Growth of Oscillations of a Ray about the Irregularly Wavy Axis of a Lens Light Guide

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If a ray is launched in a direction coincident with the axis of a lens light guide whose axis is somewhat wavy, the ray will soon begin to oscillate about the axis. The amplitude of these oscillations will grow in proportion to the square root of the product of the distance from the origin and the natural wave number of the oscillations, on the average. The growth rate is proportional to the amplitude of the components of the spectrum of the waves in the guide axis in the immediate vicinity of the natural wave number of the ray oscillations. The rest of the spectrum of waves in the guide does not contribute appreciably to the growth of ray oscillations after the first few oscillations. A tractable analytic expression with four adjustable parameters is used to approximate the wave spectrum of the shape of the guide axis. The expression is used to illustrate the relations between various factors such as mechanical stiffness of the guide, spectrum of external forces, over-all amplitude of waves in the guide, etc., and the rate of growth of the ray oscillations. Estimates of the order of magnitude of the oscillation growth rate in some realistic models of light guides are made from these relations.

I. INTRODUCTION

The possibility of guiding a beam of light over a long distance through a series of lenses or a continuous lens-like medium has recently received considerable attention because such a system might be useful in communication. One problem that has only recently come under investigation is the statistical growth rate of the oscillation of an initially paraxial ray about the axis of such a guide when the axis is crooked in a somewhat random but partially coherent way.[1,2] In this paper I will show that the ray will oscillate about the axis with ever-increasing amplitude, on the average.

The oscillations of the ray about the axis are analogous to the os-

TABLE I

Input parameters and results of computations in M.K.S. units: kilograms, (kg), meters, (m), and Newtons, (Nt). The first four input parameters describe the pipe; the next three, the nature of the support; and the last three, the optical elements.

Input	*Continuous*	*Either*	*Lenses*	*Continuous*	*Either*	*Lenses*
(1) ρ (kg/m^3)		7.8×10^3			same	
(2) M (Nt/m^2)		2.0×10^9			same	
(3) OD (m)		0.1			same	
(4) ID (m)		0.09			same	
(5) m (number)		11			same	
(6) n (number)		8			same	
(7) λ_p (m)		1.0			4.0	
(8) C (m^{-2})	0.25		—	same		—
(9) c (m^{-1})	—		0.637	—		same
(10) L (m)	—		3.14	—		same
Results						
(1) $\sqrt{C}$ (m^{-1})	0.5		—	0.5		—
(2) k_c (m^{-1})	—		0.5	—		0.5
(3) k_0 (m^{-1})		3.27			0.819	
(4) λ_y (m)		3.68			14.73	
(5) σ (Nt·m^2)		3380			same	
(6) $\bar{\bar{p}}$ (Nt/m)		114			same	
(7) $\bar{\bar{y}}$ (m)		0.407×10^{-2}			10.43×10^{-3}	
(8) P_0 (Nt/m)		222			same	
(9) $Y(\sqrt{C})$ (m)	0.572×10^{-3}		—	146.1×10^{-3}		—
(10) $S(k_c)$ (m)	—		1.040×10^{-3}	—		146.2×10^{-3}
(11) $\delta/\sqrt{x}$ (m$^{1/2}$)	0.358×10^{-3}		0.830×10^{-3}	91.6×10^{-2}		116.7×10^{-3}

cillations of an undamped mechanical oscillator subject to a particular steady "noise spectrum" of forces. Spherical aberration in the guide would be analogous to anharmonicity in the mechanical oscillator. Distance in the optical problem corresponds to time in the mechanical problem.

The dominant term in the expression for the amplitude of ray oscillations is proportional to the square root of the distance from the origin, if the amount and spectral form of the crookedness is constant and if the lenses have no spherical aberration. The oscillations of the ray about the axis have a natural wave number, k_c. The growth rate of ray oscillations is also proportional to the square root of k_c. The third factor governing the growth rate is the amount of crookedness of the axis. A major portion of this paper is devoted to a mathematical description of the amount of crookedness of the axis of the guide. Only those components of the crookedness having approximately the same wave number as the natural oscillations of the beam have an appreciable influence on the growth of the oscillations beyond the first few oscillations.

