B.S.T.J. BRIEFS

An Observation Concerning the Application of the Contraction-Mapping Fixed-Point Theorem, and a Result Concerning the Norm-Boundedness of Solutions of Nonlinear Functional Equations

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PART I

Let $\mathcal{B}$ denote a Banach space over the real or complex field $\mathfrak{F}$. Let $\Theta(\mathcal{B})$ denote the set of (not necessarily linear) operators that map $\mathcal{B}$ into itself, with I the identity operator, and let $\| T \|$ denote the "Lipshitz norm" of T for all $T \, \varepsilon \, \Theta(\mathcal{B})$ (i.e.,

$$\| T \| \triangleq \sup_{\substack{x,y \, \varepsilon \, \mathcal{B} \\ \|x-y\| \neq 0}} \frac{\| Tx - Ty \|}{\| x - y \|}).$$

Observation:

Let A and B belong to $\Theta(\mathcal{B})$, and let $g \, \varepsilon \, \mathcal{B}$. Suppose that there exists $c \, \varepsilon \, \mathfrak{F}$ such that (*i*) $(I + cA)^{-1}$ exists on $\mathcal{B}$, (ii) $\| A(I + cA)^{-1} \|$ and $\| B - cI \|$ are finite, and (iii) $\| A(I + cA)^{-1} \| \cdot \| B - cI \| < 1$. Then $\mathcal{B}$ contains exactly one element f such that $g = f + ABf$. (It can be verified that under our assumptions, $f \, \varepsilon \, \mathcal{B}$ satisfies $g = f + ABf$ if and only if f satisfies

$$g = f + A(I + cA)^{-1}[(B - cI)f + cg].)$$

For the special case in which A is a linear operator, this result is well known* and has been applied often in the engineering literature [see, for example, Ref. 2]. The fact that it can be generalized as indicated suggests that the scope of its range of applicability to engineering problems can be extended significantly.

* The linearity of A plays an essential role in all of the previous proofs known to this writer. See, for example, Ref. 1.

PART II

Let $\mathfrak{K}$ denote an abstract linear space, over the real or complex field $\mathfrak{F}$, that contains a normed linear space $\mathfrak{L}$ with norm $\| \cdot \|$. Let Ω denote a set of real numbers, and let P_y denote a linear mapping of $\mathfrak{K}$ into $\mathfrak{L}$ for each $y \in \Omega$, such that $\| P_y h \| \leqq \| h \|$ for all $h \in \mathfrak{L}$ and all $y \in \Omega$. We say that a (not necessarily linear) operator T is an element of the set Θ if and only if T maps $\mathfrak{K}$ into itself and $P_y T = P_y T P_y$ on $\mathfrak{K}$ for all $y \in \Omega$. The symbol I denotes the identity operator on $\mathfrak{K}$.

Proposition:†

Let A belong to Θ, and assume that A maps the zero-element of $\mathfrak{L}$ into itself. Let B map $\mathfrak{K}$ into itself. Let $f \in \mathfrak{K}$, and let $g = f + ABf$. Suppose that there exists $\lambda \in \mathfrak{F}$ such that

(i) $(I + \lambda A)$ is invertible on $\mathfrak{K}$, $(I + \lambda A)^{-1} \in \Theta$, and $A(I + \lambda A)^{-1}$ maps $\mathfrak{L}$ into itself

(ii) $\eta_\lambda \triangleq \sup \{ \| A(I + \lambda A)^{-1} h \| / \| h \| : h \in \mathfrak{L}, h \neq 0 \} < \infty$

(iii) there exists a nonnegative constant k_λ and a function $p_\lambda(y)$ with the property that

$$\| P_y (B - \lambda I) f \| \leqq k_\lambda \| P_y f \| + p_\lambda(y) \text{ for all } y \in \Omega$$

(iv) $\eta_\lambda k_\lambda < 1$.

Then

$$\| P_y f \| \leqq (1 - \eta_\lambda k_\lambda)^{-1} [(1 + |\lambda| \eta_\lambda) \| P_y g \| + \eta_\lambda p_\lambda(y)]$$

for all $y \in \Omega$.

Proof:

Let $y \in \Omega$. Then, since $Bf = (I + \lambda A)^{-1}[(B - \lambda I)f + \lambda g]$, we have

$$\begin{aligned} P_y f &= P_y g - P_y A (I + \lambda A)^{-1} [(B - \lambda I) f + \lambda g] \\ &= P_y g - P_y A (I + \lambda A)^{-1} P_y [(B - \lambda I) f + \lambda g], \end{aligned}$$

and hence

$$\begin{aligned} \| P_y f \| &\leqq \| P_y g \| + \eta_\lambda \| P_y [(B - \lambda I) f + \lambda g] \| \\ &\leqq \| P_y g \| + \eta_\lambda \| P_y (B - \lambda I) f \| + |\lambda| \eta_\lambda \| P_y g \| \\ &\leqq (1 + |\lambda| \eta_\lambda) \| P_y g \| + \eta_\lambda k_\lambda \| P_y f \| + \eta_\lambda p_\lambda(y), \end{aligned}$$

which establishes the proposition.

