$-boundedness and $\mathcal{L}_\infty$-boundedness of solutions of nonlinear functional equations. In this paper, these techniques are developed further and are used to prove some stability results for large classes of feedback systems and electrical networks that contain subsystems which are not necessarily representable in terms of ordinary differential equations.

I. NOTATION

Let $\mathcal{H}(0, \infty)$ denote the set of real-valued measurable functions of the real variable t defined on $[0, \infty)$. Let

$$\mathcal{L}_p(0, \infty) = \left\{ f \mid f \,\varepsilon\, \mathcal{H}(0, \infty), \int_0^\infty |f(t)|^p \, dt < \infty \right\}$$

for $p = 1$ and $p = 2$. Let

$$\mathcal{L}_\infty(0, \infty) = \{ f \mid f \,\varepsilon\, \mathcal{H}(0, \infty), \sup |f(t)| < \infty \}.$$

Let $y \,\varepsilon\, (0, \infty)$, and define f_y by

$$\begin{aligned} f_y(t) &= f(t) \qquad \text{for } t \,\varepsilon\, [0, y] \\ &= 0 \qquad \text{for } t > y \end{aligned}$$

for all $f \,\varepsilon\, \mathcal{H}(0, \infty)$, and let

$$\mathcal{E}(0, \infty) = \{ f \mid f \,\varepsilon\, \mathcal{H}(0, \infty), f_y \,\varepsilon\, \mathcal{L}_2(0, \infty) \text{ for all } y \,\varepsilon\, (0, \infty) \}$$

[i.e., $\mathcal{E}(0, \infty)$ denotes the set of real-valued *locally* square-integrable functions defined on $[0, \infty)$].

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Finally, the integral

$$\int_0^y f(t)g(t)\, dt$$

is denoted by $\langle f_y , g \rangle$ (or by $\langle g, f_y \rangle$) for all $f \varepsilon \mathcal{E}(0, \infty)$, all $g \varepsilon \mathcal{E}(0, \infty)$, and all $y \varepsilon (0, \infty)$; and $\| h \|$ denotes

$$\left(\int_0^\infty | h(t) |^2 \, dt \right)^{\frac{1}{2}}$$

for all $h \varepsilon \mathfrak{L}_2(0, \infty)$.

II. INTRODUCTION

To a considerable extent, Ref. 1 is a summary of certain results of a recent study by this writer of the input-output properties of a large class of time-varying nonlinear systems. The properties of a vector nonlinear Volterra integral equation of the second kind that frequently arises in the study of physical systems are considered in detail,* and some conditions are presented for the norm-boundedness of solutions of a functional equation of similar type defined on an abstract space. Much of the material presented in Ref. 1 is drawn from Refs. 2 and 3.

In Ref. 1, some techniques other than those of Refs. 2 and 3 are described for obtaining sufficient conditions for the $\mathfrak{L}_2$-boundedness and $\mathfrak{L}_\infty$-boundedness of solutions of functional equations. In this paper, these techniques are developed further and are used to prove some stability results, related to those of V. M. Popov,[8] for large classes of feedback systems and electrical networks that contain subsystems which are not necessarily representable in terms of ordinary differential equations.†

III. THE FEEDBACK SYSTEM AND THE MAIN RESULTS

Consider the system of Fig. 1. We shall restrict our discussion throughout to cases in which g, f, u, r, v, and w denote functions belonging to $\mathcal{E}(0, \infty)$. The block labeled ψ represents a memoryless time-invariant nonlinear element that introduces the constraint $w(t) = \psi[v(t)]$ for $t \geqq 0$.

* The results for the Volterra equation are of direct engineering interest because of the central role played by a certain "critical-disk" frequency-domain condition. "Critical-disk" frequency-domain conditions were encountered in connection with related analytical questions in Refs. 4, 5, and 6. Some material related to the results of Refs. 1, 4, 5, and 6 has been written up by G. Zames.[7]

† Some interesting "Popov-like" stability theorems for systems governed by ordinary differential equations are proved in Refs. 9, 10, and 11.

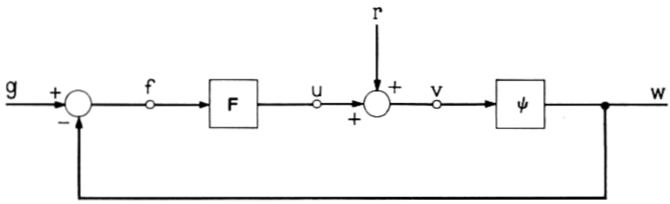

Fig. 1 — Nonlinear feedback system.

Assumption 1: $\psi(x)$ is defined and continuous on $(-\infty, \infty)$, $\psi(0) = 0$, and there exist real constants $\alpha > 0$ and $\beta < \infty$ such that

$$\alpha \leqq x^{-1}\psi(x) \leqq \beta$$

for all $x \neq 0$.

The block labeled $\mathbf{F}$ represents a (not necessarily linear or time-invariant) subsystem that introduces the constraint $(\mathbf{F}f)(t) = u(t)$ for $t \geqq 0$.

Assumption 2: The operator $\mathbf{F}$ can be written as $\mathbf{F}_2\mathbf{F}_1$ with

(1.) $\mathbf{F}_1$ a (not necessarily linear or time-invariant) mapping of $\mathcal{E}(0, \infty)$ into itself, and

(2.) $\mathbf{F}_2$ the linear mapping of $\mathcal{E}(0, \infty)$ into itself defined by the condition:

$$(\mathbf{F}_2 q)(t) = \int_0^t \exp\left[-\int_\tau^t \delta(x)dx\right] q(\tau)\, d\tau, \qquad t \geqq 0$$

for all $q \,\varepsilon\, \mathcal{E}(0, \infty)$, in which δ is a real measurable function defined on $[0, \infty)$ such that there exist real constants $c_1 > 0$ and $c_2 < \infty$ with the properties that $c_1 \leqq \delta(x) \leqq c_2$ for all $x \,\varepsilon\, [0, \infty)$.