Some numerical examples of growth rates of oscillations in guides having reasonable amounts of crookedness or waviness are given at the end of the paper and in Table I. The amount of crookedness in the examples was obtained by estimating the variation of forces on a pipe of reasonable stiffness lying on a rough surface. See Fig. 1. The pipe was assumed to be relatively straight before the irregular forces due to the rough surface were applied. This specific method of estimating crookedness does not limit the generality of the relations between the crookedness spectrum and the growth rate of the oscillations.

The results indicate that, unless such a light guide could be kept extremely straight or free of waves on a scale approximating the wavelength of natural oscillations of the beam, the beam would have to be recentered at frequent intervals* or the guide would have to have a very large aperture to avoid vignetting of an initially paraxial ray.

II. EQUATIONS OF THE RAY TRAJECTORY

In order to keep the analysis simple, I will suppose that the axis of the guide is wavy only in one dimension, y. The distance from a straight

* There is no physical reason why we could not redirect the beam down the axis of the guide at intervals with almost no loss of beam energy. For example, the position and direction of the main beam could be located at one point by reflecting a very small fraction of it into a photoelectric analyzing device with a very weakly reflecting mirror. The analyzing device could be used to control some prisms which would change the direction and displacement of the beam. The analyzing device could be a simple null device if the beam passed through the prisms shortly before reaching the analyzer. L. U. Kibler of Bell Telephone Laboratories described this idea in an unpublished memorandum in March, 1962.

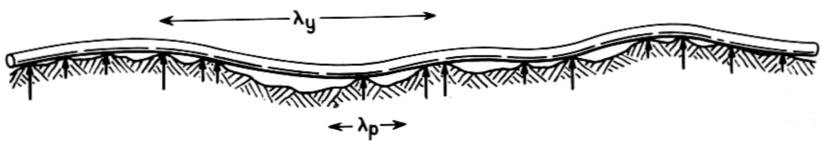

Fig. 1 — A guide on a rough surface that produces an irregular distribution of forces with a characteristic separation λ_p. This results in waves with a characteristic length λ_y .

coordinate axis, x, to the axis of the guide will be called $y(x)$ and the distance from the axis of the guide to the ray or any point in the guide will be called $\delta(x)$. See Fig. 2.

For the case of the continuous lens-like medium, suppose that the refractive index can be described as a function only of distance, δ, from the axis of the guide. A medium free of spherical aberration will be defined as one obeying the relation

$$n(\delta) = n_0 \exp(-C\delta^2/2) \approx n_0(1 - C\delta^2/2). \tag{1}$$

C is the *specific convergence*[3] of the guide, which is approximately the convergence, in diopters or reciprocal meters of a one-meter segment of the guide if that convergence is small compared to one diopter.

The equation of the trajectory of a ray in a medium of slowly, smoothly varying isotropic refractive index is[4]

$$\frac{d}{ds}\left(n \frac{d\mathbf{r}}{ds}\right) = \text{grad } n. \tag{2}$$

In this equation ds is an element of length along the path of the ray, $\mathbf{r}$ is the vector position of a point on the ray and n is the (isotropic) refractive index at that point.

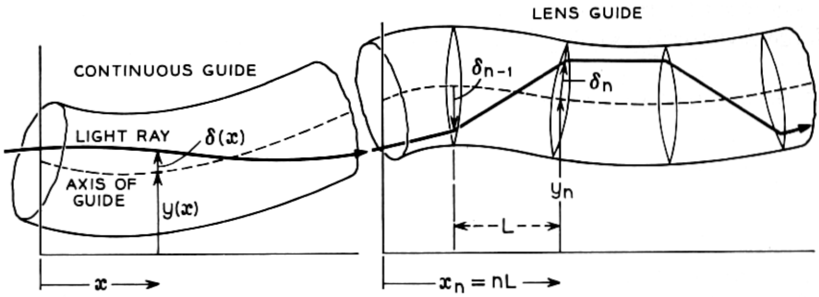

Fig. 2 — Symbols used to describe a continuous guide (left) and a thin lens guide (right).

If the slope of the ray, $d\delta/dx$, and the slope of the guide axis, dy/dx, both remain small, then (1) and (2) may be combined to give the following differential equation for the trajectory of the ray,

$$\frac{d^2\delta}{dx^2} + C\delta = -\frac{d^2y}{dx^2}. \tag{3}$$

This is like the familiar forced, undamped harmonic oscillator equation.