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† This proposition is a generalization of a result proved in Ref. 3, and is of considerable utility in stability studies of nonlinear physical systems.

Comments:

Consider the important special case in which: $\mathcal{K}$ denotes the set of real-valued locally-square-integrable functions on $[0, \infty)$, $\mathcal{L}$ denotes the space of real-valued square-integrable functions x on $[0, \infty)$ with norm

$$\| x \| = \left(\int_0^\infty x(t)^2 dt \right)^{\frac{1}{2}},$$

$\Omega = [0, \infty)$, and P_y is defined by

$$\begin{aligned} (P_y h)(t) &= h(t), \qquad t \,\varepsilon\, [0, y] \\ &= 0, \qquad t > y \end{aligned}$$

for all $h \,\varepsilon\, \mathcal{K}$. Suppose that A is defined on $\mathcal{K}$ by

$$(Ah)(t) = k_0 h(t) + \int_0^t [k_1(t - \tau) + k_2(t - \tau)] h(\tau) d\tau$$

for all $h \,\varepsilon\, \mathcal{K}$, where k_0 is a real constant, k_1 and k_2 are real-valued measurable functions on $[0, \infty)$, with k_1 bounded on $[0, \infty)$ and k_2 integrable on $[0, \infty)$.

Let

$$K(s) = k_0 + \int_0^\infty [k_1(t) + k_2(t)] e^{-st}\, dt$$

for $\sigma \triangleq \operatorname{Re}[s] > 0$, and, with λ a real constant, assume that

$$\sup_{\sigma > 0} \left| \frac{K(s)}{1 + \lambda K(s)} \right| < \infty.$$

Then, with the aid of some known results[4] from the theory of Fourier transforms, it can be proved that

(i) $(I + \lambda A)^{-1} \,\varepsilon\, \Theta$, and $A(I + \lambda A)^{-1}$ maps $\mathcal{L}$ into itself,
(ii) there exists a zero-measure subset $\mathfrak{N}$ of $[0, \infty)$ such that

$$\lim_{\sigma \to 0+} \frac{K(\sigma + i\omega)}{1 + \lambda K(\sigma + i\omega)}$$

exists for all $\omega \,\varepsilon\, \tilde{\mathfrak{N}} \triangleq [0, \infty) - \mathfrak{N}$,

and

(iii) $$\eta_\lambda \triangleq \| A(I + \lambda A)^{-1} \| = \operatorname*{ess\,sup}_{\omega \,\varepsilon\, \tilde{\mathfrak{N}}} \left| \lim_{\sigma \to 0+} \frac{K(\sigma + i\omega)}{1 + \lambda K(\sigma + i\omega)} \right|.$$

These facts can be used to extend some of the results of Ref. 3 to a more general class of integral equations. For example, let B denote the mapping of $\mathcal{K}$ into itself defined by the condition that $(Bh)(t) = b(t)h(t)$ for all $t \geqq 0$ and all $h \varepsilon \mathcal{K}$, where $b(\cdot)$ is a real-valued measurable function with the property that there exist real numbers α and β such that $\alpha \leqq b(t) \leqq \beta$ for all $t \geqq 0$. With $g \varepsilon \mathcal{L}$, let $g = f + ABf$ with $f \varepsilon \mathcal{K}$. Let k_1 be a constant, and let

$$K(s) = k_0 + s^{-1}k_1 + \int_0^\infty k_2(t)e^{-st}dt$$

for all $s \varepsilon \mathcal{S} \triangleq \{s: s \neq 0, \sigma \geqq 0\}$. Suppose that

$$\begin{aligned} &1 + \tfrac{1}{2}(\alpha + \beta)k_0 \neq 0 \\ &1 + \tfrac{1}{2}(\alpha + \beta)K(s) \neq 0 \qquad \text{for all } s \varepsilon \mathcal{S}, \end{aligned} \tag{1}$$

and

$$\tfrac{1}{2}(\beta - \alpha) \sup_{\omega > 0} \left| \frac{K(i\omega)}{1 + \tfrac{1}{2}(\alpha + \beta)K(i\omega)} \right| < 1. \tag{2}$$

Then, an application of the proposition shows that $f \varepsilon \mathcal{L}$. This result, which is concerned with feedback loops containing a pure integrator, cannot be proved as an application of the result similar to our propositions given in Ref. 3, because there A is assumed to map $\mathcal{L}$ into itself.

REFERENCES

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