We note that $\mathbf{F}_2 q$ denotes the convolution of q with the impulse-response function of a positive-element parallel resistor-capacitor combination with time-varying resistance.

In Fig. 1, g denotes an input and r takes into account the effect of initial conditions at $t = 0$. The relation between f, g, and r is

$$g = f + \psi[\mathbf{F}_2\mathbf{F}_1 f + r]. \tag{1}$$

Equation (1) also governs the behavior of a large class of active time-varying nonlinear networks. A network analog of the feedback system of Fig. 1 is shown in Fig. 2, where ψ denotes a nonlinear conductance.

Assumption 3: $r \,\varepsilon\, \mathcal{E}(0, \infty)$, $\dot{r}$ exists on $[0, \infty)$ and $\dot{r} \,\varepsilon\, \mathcal{E}(0, \infty)$.

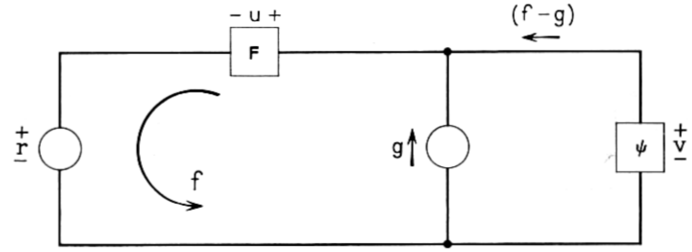

Fig. 2 — Equivalent network.

3.1 *The Main Results*

The principal contributions of this paper are believed to be the techniques used to prove the following results.

Theorem 1: Let Assumptions 1, 2, and 3 be satisfied. Let $\mathbf{F}_1$ *be such that there exist a real constant k, a nonnegative constant σ, and a positive constant c with the properties that*

$$(i) \qquad \sigma < 2\alpha^2\beta^{-1}(\beta^{-1}c_1 + k)$$

$$(ii) \qquad \langle e^{\sigma t}(\mathbf{F}_1 q)_y\,, q\rangle \geqq k \,\|\, e^{\frac{1}{2}\sigma t} q_y \,\|^2$$

$$\|\, e^{\frac{1}{2}\sigma t}(\mathbf{F}_1 q)_y \,\| \leqq c \,\|\, e^{\frac{1}{2}\sigma t} q_y \,\|$$

for all $q \,\varepsilon\, \mathcal{E}(0, \infty)$ *and all* $y \,\varepsilon\, (0, \infty)$.

Let $f \,\varepsilon\, \mathcal{E}(0, \infty)$, *and let*

$$g = f + \psi[\mathbf{F}_2\mathbf{F}_1 f + r].$$

Then there exists a positive constant λ, depending only on k, σ, c_1 *, α, β, and c, such that*

$$\lambda \,\|\, e^{\frac{1}{2}\sigma t} f_y \,\| \leqq \|\, e^{\frac{1}{2}\sigma t} g_y \,\| + \|\, e^{\frac{1}{2}\sigma t}(\dot{r} + \delta r)_y \,\| + \left[\int_0^{r(0)} \psi(\eta)\, d\eta\right]^{\frac{1}{2}}$$

for all $y \,\varepsilon\, (0, \infty)$.

Corollary 1: If the hypotheses of Theorem 1 are satisfied with $\sigma = 0$, *i* $g \,\varepsilon\, \mathcal{L}_2(0, \infty)$, *and if* $(\dot{r} + \delta r) \,\varepsilon\, \mathcal{L}_2(0, \infty)$, *then* $f \,\varepsilon\, \mathcal{L}_2(0, \infty)$, *and there exists a constant* $\lambda > 0$, *depending only on k,* c_1 *, α, β, and c such that*

$$\lambda \,\| f \| \leqq \| g \| + \| \dot{r} + \delta r \| + \left[\int_0^{r(0)} \psi(\eta) d\eta\right]^{\frac{1}{2}}.$$

Remarks: If $\mathbf{F}_1$ denotes the mapping of $\mathcal{E}(0,\infty)$ into itself defined by

$$(\mathbf{F}_1 u)(t) = v_0 u(t) + \int_0^t v(t-\tau)u(\tau)\,d\tau, \qquad t \geqq 0$$

for all $u \,\varepsilon\, \mathcal{E}(0,\infty)$, where v_0 is a real constant and $v \,\varepsilon\, \mathcal{L}_1(0,\infty)$, then

$$\| (\mathbf{F}_1 q)_y \| \leqq \left[|v_c| + \int_0^\infty | v(t) |\, dt \right] \| q_y \|$$

for all $y \,\varepsilon\, (0,\infty)$ and all $q \,\varepsilon\, \mathcal{E}(0,\infty)$, and $\langle (\mathbf{F}_1 q)_y , q \rangle \geqq k \| q_y \|^2$ for all $y \,\varepsilon\, (0,\infty)$ and all $q \,\varepsilon\, \mathcal{E}(0,\infty)$ provided that

$$v_0 + \mathrm{Re} \int_0^\infty e^{-i\omega t} v(t)\, dt \geqq k$$

for all $\omega \,\varepsilon\, (-\infty,\infty)$.

Corollary 2: If the hypotheses of Corollary 1 are satisfied, if $g(t) \to 0$ as $t \to \infty$, and if $r(t) \to 0$ as $t \to \infty$, then $f(t) \to 0$ as $t \to \infty$.