For a guide composed of a series of thin, aberration-free lenses of equal convergence, c, separated by a distance L, we can obtain the following analogous difference equation for the trajectory from simple geometrical construction.

$$\tfrac{1}{2}(\delta_{n+1} + \delta_{n-1}) - \delta_n + \frac{cL}{2}\delta_n = -(\tfrac{1}{2}(y_{n+1} + y_{n-1}) - y_n) \tag{4}$$

δ_n is the displacement of the ray from the axis of the nth lens and y_n is the displacement of that lens axis from the straight x-coordinate axis. (See Fig. 2.) Note that the product nL is equal to x. Hence, once we define a shape, $y(x)$, for the guide axis, the right-hand side of either (3) or (4) can be written down.

III. SOLUTIONS IN TERMS OF GUIDE SHAPE

Suppose we pretend that the shape of the guide is periodic with a length, Λ, for the periodicity. The path of the axis of the pipe can be represented by the Fourier series:

$$y(x) = \sum_{j=1}^{\infty} Y\left(\frac{2\pi j}{\Lambda}\right) \sin\left(\frac{2\pi j x}{\Lambda} - \varphi_j\right). \tag{5}$$

The phase factor φ_j is used to avoid writing cosine terms in the series. We can avoid a constant term ($j = 0$) by proper choice of the position $y = 0$. Making the substitution $k_j = 2\pi j/\Lambda$, we may write

$$y(x) = \sum_{j=1}^{\infty} Y(k_j) \sin(k_j x - \varphi_j). \tag{6}$$

3.1 *The Case of a Continuous Lens-Like Medium*

By substituting a trial solution of the form

$$\delta(x) = \sum_{j=1}^{\infty} A(k_j) \sin(k_j x - \psi_j) + B(k_j) \sin(\sqrt{C}x - \xi_j) \tag{7}$$

into (3), along with the shape spectrum (6), we obtain the results

$$A(k_j) = \frac{k_j^2 Y(k_j)}{C - k_j^2} \quad \text{and} \quad \psi_j = \varphi_j . \tag{8}$$

If we add the boundary conditions

$$\delta(0) = (d\delta/dx)_{x=0} = 0 \tag{9}$$

we find that

$$B(k_j) = -A(k_j) \sin \varphi_j / \sin \xi_j \tag{10}$$

where

$$\cot \xi_j = \frac{k_j}{\sqrt{C}} \cot \varphi_j . \tag{11}$$

Note that $\sqrt{C}$ is the natural angular wave number of oscillations of the ray about the axis of the guide if the guide is straight.

Inserting these results into (7) we finally obtain

$$\delta(x) = \sum_{j=1}^{\infty} \frac{k_j^2 Y(k_j)}{C - k_j^2} \left[\cos \varphi_j \left(\sin k_j x - \frac{k_j}{\sqrt{C}} \sin \sqrt{C} x \right) - \sin \varphi_j (\cos k_j x - \cos \sqrt{C} x) \right]. \tag{12}$$

At this point in the derivation we introduce the random, statistical character of the problem by asserting that the phase factors, φ_j, are random. Then we can no longer specify $y(x)$ or $\delta(x)$ but we can specify a mean square value of each and relate one to the other. Since the mean square value of a sinusoidal function is half the square of its absolute value, we can rewrite (5) as

$$\langle y^2(x) \rangle \equiv \bar{\bar{y}}^2 = \tfrac{1}{2} \sum_{j=1}^{\infty} Y^2(k_j). \tag{13}$$

Similarly, from (12) we obtain

$$\langle \delta^2(x) \rangle \equiv \bar{\bar{\delta}}^2(x) = \frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{k_j^2 Y(k_j)}{C - k_j^2} \right)^2 \left[\left(\sin k_j x - \frac{k_j}{\sqrt{C}} \sin \sqrt{C} x \right)^2 + (\cos k_j x - \cos \sqrt{C} x)^2 \right]. \tag{14}$$

If we make the substitution

$$\epsilon_j = \frac{k_j}{\sqrt{C}} - 1 \tag{15}$$

and do quite a lot of algebra and trigonometry, we find that (14) can be written in the following form: (Remember that ϵ_j ranges from -1 to $+\infty$.)

$$\bar{\bar{\delta}}^2(x) = \frac{1}{2}\sum_{j=1}^{\infty} Y^2(\epsilon_j)\,\frac{(1+\epsilon_j)^4}{(2+\epsilon_j)^2}\left[\frac{4\sin^2(\epsilon_j\sqrt{C}x/2)}{\epsilon_j^{\,2}} + \frac{4\sin(\epsilon_j\sqrt{C}x/2)}{\epsilon_j} - \frac{\sin\epsilon_j\sqrt{C}x\,\sin 2\sqrt{C}x}{\epsilon_j} + \sin^2\sqrt{C}x\right]. \tag{16}$$

If the length, Λ, of the periodicity of the shape of the guide is allowed to become very large, we may replace the summation signs in (13) and (16) by integrations over the variable ϵ, and we may drop the index j.