Corollary 3: If the hypotheses of Theorem 1 are satisfied with $\sigma > 0$, if $g \,\varepsilon\, \mathcal{L}_\infty(0,\infty)$, if r and $\dot{r}$ belong to $\mathcal{L}_\infty(0,\infty)$, and if there exists a constant γ such that, with $\mathbf{F} = \mathbf{F}_2\mathbf{F}_1$,

$$|(\mathbf{F}q)(y)| \leqq \gamma e^{-\frac{1}{2}\sigma y} \| e^{\frac{1}{2}\sigma t} q_y \|$$

for all $q \,\varepsilon\, \mathcal{E}(0,\infty)$ and all $y \,\varepsilon\, (0,\infty)$, then $f \,\varepsilon\, \mathcal{L}_\infty(0,\infty)$, and there exists a positive constant λ_1, depending only on k, σ, γ, c_1, α, β, and c such that

$$\lambda_1 \sup_{t \geqq 0} | (\mathbf{F}_2\mathbf{F}_1 f)(t) | \leqq \sup_{t \geqq 0} | g(t) | + \sup_{t \geqq 0} | (\dot{r} + \delta r)(t) | + \left[\int_0^{r(0)} \psi(\eta)\, d\eta \right]^{\frac{1}{2}}.$$

Remarks: Suppose that $\mathbf{F}_1$ is the mapping of $\mathcal{E}(0,\infty)$ into itself defined by

$$(\mathbf{F}_1 u)(t) = v_0 u(t) + \int_0^t v(t-\tau)u(\tau)\,d\tau, \qquad t \geqq 0$$

for all $u \,\varepsilon\, \mathcal{E}(0,\infty)$, where v_0 is a real constant and $e^{\frac{1}{2}\sigma t} v \,\varepsilon\, \mathcal{L}_1(0,\infty)$ for some positive constant σ. Then

$$\langle e^{\sigma t}(\mathbf{F}_1 q)_y, q\rangle = \int_0^\infty e^{\frac{1}{2}\sigma t} q_y(t) \Big[v_0 e^{\frac{1}{2}\sigma t} q_y(t) + \int_0^t v(t-\tau) e^{\frac{1}{2}\sigma(t-\tau)} e^{\frac{1}{2}\sigma\tau} q_y(\tau) d\tau \Big] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \Big[v_0 + \int_0^\infty v(t) e^{-(i\omega - \frac{1}{2}\sigma)t} dt \Big] \Big| \int_0^\infty e^{\frac{1}{2}\sigma t} q_y(t) e^{-i\omega t} dt \Big|^2 d\omega$$

and

$$\| e^{\frac{1}{2}\sigma t}(\mathbf{F}_1 q)_y \| \leqq \| e^{\frac{1}{2}\sigma t}\mathbf{F}_1 q_y \| = \Big\| e^{\frac{1}{2}\sigma t} v_0 q_y + \int_0^t v(t-\tau) e^{\frac{1}{2}\sigma(t-\tau)} e^{\frac{1}{2}\sigma\tau} q_y(\tau)\, d\tau \Big\|$$

$$\leqq \Big[| v_0 | + \int_0^\infty | v(t) e^{\frac{1}{2}\sigma t} | \, dt \Big] \| e^{\frac{1}{2}\sigma t} q_y \|$$

for all $y \ \varepsilon \ (0, \infty)$ and all $q \ \varepsilon \ \mathcal{E}(0, \infty)$. Thus Assumption (ii) of Theorem 1 is satisfied if

$$v_0 + \mathrm{Re} \int_0^\infty v(t) e^{-(i\omega - \frac{1}{2}\sigma)t} dt \geqq k$$

for all $\omega \ \varepsilon \ (-\infty, \infty)$.

Concerning the key hypothesis of Corollary 3, if $\mathbf{F} = \mathbf{F}_2\mathbf{F}_1$ is the mapping of $\mathcal{E}(0, \infty)$ into itself defined by

$$(\mathbf{F}u)(t) = \int_0^t w(t-\tau) u(\tau) d\tau, \qquad t \geqq 0$$

for all $u \ \varepsilon \ \mathcal{E}(0, \infty)$, where $e^{\frac{1}{2}\sigma t} w \ \varepsilon \ \mathfrak{L}_2(0, \infty)$, then

$$| (\mathbf{F}q)(y) | = e^{-\frac{1}{2}\sigma y} \Big| \int_0^y w(y-\tau) e^{\frac{1}{2}\sigma(y-\tau)} e^{\frac{1}{2}\sigma\tau} q_y(\tau)\, d\tau \Big|$$

$$\leqq e^{-\frac{1}{2}\sigma y} \Big(\int_0^\infty | w(t) e^{\frac{1}{2}\sigma t} |^2 \, dt \Big)^{\frac{1}{2}} \| e^{\frac{1}{2}\sigma t} q_y \|$$

for all $q \ \varepsilon \ \mathcal{E}(0, \infty)$ and all $y \ \varepsilon \ (0, \infty)$.

3.2 *Related Results*

The results stated above can be extended in many different directions by exploiting the techniques of Section 4.1. For example, similar results can be obtained (see Section 4.5) for the case in which the nonlinear element ψ of Fig. 1 is replaced by a linear time-varying element that introduces the constraint $w(t) = m(t)v(t)$ for $t \geqq 0$, in which $m(\cdot)$ is a positive bounded measurable function. In that case

$$\tilde{g} = f + m\mathbf{F}_2\mathbf{F}_1 f,$$

in which $\tilde{g} = g - mr$.

A specific application of the material of Section 3.1 is considered in the appendix. In particular, the result proved there implies that a rather general type of (not necessarily lumped) time-invariant physical system containing a single nonlinear element is "bounded-input bounded-output stable" if the so-called Popov inequality[8] is satisfied.

Some material related to the content of Theorem 1 can be found in Ref. 7. Our results differ in many respects from those stated in Ref. 7. In particular, there the effect of the initial condition function r is not taken into consideration.

The idea of using an inequality of the form stated in Corollary 3 in order to establish the boundedness of solutions of nonlinear functional equations evolved from the techniques of Ref. 3 and was presented in Ref. 1. This idea has also been considered by G. Zames in very recent independent unpublished work.