If $\epsilon^4 Y^2(\epsilon)$ remains finite as ϵ approaches infinity, then large values of ϵ will not cause the integral for $(\bar{\bar{\delta}}(x))^2$ to diverge. (Slightly weaker conditions could be applied.)

For large values of x, the first of the four terms in square brackets in (16) will dominate if $Y(\epsilon = 0)$ is nonzero. Thus we obtain the following result

$$\bar{\bar{\delta}}^2(x) \approx \frac{1}{2}\int_{-1}^{\infty} Y^2(\epsilon)\,\frac{(1+\epsilon)^4}{(2+\epsilon)^2}\cdot\frac{4\sin^2\epsilon\sqrt{C}x/2}{\epsilon^2}\,d\epsilon \rightarrow x\cdot\frac{\pi\sqrt{C}}{4}\,Y^2(\epsilon=0) \quad \text{for large } x. \tag{17}$$

The proper normalization of the function $Y(\epsilon)$ can be obtained from the following relation:

$$\bar{\bar{y}}^2 = \frac{1}{2}\int_{-1}^{\infty} Y^2(\epsilon)d\epsilon\,. \tag{18}$$

(Compare (13). It is assumed that the relative spectral distribution for $Y(\epsilon)$ is known, and we may also assume that the mean square value of y is known. See Section V for examples.)

3.2 *The Case of a Series of Lenses*

An exactly analogous but slightly more complicated procedure can be followed for the case of a series of lenses. We can keep the same function, (6), to describe the displacements of the lens centers if we let x be an integral multiple, n, of the lens separation, L. (See Fig. 2.) Equation (7) for the ray trajectory may similarly be retained with the same expression replacing x.

However, when (7) is substituted into (4) rather than (3), we obtain a somewhat different expression for $A(k_j)$. The phase factors ψ_n are the same.

$$A(k_j) = \frac{(1 - \cos k_jL)Y(k_j)}{\frac{cL}{2} - (1 - \cos k_jL)} \quad \text{and} \quad \psi_j = \varphi_j . \tag{8'}$$

We use boundary conditions analogous to (9),

$$\delta(n = 0) = \delta(n = 1) = 0 \tag{9'}$$

and we obtain expressions for B and ξ_j :

$$B(k_j) = -A(k_j)\frac{\sin \varphi_j}{\sin \xi_j} \tag{10}$$

and

$$\cot \xi_j = \frac{\cos kL - \cos k_cL + \sin kL \cot \varphi_j}{\sin k_cL} . \tag{11'}$$

In (11′), k_c is the natural angular wave number of oscillations of the ray about the axis, corresponding to $\sqrt{C}$ in the case of a continuous lens. In this case it is the first positive real solution to the following equation, if such a solution exists.

$$\cos k_cL = 1 - \frac{cL}{2} \tag{19}$$

The denominator in (8′) may be written $\cos k_jL - \cos k_cL$. (When $cL > 4$ the lenses are separated by more than four times their focal length and they will not guide the beam.)

The analog of (12) would be like (12) except that terms from (8′), (11′) and (19) replace the terms from (8) and (11). It is not worthwhile to write it out here.

Equation 19 has an infinite number of real roots if $cL < 4$. We have chosen to describe the ray trajectory using only the first root, but we cannot ignore the rest entirely. If the pipe containing the lenses has an appreciable amplitude of waviness near one or more of these higher wave-number roots, the ray oscillations will grow because the lenses will be displaced as if there were additional amplitude in the fundamental, long wave-length component. These higher roots correspond to waves in the pipe with length less than $2L$ or twice the lens separation. In choosing the lowest root for k_c we are following Brillouin's example[5] and restricting values of k to the "first Brillouin zone" of the periodic array of lenses. Remember that $k_j = 2\pi j/\Lambda$ where Λ is the length of the periodicity in the shape of the guide, which will ultimately be made very large. We can rewrite (6) in the following form in order to limit the range of k_j from zero to π/L.

$$y(nL) = \sum_{j=1}^{\Lambda/2L} \sum_{h=0}^{\infty} \left[Y\left(\frac{2\pi h}{L} + k_j\right) \sin\,(k_j nL + \varphi_{h,j,1}) + Y\left(\frac{2\pi(h+1)}{L} - k_j\right) \sin\,(k_j nL + \varphi_{h,j,2}) \right] \tag{6'}$$

(nL is the x coordinate of lens number n. See Fig. 2.) Then the analog of (13) may be written as

$$\bar{\bar{y}}^2 = \frac{1}{2} \sum_{j=1}^{\Lambda/2L} \sum_{h=0}^{\infty} \left[Y^2\left(\frac{2\pi h}{L} + k_j\right) + Y^2\left(\frac{2\pi(h+1)}{L} - k_j\right)\right] \equiv \frac{1}{2} \sum_{j=1}^{\Lambda/2L} S^2(k_j). \tag{13'}$$

When written in terms of S rather than Y, the analog of (14) looks like this:

$$\bar{\bar{\delta}}^2(x) = \frac{1}{2} \sum_{j=1}^{\Lambda/2L} S^2(k_j) \left(\frac{1 - \cos k_j L}{\cos k_j L - \cos k_c L}\right)^2 \cdot \left[\left(\sin k_j x - \frac{\sin k_j L \sin k_c x}{\sin k_c L}\right)^2 + \left(\cos k_j x - \cos k_c x + \frac{\sin k_c x}{\sin k_c L}(\cos k_c L - \cos k_j L)\right)^2\right]. \tag{14'}$$

It is easy to show yourself that if L approaches zero then $k_c \to \sqrt{C}$ and (14′) approaches (14).

Next, we make the substitution

$$\epsilon_j = \frac{k_j}{k_c} - 1 \tag{15'}$$

in (14′) and neglect terms of order $(\epsilon_j k_c L)^2$ or smaller in the result. The neglect of these terms is justified because the result we are seeking, as before, turns out to depend only on extremely small values of ϵ when x is large. The result of this and a large amount of manipulation is the analog of (16) which follows:

$$\bar{\bar{\delta}}^2(x) \approx \frac{1}{2} \sum_{j=1}^{\Lambda/2L} \frac{S^2(\epsilon_j)}{k_c^2L^2} \left(\frac{1 - \cos k_cL}{1 + \cos k_cL}\right) \cdot \left[\frac{4 \sin^2 (\epsilon_j k_c x/2)}{\epsilon_j^2} + \frac{k_cL}{\epsilon_j}\left(\cot k_cL\left(1 + \cos\frac{k_cL}{2} - 2 \sin k_cx\right) - \tan k_cL\left(1 + \cos\frac{k_cL}{2} - 2 \cos k_cx\right)\right)\right] \tag{16'}$$

$$\equiv \frac{1}{2k_c^2L^2}\left(\frac{1 - \cos k_cL}{1 + \cos k_cL}\right) \sum_{j=1}^{\Lambda/2L} S^2(\epsilon_j)\left(\frac{u(\epsilon_j)}{\epsilon_j^2} + \frac{v}{\epsilon_j}\right).$$

As before, we let Λ go to infinity and change the sum to an integral.

$$\bar{\bar{\delta}}^2(x) \approx \frac{1}{2k_c^2L^2}\left(\frac{1 - \cos k_cL}{1 + \cos k_cL}\right) \int_{-1}^{(\pi/k_cL)-1} S^2(\epsilon)\left(\frac{u(\epsilon)}{\epsilon^2} + \frac{v}{\epsilon}\right) d\epsilon$$

Although the integral of v/ϵ has a logarithmic divergence at $\epsilon = 0$, the divergent part is an odd function of ϵ and the integrand over a small finite range around zero is small. The dominant term in the integration, for large values of x, comes from the $u(\epsilon)/\epsilon^2$ part. Thus we find that for large x

$$\bar{\bar{\delta}}^2(x) \approx \frac{1}{2k_c^2L^2}\left(\frac{1 - \cos k_cL}{1 + \cos k_cL}\right) \int_{-1}^{(\pi/k_cL)-1} S^2(\epsilon) \cdot \frac{4 \sin^2(\epsilon k_c x/2)}{\epsilon^2}\, d\epsilon$$
$$\approx \frac{1}{2k_c^2L^2}\left(\frac{1 - \cos k_cL}{1 + \cos k_cL}\right) \cdot 2\pi k_c x S^2(\epsilon = 0) \tag{17'}$$

where, from (13′),

$$S^2(\epsilon = 0) = \sum_{h=0}^{\infty} Y^2\left(\frac{2\pi}{L} h + k_c\right) + Y^2\left(\frac{2\pi}{L}(h + 1) - k_c\right).$$

Again, when $L \to 0$ we obtain the same result as from (17).