IV. PROOFS

4.1 *Proof of Theorem 1*

Lemma 1: Suppose that Assumptions 1, 2, and 3 are satisfied. Let σ be a nonnegative constant. Then

$$\langle e^{\sigma t}(\psi[\mathbf{F}_2 q_y + r])_y, q_y\rangle \geqq \left(\frac{c_1}{\beta} - \frac{\sigma\beta}{2\alpha^2}\right) \| e^{\frac{1}{2}\sigma t}(\psi[\mathbf{F}_2 q_y + r])_y \|^2$$

$$- \| e^{\frac{1}{2}\sigma t}(\dot{r} + \delta r)_y \| \cdot \| e^{\frac{1}{2}\sigma t}(\psi[\mathbf{F}_2 q_y + r])_y \| - \int_0^{r(0)} \psi[\eta]\, d\eta$$

for all $y \,\varepsilon\, (0,\infty)$ and all $q \,\varepsilon\, \mathcal{E}(0,\infty)$.

Proof of Lemma 1: Let $y \,\varepsilon\, (0,\infty)$, let $q \,\varepsilon\, \mathcal{E}(0,\infty)$, and let $z = \mathbf{F}_2 q_y$. Then $\dot{z}(t) + \delta(t)z(t) = q_y(t)$ for almost all $t \,\varepsilon\, (0,\infty)$, and, with $\sigma \,\varepsilon\, [0,\infty)$,

$$\langle e^{\sigma t}(\psi[\mathbf{F}_2 q_y + r])_y, q_y\rangle = \langle e^{\sigma t}(\psi[z + r])_y, \delta(z + r)\rangle$$

$$+ \langle e^{\sigma t}(\psi[z + r])_y, \dot{z} + \dot{r}\rangle$$

$$- \langle e^{\sigma t}(\psi[z + r])_y, \dot{r} + \delta r\rangle.$$

Thus, since

$$\langle e^{\sigma t}(\psi[z + r])_y, \delta(z + r)\rangle \geqq (c_1/\beta) \| e^{\frac{1}{2}\sigma t}(\psi[z + r])_y \|^2$$

(we have used the fact that $x/\psi(x) \geqq \beta^{-1}$ for all real $x \neq 0$), and (by the Schwarz inequality)

$$| \langle e^{\sigma t}(\psi[z + r])_y, \dot{r} + \delta r\rangle | \leqq \| e^{\frac{1}{2}\sigma t}(\dot{r} + \delta r)_y \| \cdot \| e^{\frac{1}{2}\sigma t}(\psi[z + r])_y \|,$$

we have

$$\begin{aligned}\langle e^{\sigma t}(\psi[\mathbf{F}_2 q_y + r])_y, q_y\rangle \geqq (c_1/\beta) \| e^{\frac{1}{2}\sigma t}(\psi[z + r])_y \|^2 \\ - \| e^{\frac{1}{2}\sigma t}(\dot{r} + \delta r)_y \| \cdot \| e^{\frac{1}{2}\sigma t}(\psi[z + r])_y \| \\ + \langle e^{\sigma t}(\psi[z + r])_y, \dot{z} + \dot{r}\rangle.\end{aligned}$$

With $\eta = z + r$, we find that

$$\begin{aligned}\langle e^{\sigma t}(\psi[z + r])_y, \dot{z} + \dot{r}\rangle &= \langle e^{\sigma t}(\psi[\eta])_y, \dot{\eta}\rangle = \int_0^y e^{\sigma t}\psi[\eta]\dot{\eta}\, dt \\ &= e^{\sigma t}\int_{\eta(0)}^{\eta(t)} \psi[\eta]d\eta \Big|_0^y - \sigma \int_0^y \int_{\eta(0)}^{\eta(t)} \psi[\eta]d\eta e^{\sigma t}\, dt \\ &= e^{\sigma y}\int_0^{\eta(y)} \psi[\eta]d\eta - \int_0^{\eta(0)} \psi[\eta]d\eta \\ &\quad - \sigma \int_0^y \int_0^{\eta(t)} \psi[\eta]d\eta e^{\sigma t}\, dt.\end{aligned}$$

Thus,

$$\langle e^{\sigma t}(\psi[\eta])_y, \dot{\eta}\rangle \geqq -\int_0^{\eta(0)} \psi[\eta]d\eta - \sigma \int_0^y \int_0^{\eta(t)} \psi[\eta]d\eta e^{\sigma t}\, dt$$

Since

$$0 \leqq \int_0^y \int_0^{\eta(t)} \psi[\eta]d\eta e^{\sigma t}dt \leqq \beta \int_0^y \int_0^{\eta(t)} \eta d\eta e^{\sigma t}\, dt,$$

and

$$\begin{aligned}\beta \int_0^y \tfrac{1}{2}[\eta(t)]^2 e^{\sigma t}dt &\leqq \frac{\beta}{2\alpha^2}\int_0^y \{\psi[\eta(t)]\}^2 e^{\sigma t}\, dt \\ &= \frac{\beta}{2\alpha^2} \| e^{\frac{1}{2}\sigma t}(\psi[\eta])_y \|^2,\end{aligned}$$

we have, using the fact that $\eta(0) = r(0)$,

$$\langle e^{\sigma t}(\psi[z+r])_y, \dot{z}+\dot{r}\rangle \geqq -\int_0^{r(0)} \psi[\eta]d\eta - \frac{\sigma\beta}{2\alpha^2}\| e^{\frac{1}{2}\sigma t}(\psi[z+r])_y \|^2.$$

Upon combining our bounds, we obtain the inequality stated in the lemma.

Lemma 2: Let **A** *and* **B** *denote mappings of* $\mathcal{E}(0,\infty)$ *into itself. Let* σ *be a real constant. Let* $f \,\varepsilon\, \mathcal{E}(0,\infty)$, $h = \mathbf{B}f$, *and* $g = f + \mathbf{A}h$. *Then*

$$|\langle e^{\sigma t}(\mathbf{A}h)_y, h_y\rangle + \langle e^{\sigma t}(\mathbf{B}f)_y, f_y\rangle| \leqq \| e^{\frac{1}{2}\sigma t} g_y \| \cdot \| e^{\frac{1}{2}\sigma t} h_y \|$$

for all $y \,\varepsilon\, (0,\infty)$.