IV. THE SHAPE OF THE AXIS OF THE GUIDE

We might arbitrarily guess a spectral distribution for the Fourier components, $Y(k)$, of the shape of the somewhat irregular pipe. It may be somewhat more meaningful, however, to guess a spectrum of forces on the pipe and to obtain the shape from that. The latter procedure enables us to see the effect of stiffness of the pipe, and of the spectrum of applied forces, on its shape. For simplicity we will assume that the pipe would be perfectly straight, or at least relatively very straight, in the absence of the forces that are to be applied to it.

Suppose the pipe has a modulus of rigidity $\sigma = EI$ where E is Young's modulus and I is the geometric second moment of a cross section of the pipe about a transverse axis. The curvature of the pipe is related to the local moment of torque, $\mathbf{M}(x)$, through the equation[6]

$$\frac{d^2y}{dx^2} = \frac{\mathbf{M}(x)}{\sigma}. \tag{20}$$

The second derivative of the torque moment is the transverse force per unit length, p, applied to the pipe, so that

$$\frac{d^4y(x)}{dx^4} = \frac{1}{\sigma}\, p(x). \tag{21}$$

If the net force per unit length on the pipe is represented by the expression

$$p(x) = \sum_{j=1}^{\infty} P(k_j) \sin (k_jx - \varphi_j), \tag{22}$$

we obtain the following expression for the Fourier transform of the shape of the pipe by comparing (6), (21) and (22):

$$Y(k) = \frac{P(k)}{k^4\sigma}. \tag{23}$$

A convenient analytic expression that can be used to approximate a reasonable spectrum of forces is the following, where P_0, k_0, m and n are adjustable parameters.

$$P(k) = P_0((k/k_0)^n/(1 + (k/k_0)^m))^{\frac{1}{2}}. \tag{24}$$

In this case (23) becomes

$$Y(k) = (P_0/k_0^4\sigma)((k/k_0)^{n-8}/(1 + (k/k_0)^m))^{\frac{1}{2}}. \tag{23$'$}$$

Curves of Y and P as functions of k/k_0 are shown in Fig. 3 for the specific values $m = 11$ and $n = 8$. The factors outside the square roots are omitted. (If necessary we could use a sum of several such expressions with different parameters in each without greatly complicating the results, but we shall restrict the analysis to one.)

We will make use of the following formula several times in the following analysis.[7]

$$\int_0^{\infty} \frac{K^r}{1 + K^s}\, dK = \frac{\pi}{s \sin\left(\dfrac{(r+1)\pi}{s}\right)}. \tag{25}$$

Let K represent the ratio k/k_0.

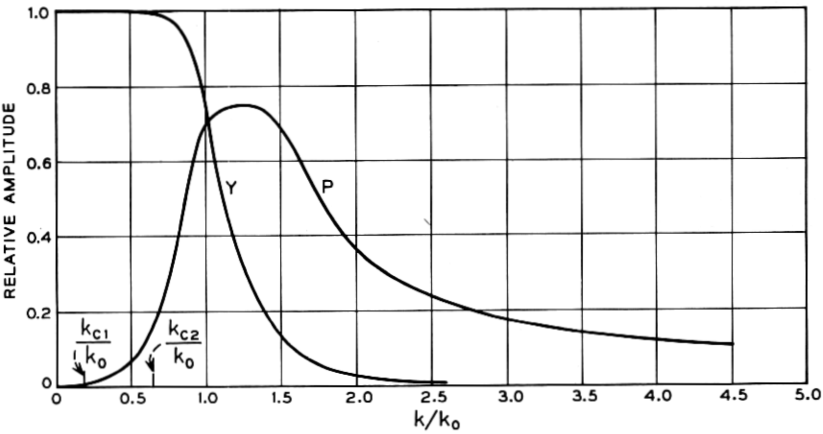

Fig. 3 — Relative amplitudes of sinusoidal components of forces applied to the guide (P), and of the shape of the pipe (Y), as a function of relative wave number, k/k_0. k_{c1}/k_0 is the ratio of k_c to k_0 used in the examples on the left side of Table I, while k_{c2}/k_0 is the ratio used in the examples on the right.