Proof of Lemma 2: It is clear that

$$\begin{aligned}\langle e^{\sigma t}(\mathbf{A}h)_y, h_y\rangle + \langle e^{\sigma t}(\mathbf{B}f)_y, f_y\rangle &= \langle e^{\sigma t}(\mathbf{A}h)_y + e^{\sigma t} f_y, h_y\rangle \\ &= \langle e^{\sigma t} g_y, h_y\rangle \\ &= \langle e^{\frac{1}{2}\sigma t} g_y, e^{\frac{1}{2}\sigma t} h_y\rangle\end{aligned}$$

for all $y \,\varepsilon\, (0,\infty)$. Therefore, by the Schwarz inequality,

$$|\langle e^{\sigma t}(\mathbf{A}h)_y, h_y\rangle + \langle e^{\sigma t}(\mathbf{B}f)_y, f_y\rangle| \leqq \| e^{\frac{1}{2}\sigma t} g_y \| \cdot \| e^{\frac{1}{2}\sigma t} h_y \|$$

for all $y \,\varepsilon\, (0,\infty)$.

Lemma 3: Let **A** *and* **B** *denote mappings of* $\mathcal{E}(0,\infty)$ *into itself. Let* σ *be a real constant. Let* $f \,\varepsilon\, \mathcal{E}(0,\infty)$, *and let* $g = f + \mathbf{AB}f$. *Suppose that*

(*i*) *there exists a real constant* k_1' *such that*

$$\langle e^{\sigma t}(\mathbf{B}q)_y, q_y\rangle \geqq k_1' \| e^{\frac{1}{2}\sigma t} q_y \|^2$$

for all $q \,\varepsilon\, \mathcal{E}(0,\infty)$ *and all* $y \,\varepsilon\, (0,\infty)$

(*ii*) *there exist a positive constant* k_1, *and nonnegative functions* $k_2(y)$ *and* $k_3(y)$ *such that*

$$\langle e^{\sigma t}(\mathbf{A}q)_y, q_y\rangle \geqq (k_1 - k_1')\| e^{\frac{1}{2}\sigma t}(\mathbf{A}q)_y \|^2 - k_2(y)\| e^{\frac{1}{2}\sigma t}(\mathbf{A}q)_y \| - k_3(y)$$

for all $q \,\varepsilon\, \mathcal{E}(0,\infty)$ *and all* $y \,\varepsilon\, (0,\infty)$

(*iii*) *there exists a constant* $k_4 > 0$ *such that* $\| e^{\frac{1}{2}\sigma t}(\mathbf{B}q)_y \| \leqq k_4 \| e^{\frac{1}{2}\sigma t} q_y \|$ *for all* $q \,\varepsilon\, \mathcal{E}(0,\infty)$ *and all* $y \,\varepsilon\, (0,\infty)$.

Then there exists a positive constant λ, *depending only on* k_1', k_1 *and* k_4 *such that*

$$\lambda \| e^{\frac{1}{2}\sigma t} f_y \| \leqq \| e^{\frac{1}{2}\sigma t} g_y \| + k_2(y) + [k_3(y)]^{\frac{1}{2}}$$

for all $y \,\varepsilon\, (0,\infty)$.

Proof of Lemma 3: Let $y \,\varepsilon\, (0, \infty)$. Using Lemma 2, we have, with $h = \mathbf{B}f$,

$$\langle e^{\sigma t}(\mathbf{A}h)_y, h_y\rangle + \langle e^{\sigma t}(\mathbf{B}f)_y, f_y\rangle \leqq |\langle e^{\sigma t}(\mathbf{A}h)_y, h_y\rangle + \langle e^{\sigma t}(\mathbf{B}f)_y, f_y\rangle| \leqq \| e^{\frac{1}{2}\sigma t} g_y \| \cdot \| e^{\frac{1}{2}\sigma t} h_y \|.$$

Thus,

$$(k_1 - k_1')\| e^{\frac{1}{2}\sigma t}(\mathbf{A}h)_y \|^2 - k_2(y) \| e^{\frac{1}{2}\sigma t}(\mathbf{A}h)_y \| - k_3(y) + k_1' \| e^{\frac{1}{2}\sigma t} f_y \|^2 \leqq k_4 \| e^{\frac{1}{2}\sigma t} g_y \| \cdot \| e^{\frac{1}{2}\sigma t} f_y \|.$$

Using the fact that $(\mathbf{A}h)_y = g_y - f_y$, we have

$$k_1 \| e^{\frac{1}{2}\sigma t}(g_y - f_y) \|^2 + 2k_1'\langle e^{\frac{1}{2}\sigma t} g_y, e^{\frac{1}{2}\sigma t} f_y\rangle - k_1' \| e^{\frac{1}{2}\sigma t} g_y \|^2 - k_2(y) \| e^{\frac{1}{2}\sigma t}(g_y - f_y) \| - k_3(y) \leqq k_4 \| e^{\frac{1}{2}\sigma t} g_y \| \cdot \| e^{\frac{1}{2}\sigma t} f_y \|.$$

Therefore,

$$\begin{aligned} k_1 \| e^{\frac{1}{2}\sigma t}(g_y - f_y) \|^2 &\leqq k_2(y) \| e^{\frac{1}{2}\sigma t}(g_y - f_y) \| + k_3(y) + k_1' \| e^{\frac{1}{2}\sigma t} g_y \|^2 \\ &\quad + (2 | k_1' | + k_4) \| e^{\frac{1}{2}\sigma t} g_y \| \cdot \| e^{\frac{1}{2}\sigma t} f_y \| \\ &\leqq k_2(y) \| e^{\frac{1}{2}\sigma t}(g_y - f_y) \| + k_3(y) \\ &\quad + (2 | k_1' | + k_4) \| e^{\frac{1}{2}\sigma t}(g_y - f_y) \| \cdot \| e^{\frac{1}{2}\sigma t} g_y \| \\ &\quad + (2 | k_1' | + k_1' + k_4) \| e^{\frac{1}{2}\sigma t} g_y \|^2. \end{aligned}$$

Let $\rho = \| e^{\frac{1}{2}\sigma t}(g_y - f_y) \|$. Then

$$k_1\rho^2 \leqq [k_2(y) + (2 | k_1' | + k_4) \| e^{\frac{1}{2}\sigma t} g_y \|]\rho + k_3(y) + (2 | k_1' | + k_1' + k_4) \| e^{\frac{1}{2}\sigma t} g_y \|^2,$$

and hence,

$$\begin{aligned} 2\rho &\leqq [k_1^{-1}k_2(y) + k_1^{-1}(2 | k_1' | + k_4) \| e^{\frac{1}{2}\sigma t} g_y \|] \\ &\quad + \{[k_1^{-1}k_2(y) + k_1^{-1}(2 | k_1' | + k_4) \| e^{\frac{1}{2}\sigma t} g_y \|]^2 \\ &\quad + 4[k_1^{-1}k_3(y) + k_1^{-1}(2 | k_1' | + k_1' + k_4) \| e^{\frac{1}{2}\sigma t} g_y \|^2]\}^{\frac{1}{2}}. \end{aligned}$$

Since $(a^2 + b)^{\frac{1}{2}} \leqq a + b^{\frac{1}{2}}$ for any positive constants a and b,

$$\begin{aligned} \rho &\leqq [k_1^{-1}k_2(y) + k_1^{-1}(2 | k_1' | + k_4) \| e^{\frac{1}{2}\sigma t} g_y \|] \\ &\quad + [k_1^{-1}k_3(y) + k_1^{-1}(2 | k_1' | + k_1' + k_4) \| e^{\frac{1}{2}\sigma t} g_y \|^2]^{\frac{1}{2}} \\ &\leqq [k_1^{-1}k_2(y) + k_1^{-1}(2 | k_1' | + k_4) \| e^{\frac{1}{2}\sigma t} g_y \|] \\ &\quad + [k_1^{-1}k_3(y)]^{\frac{1}{2}} + [k_1^{-1}(2 | k_1' | + k_1' + k_4)]^{\frac{1}{2}} \| e^{\frac{1}{2}\sigma t} g_y \|. \end{aligned}$$

Using $\| e^{\frac{1}{2}\sigma t} f_y \| \leqq \rho + \| e^{\frac{1}{2}\sigma t} g_y \|$, we see that

$$\| e^{\frac{1}{2}\sigma t} f_y \| \leqq \{1 + k_1^{-1}(2 | k_1' | + k_4) + [k_1^{-1}(2 | k_1' | + k_1' + k_4)]^{\frac{1}{2}}\} \| e^{\frac{1}{2}\sigma t} g_y \| + k_1^{-1} k_2(y) + k_1^{-\frac{1}{2}} k_3(y)^{\frac{1}{2}}.$$

This proves the lemma.*

Theorem 1 follows at once from Lemmas 1, 2, and 3 with $\mathbf{B} = \mathbf{F}_1$, $\mathbf{A}$ defined by

$$\mathbf{A}h = \psi[\mathbf{F}_2 h + r]$$

for all $h \,\varepsilon\, \mathcal{E}(0, \infty)$, $k_1' = k$, $k_4 = c$, $k_1 = c_1\beta^{-1} - \frac{1}{2}\sigma\beta\alpha^{-2} + k$, $k_2(y) = \| e^{\frac{1}{2}\sigma t}(\dot{r} + \delta r)_y \|$, and

$$k_3 = \int_0^{r(0)} \psi(\eta)\, d\eta .$$

4.2 *Proof of Corollary 1*

Since $\| g_y \| \leqq \| g \|$ and $\| (\dot{r} + \delta r)_y \| \leqq \| \dot{r} + \delta r \|$,

$$\lambda \| f_y \| \leqq \| g \| + \| \dot{r} + \delta r \| + \left[\int_0^{r(0)} \psi(\eta) d\eta \right]^{\frac{1}{2}} .$$

The right-hand side is finite and is independent of y. The conclusion of the corollary follows at once.

An equally simple argument establishes the following useful result.

Proposition 1: Suppose that the hypotheses of Lemma 3 are satisfied with $\sigma = 0$. Let $g \,\varepsilon\, \mathfrak{L}_2(0, \infty)$, and let $k_2(y)$ and $k_3(y)$ be uniformly bounded on $[0, \infty)$. Then $f \,\varepsilon\, \mathfrak{L}_2(0, \infty)$, and there exists a positive constant λ, depending only on k_1', k_1 and k_4, such that

$$\lambda \| f \| \leqq \| g \| + \sup_{y \geqq 0} k_2(y) + \sup_{y \geqq 0} [k_3(y)]^{\frac{1}{2}}.$$

4.3 *Proof of Corollary 2*

Since $f \,\varepsilon\, \mathfrak{L}_2(0, \infty)$, we have $\mathbf{F}_1 f \,\varepsilon\, \mathfrak{L}_2(0, \infty)$. Thus, for $t \geqq 0$,

$$| (\mathbf{F}_2\mathbf{F}_1 f)(t) | \leqq \int_0^t \exp\left[-\int_\tau^t \delta(x) dx \right] | (\mathbf{F}_1 f)(\tau) | \, d\tau$$

$$\leqq \int_0^t e^{-c_1(t-\tau)} | (\mathbf{F}_1 f)(\tau) | \, d\tau$$

* For results directly related to Lemmas 2 and 3, see Section 5.3 of Ref. 1.

in which the last integral approaches zero as $t \to \infty$ (see the proof of Theorem 6 of Ref. 2). Hence,

$$g(t) - (\psi[\mathbf{F}_2\mathbf{F}_1 f + r])(t) \to 0 \quad \text{as} \quad t \to \infty .$$