The mean square value of $P(K)$ is

$$\bar{\bar{p}}^2 = \frac{1}{2}\int_0^\infty P^2(K)dK = \frac{P_0^2}{2}\int_0^\infty \frac{K^n}{1+K^m}\,dK = \frac{\pi P_0^2}{2m \sin\left((n+1)\pi/m\right)}. \tag{26}$$

The mean square displacement on the guide axis from the straight x axis is (cf. (23))

$$\bar{\bar{y}}^2 = \frac{1}{2}\int_0^\infty Y^2(K)dK = \frac{1}{2}\left(\frac{P_0}{k_0^4\sigma}\right)^2\int_0^\infty \frac{K^{n-8}}{1+K^m}\,dK = \left(\frac{P_0}{k_0^4\sigma}\right)^2 \frac{\pi}{2m\sin\left((n-7)\pi/m\right)}. \tag{27}$$

Hence $\bar{\bar{y}}$ and $\bar{\bar{p}}$ are related through the expression

$$\bar{\bar{y}} = \frac{\bar{\bar{p}}}{k_0^4\sigma}\sqrt{\sin\frac{(n+1)\pi}{m}\Big/\sin\frac{(n-7)\pi}{m}}. \tag{28}$$

We will also define characteristic wavelengths, λ_p and λ_y for $p(x)$ and $y(x)$ since they are more tangible than the parameter k_0. The characteristic wavelengths will be taken as 2π divided by the angular wave number corresponding to the "center of gravity" of the integrands of

(26) and (27). Thus we obtain

$$\lambda_p = \frac{2\pi}{k_0} \int_0^\infty \frac{K^n}{1 + K^m} \, dK \Big/ \int_0^\infty \frac{K^n}{1 + K^m} \cdot K \, dK = \frac{2\pi}{k_0} \frac{\sin\left((n + 2)\pi/m\right)}{\sin\left((n + 1)\pi/m\right)} \tag{29}$$

and similarly

$$\lambda_y = \frac{2\pi}{k_0} \frac{\sin \frac{(n - 6)\pi}{m}}{\sin \frac{(n - 7)\pi}{m}}. \tag{30}$$

For very narrow force spectra λ_y and λ_p are both equal to $2\pi/k_0$, but when the spectra are broad, λ_y tends to be larger than λ_p, since short wavelength force components are unable to bend the stiff pipe much. See Fig. 1.

V. NUMERICAL ILLUSTRATION

The following numerical illustration serves mainly to show how the preceding results could be used. The parameters chosen are the ones that one is likely to know in a real situation. The actual numbers might vary considerably, but it is not hard to estimate how much changes in various parameters would affect the results. What is most important is to know the order of magnitude of the effects using reasonable parameters. The following example forms the left half of Table I.

5.1 *Shape of the Guide Axis*

Let us suppose we have a guide composed of a series of lenses or a lens-like medium perfectly centered in a cylindrical pipe. We will suppose that curvature of the pipe axis is due largely to strain from externally applied forces. For purposes of illustration, we will suppose that the main force on the pipe is due to its own weight. We shall suppose that it is lying on an irregular bed which supports the weight of the pipe at more or less random intervals averaging about λ_p meters apart. See Fig. 1. (The spread in the intervals is determined by m and n in (24).)

Suppose the pipe is made of steel whose density is $\rho = 7.8 \times 10^3$ kg/meter^3 and whose Young's modulus of rigidity is $E = 2.0 \times 10^9$ newtons/meter^2.

Let the pipe be 10 cm OD and 9 cm ID. Let the distance λ_p be one meter.

Let us use the smallest integer values of m and n that are consistent with the conditions on (25) for all the applications of that equation (see (26) to (29)). We find that these values are $m = 11$ and $n = 8$. (This gives a rather broad spectrum of wavelengths of applied force. See Fig. 3. Larger values would give narrower spectra, which would be appropriate if the pipe were supported at regular intervals.)

Using these numbers in (29) we find that $k_0 = 3.27$ meters^{-1}.

Equation (30) then gives $\lambda_y = 3.68$ meters.

The stiffness of a cylindrical pipe is related to its OD and ID through the equation

$$\sigma = \frac{\pi}{64} E((\text{OD})^4 - (\text{ID})^4) \tag{31}$$

which gives $\sigma = 3.38 \times 10^3$ newton meters^2 for our example.