4.4 *Proof of Corollary 3*

Since $\| e^{\frac{1}{2}\sigma t} g_y \| \leqq \sigma^{-\frac{1}{2}} e^{\frac{1}{2}\sigma y} \sup_{t \geqq 0} | g(t) |$, and

$$\| e^{\frac{1}{2}\sigma t} (\dot{r} + \delta r)_y \| \leqq \sigma^{-\frac{1}{2}} e^{\frac{1}{2}\sigma y} \sup_{t \geqq 0} | (\dot{r} + \delta r)(t) |,$$

we have

$$| (\mathbf{F}_2\mathbf{F}_1 f)(y) | \leqq \frac{\gamma}{\lambda} \left[\sigma^{-\frac{1}{2}} \sup_{t \geqq 0} | g(t) | + \sigma^{-\frac{1}{2}} \sup_{t \geqq 0} | (\dot{r} + \delta r)(t) | + \left[\int_0^{r(0)} \psi(\eta) d\eta \right]^{\frac{1}{2}} \right]$$

for all $y \; \varepsilon \; (0, \infty)$. This establishes the last inequality of the corollary. Since

$$| f(t) | \leqq \sup_{t \geqq 0} | g(t) | + \sup_{t \geqq 0} | (\psi[\mathbf{F}_2\mathbf{F}_1 f + r])(t) |,$$

it is evident that $f \; \varepsilon \; \mathfrak{L}_\infty(0, \infty)$.

4.5 *Proof of Proposition 2*

As was stated in Section 3.2, results similar to those of Section 3.1 can be obtained for the case in which the nonlinear element ψ in Fig. 1 is replaced by a linear time-varying element that introduces the constraint $w(t) = m(t)v(t)$ for $t \geqq 0$, in which $m(\cdot)$ is a positive bounded function. For that case the proposition that plays the role of Lemma 1 is

Proposition 2: Let $m(\cdot)$ denote a positive bounded measurable function defined on $[0, \infty)$. Let $\dot{m}(t)$ exist on $[0, \infty)$ with $\dot{m} \; \varepsilon \; \mathfrak{L}_\infty(0, \infty)$, and let $\mathbf{F}_2$ be as defined in Assumption 2. Let σ be a real constant. Then

$$\langle e^{\sigma t} m(\mathbf{F}_2 q)_y , q \rangle \geqq \inf_{t \geqq 0} \left[\frac{m(t)\delta(t) - \frac{1}{2}\dot{m}(t) - \frac{1}{2}\sigma m(t)}{m(t)^2} \right] \| e^{\frac{1}{2}\sigma t} m(\mathbf{F}_2 q)_y \|^2$$

for all $y \; \varepsilon \; (0, \infty)$ and all $q \; \varepsilon \; \mathcal{E}(0, \infty)$.

Proof of Proposition 2: Let $z = \mathbf{F}_2 q_y$ with $y \; \varepsilon \; (0, \infty)$. Then

$$\langle e^{\sigma t} m(\mathbf{F}_2 q)_y , q\rangle = \langle e^{\sigma t} m z_y , \dot{z} + \delta z\rangle = \langle e^{\sigma t} m z_y , \delta z\rangle + \langle e^{\sigma t} m z_y , \dot{z}\rangle,$$

and

$$\langle e^{\sigma t} m z_y , \dot{z}\rangle = \int_0^y m(t) e^{\sigma t} z(t)\dot{z}(t) dt$$

$$= \tfrac{1}{2} m(t) e^{\sigma t} z(t)^2 \big|_0^y - \tfrac{1}{2} \int_0^y [\dot{m}(t) + \sigma m(t)] e^{\sigma t} z(t)^2 \, dt.$$

Therefore, since $z(0) = 0$,

$$\langle e^{\sigma t} m(\mathbf{F}_2 q)_y , q\rangle = \tfrac{1}{2} m(y) e^{\sigma y} z(y)^2 + \int_0^y e^{\sigma t}[m(t)\delta(t) - \tfrac{1}{2}\dot{m}(t) - \tfrac{1}{2}\sigma m(t)] z(t)^2 \, dt.$$

This establishes Proposition 2.

Comment:

The case in which $\mathbf{F}_2$ is the *identity* operator, $m(\cdot)$ is a positive bounded measurable function, and $\dot{m}(t)$ does not necessarily exist, is also of some interest.[1,2] Then

$$\langle e^{\sigma t} m(\mathbf{F}_2 q)_y , q\rangle \geqq \inf_{t \geqq 0} m(t)^{-1} \, \| e^{\frac{1}{2}\sigma t} m(\mathbf{F}_2 q)_y \|^2$$

for all $y \, \varepsilon \, (0, \infty)$ and all $q \, \varepsilon \, \mathcal{E}(0, \infty)$.

V. APPENDIX

As a specific application of the material of Section 3.1, we shall prove the following result.

Theorem 2: Let ψ satisfy Assumption 1 of Section III. Let g and r belong to $\mathcal{E}(0, \infty)$. Let w and $\dot{w}$ belong to $\mathcal{L}_1(0, \infty)$ with $w(t) \to 0$ as $t \to \infty$. Suppose that there exists a positive constant ξ such that

$$\inf_{0 \leqq \omega < \infty} \operatorname{Re}\,[(1 + \xi i\omega) W(i\omega) + \beta^{-1}] > 0,$$

in which $W(i\omega) = \int_0^\infty w(t) e^{-i\omega t} dt$. Let $f \, \varepsilon \, \mathcal{E}(0, \infty)$ satisfy

$$g(t) = f(t) + \psi\left[\int_0^t w(t - \tau) f(\tau) d\tau + r(t)\right], \qquad t \geqq 0$$

Then

(1.) *if* $g \varepsilon \mathcal{L}_2(0,\infty) \cap \mathcal{L}_\infty(0,\infty)$ *with* $g(t) \to 0$ *as* $t \to \infty$,
if $r \varepsilon \mathcal{L}_2(0,\infty) \cap \mathcal{L}_\infty(0,\infty)$ *with* $r(t) \to 0$ *as* $t \to \infty$,
and if $\dot{r} \varepsilon \mathcal{L}_2(0,\infty)$, *then* $f \varepsilon \mathcal{L}_2(0,\infty) \cap \mathcal{L}_\infty(0,\infty)$,
there exists a positive constant λ, *depending only on* ξ, α, β, *and* w, *such that*

$$\lambda \| f \| \leqq \| g \| + \|(\dot{r} + \xi^{-1} r)\| + \left[\int_0^{r(0)} \psi(\eta) d\eta \right]^{\frac{1}{2}},$$

and $f(t) \to 0$ *as* $t \to \infty$

(2.) *if there exists a positive constant* ρ *such that* $e^{\rho t} w \varepsilon \mathcal{L}_1(0,\infty) \cap \mathcal{L}_2(0,\infty)$ *and* $e^{\rho t} \dot{w} \varepsilon \mathcal{L}_1(0,\infty)$ *with* $e^{\rho t} w(t) \to 0$ *as* $t \to \infty$, *if* g, r, *and* $\dot{r}$ *belong to* $\mathcal{L}_\infty(0,\infty)$, *then* $f \varepsilon \mathcal{L}_\infty(0,\infty)$, *and there exists a positive constant* λ_1, *depending only on* ρ, ξ, α, β, *and* w *such that*

$$\lambda_1 \sup_{t \geqq 0} \left| \int_0^t w(t - \tau) f(\tau) d\tau \right| \leqq \sup_{t \geqq 0} | g(t) | + \sup_{t \geqq 0} | (\dot{r} + \xi^{-1} r)(t) | + \left[\int_0^{r(0)} \psi(\eta) d\eta \right]^{\frac{1}{2}}.$$

Proof of Theorem 2:

Let $\mathbf{F}$ be defined by

$$(\mathbf{F}q)(t) = \int_0^t w(t - \tau) q(\tau) d\tau, \qquad t \geqq 0.$$

Then $\mathbf{F} = \mathbf{F}_2 \mathbf{F}_1$, where $\mathbf{F}_2$ is as defined in Assumption 2 of Section III with $\delta(x) = \xi^{-1}$, and

$$(\mathbf{F}_1 q)(t) = \int_0^t [\dot{w}(t - \tau) + \xi^{-1} w(t - \tau)] q(\tau) d\tau + w(0+) q(t), \qquad t \geqq 0.$$

Let

$$\zeta = \inf_{0 \leqq \omega < \infty} \operatorname{Re} [(1 + \xi i\omega) W(i\omega) + \beta^{-1}].$$

Then

$$\langle (\mathbf{F}_1 q)_y, q \rangle \geqq \xi^{-1} (\zeta - \beta^{-1}) \| q_y \|^2$$

for all $q \varepsilon \mathcal{E}(0,\infty)$ and all $y \varepsilon (0,\infty)$ [see the remark following Corollary 1]. Thus conditions (i) and (ii) of Theorem 1 are satisfied for $\sigma = 0$.

Therefore, by Corollary 1, the hypotheses of (i) of Theorem 2 imply that $f \, \varepsilon \, \mathfrak{L}_2(0,\infty)$ and that $\| f \|$ is bounded as indicated. By Corollary 2, we have $f(t) \to 0$ as $t \to \infty$. Further, since $f \, \varepsilon \, \mathfrak{L}_2(0,\infty)$, we have $\mathbf{F}_1 f \, \varepsilon \, \mathfrak{L}_2(0,\infty)$, and, by the Schwarz inequality, $\mathbf{F}_2\mathbf{F}_1 f \, \varepsilon \, \mathfrak{L}_\infty(0,\infty)$. Therefore $g - \psi[\mathbf{F}f + r] \, \varepsilon \, \mathfrak{L}_\infty(0,\infty)$.

Suppose now that the hypotheses of (ii) are satisfied. Then since both $| W(i\omega - x) - W(i\omega) |$ and $| (i\omega - x)W(i\omega - x) - i\omega W(i\omega) |$ approach zero uniformly in ω as $x \to 0+$, there exists a positive constant σ such that $\sigma < \min [2\rho,\alpha^2\zeta(\beta\xi)^{-1}]$ and

$$\inf_{0 \leqq \omega < \infty} \mathrm{Re} \{[1 + \xi(i\omega - \tfrac{1}{2}\sigma)]W(i\omega - \tfrac{1}{2}\sigma) + \beta^{-1}\} > \tfrac{1}{2}\zeta.$$

Hence,

$$\langle e^{\sigma t}(\mathbf{F}_1 q)_y, q \rangle \geqq \xi^{-1}(\tfrac{1}{2}\zeta - \beta^{-1}) \, \| e^{\frac{1}{2}\sigma t} q_y \|^2$$

for all $y \, \varepsilon \, (0,\infty)$ and all $q \, \varepsilon \, \mathcal{E}(0,\infty)$. Thus, by Corollary 3 and the remarks following Corollary 3, we have $f \, \varepsilon \, \mathfrak{L}_\infty(0,\infty)$ with

$$\sup_{t \geqq 0} \left| \int_0^t w(t - \tau) f(\tau) d\tau \right|$$

bounded as stated in Theorem 2.

Comments:

Our assumption that $f \, \varepsilon \, \mathcal{E}(0,\infty)$ is satisfied if f is locally (Lebesgue) integrable on $(0,\infty)$, since then (under the stated assumptions on g, ψ, w, and r):

$$\int_0^t w(t - \tau) f(\tau) d\tau$$

is continuous on $[0,\infty)$ and hence,

$$g - \psi\left[\int_0^t w(t - \tau) f(\tau) d\tau + r\right] \varepsilon \, \mathcal{E}(0, \infty).$$

A result closely related to the first part of Theorem 2 has been proved by Desoer.[12]

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