We shall assume for simplicity that the mean square amplitude of force variations, $\overline{\overline{p}}$, is equal to the linear density of the pipe times the gravity constant, which is correct to within a small numerical factor when the weight of the pipe determines all the forces. We thus obtain

$$\overline{\overline{p}} = \rho g((\text{OD})^2 - (\text{ID})^2)\,\pi/4. \tag{32}$$

For our pipe we get $\overline{\overline{p}} = 114$ newtons/meter.

Equation (28) then gives us $\overline{\overline{y}} = 0.407 \times 10^{-3}$ meters.

We shall need to know P_0 when we compute the beam oscillation growth rate. Equation (26) gives us $P_0 = 222$ newtons/meter.

5.2 *Beam Oscillation Growth Rate*

Let us first suppose we have a continuous lens-like guide with specific convergence C and the shape defined in part A of this section. We use (23′) to evaluate Y at $k = \sqrt{C} = 0.5$ meters^{-1}, which is the value at $\epsilon = 0$. If we let $C = 0.25$ diopters per meter, we get $Y(\sqrt{C}) = 0.572 \times 10^{-3}$ meters. The ratio $\sqrt{C}/k_0$ is 0.153 in this example. It is labeled k_{c1}/k_0 on Fig. 3.

Inserting the value of Y into (17) gives $\overline{\overline{\delta}}(x) = \sqrt{x} \cdot (0.358 \times 10^{-3})$ meters. In 100 meters, the root mean square amplitude of oscillation is 3.58 mm and in 10 kilometers it is 3.58 cm.

Next let us consider a series of thin lenses that are separated by twice their individual focal lengths, ($L = 2/c$), and that give the same angular wave number for ray oscillations as in the preceding example,

$k_c = 0.5$ meters^{-1}. From (19) we find that $\cos k_c L = 0$, which gives $L = \pi$ or 3.14 meters and $c = 2/\pi$ or 0.637 diopters.

Now we evaluate the sum for $S^2(\epsilon = 0)$ in (17′), using (23′), as in the first example. The first term of the sum is equal to Y^2 at 0.5 meters^{-1}, whose square root we already evaluated in the first example. The next term is Y^2 evaluated at 1.5 meters^{-1}, the next at 2.5 meters^{-1}, etc. The terms rapidly diminish in size. Equation (17′) gives $\bar{\bar{\delta}} = \sqrt{x} \cdot (0.830 \times 10^{-3})$ meters, which is not much larger than the result for the continuous lens-like medium with the same ray oscillation wave number.

The preceding examples illustrate the fact that the results are essentially the same for a series of lenses or a continuous lens-like medium of equal k_c when the characteristic wavelength, λ_y, of irregularities in the pipe axis is longer than the separation of the lenses and when the lens separation is not too near to the limiting value of four focal lengths.

The input parameters and results of the preceding examples are listed on the left side of Table I.

The right side of Table I shows what happens when the characteristic distance between bumps on the ground, λ_p, is increased to 4 meters, keeping other parameters the same. There is a catastrophic increase in rate of growth of beam oscillations because the root mean square amplitude of waves in the pipe is greatly increased while the value of Y at k_c remains near its maximum. See Fig. 3. The ratio $\sqrt{C}/k_0$ or k_c/k_0 in these examples is 0.611 and is labeled k_{c2}/k_0 on Fig. 3.

A short FORTRAN computer program is available upon request to anyone who may wish to enlarge on Table I using other values of the input parameters.

VI. CONCLUSIONS

The results are strongly dependent on some of the input parameters, but we hope the examples represent the general magnitude of the factors one might have to work with if such a light guide were built. By using a suitably spaced periodic set of supports for the guide, we could probably make Y^2 or S^2 very small at $\sqrt{C}$ or k_c. The peak or peaks in the function $Y^2(k)$ or $S^2(k)$ could lie elsewhere. Thus, the contribution to wave growth due to forces on the pipe might be considerably reduced.

The computations did not consider waves in the pipe axis due to tolerance limits that would arise in manufacture of the guide or in linking sections, but these waves could obviously be incorporated as an additional term in $Y^2(k)$ if they could be estimated or measured.

It seems probable that the beam would have to be recentered at rather frequent intervals along the guide unless the aperture were very large or the pipe were extremely stiff and straight.

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