Cosine Sum Approximation and Synthesis of Array Antennas

By D. JAGERMAN

(Manuscript received May 10, 1965)

The problem of approximating a band-limited function, $H(t)$, by a sum of cosines arises in the design of phased array antennas. Three methods of synthesis are presented for establishing such designs. Error formulae are deduced for each method, including a new error formula for Tchebycheff quadrature. The existence of grating lobes is proved, and lower bounds for their location are developed.

I. INTRODUCTION

This paper is concerned with the problem of approximating a function $H(t)$ $(-\infty < t < \infty)$ by a cosine sum of the form

$$S_N(t) = \frac{1}{N} \sum_{j=1}^{N} \cos tx_j\,, 0 \leqq x_1 < x_2 < \cdots < x_N \leqq 1. \tag{1}$$

The synthesis of array antennas is an application of the problem of this paper. Let isotropic radiating elements of strength $1/N$ be located along the x-axis at the points $x_j (1 \leqq j \leqq N)$ providing planar radiation of wavelength λ, and let θ designate the angle between the positive y-axis and a line passing through the origin and a far-field point, then, setting

$$t = \frac{2\pi \sin \theta}{\lambda}, \tag{2}$$

the far-field radiation pattern of the linear array is given by $S_N(t)$. The requirement that all the coefficients of the sum in (1) be equal generally stems out of the use of identical radiating elements, and out of the desire to employ identical feed for each element.

The function $H(t)$ represents the desired far-field radiation pattern; it will be required to satisfy the condition

$$H(t) = \int_0^1 F(x) \cos tx \, dx, \tag{3}$$

for some function $F(x) \in L(0,1)$, and the normalization condition

$$H(0) = 1. \tag{4}$$

When required, the function $F(x)$ will be extended to the interval $(-1,1)$ by

$$F(-x) = F(x). \tag{5}$$

The function $F(x)$ is, thus, the illumination required for a continuous aperture to produce the far-field pattern $H(t)$. Equation (3) defines the array aperture as one, and the function $H(t)$ to be bandlimited with bandwidth one.

The approximation or synthesis problem consists in the determination of the quantities $x_1, \cdots, x_N$ subject to the condition of (1) so that $S_N(t)$ shall approximate $H(t)$.

In this form, the problem is that of numerical quadrature by means of an equal-coefficient rule. Sections II and III present methods for accomplishing this. Section IV drops the restriction of equal coefficients and applies the well-known Gaussian quadrature rule. Section V discusses the existence of grating lobes and presents estimates for their location.

II. A RIEMANNIAN SUM METHOD

Let $H(t)$ be a characteristic function, that is, $H(t)$ satisfies the normalization condition (4) and the additional requirement

$$F(x) \geqq 0, \qquad 0 < x < 1, \tag{6}$$

then the function

$$L(x) = \int_0^x F(u)\, du \tag{7}$$

satisfies

$$L(0) = 0, \qquad L(1) = 1 \tag{8}$$

and is monotonic increasing. Let

$$y = L(x), \qquad x = G(y) \tag{9}$$

in which $G(y)$ is the function inverse to $L(x)$, then the required numbers x_j are given explicitly by

$$x_j = G\left(\frac{2j-1}{2N}\right). \qquad 1 \leqq j \leqq N. \tag{10}$$

The sum

$$S_N(t) = \frac{1}{N} \sum_{j=1}^{N} \cos\left[tG\left(\frac{2j-1}{2N}\right)\right] \tag{11}$$

is clearly a Riemannian sum for

$$I = \int_0^1 \cos\,[tG(y)]\,dy, \tag{12}$$

and hence

$$\lim_{N\to\infty} S_N(t) = I; \tag{13}$$

however,

$$I = \int_0^1 F(x)\cos tx\,dx = H(t) \tag{14}$$

and hence the approximation is secured. The error $R_N(t)$ given by

$$R_N(t) = H(t) - S_N(t) \tag{15}$$

will now be studied. For this purpose consider

Lemma 1:

$$c_n = \int_0^1 \sin\,[2\pi nL(x)]\sin tx\,dx$$

$$\Rightarrow R_N(t) = \frac{t}{\pi N}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{c_{Nk}}{k}.$$

Proof: It will be convenient to introduce the function

$$S_N(t,y) = \frac{1}{N}\sum_{j=1}^{N}\cos\left[tG\left(y + \frac{2j-1}{2N}\right)\right]; \tag{16}$$

thus

$$S_N(t,0) = S_N(t). \tag{17}$$

The function $\cos\,[tG(y)]$ may be expanded into a Fourier series on the interval (0,1); one has,

$$\cos\,[tG(y)] = H(t) + \sum_{n=1}^{\infty} a_n \cos 2\pi ny + \sum_{n=1}^{\infty} b_n \sin 2\pi ny, \tag{18}$$

in which

$$a_n = 2\int_0^1 \cos\,[tG(y)]\cos 2\pi ny\,dy,$$

$$b_n = 2\int_0^1 \cos\,[tG(y)]\sin 2\pi ny\,dy. \tag{19}$$

Define c_n, d_n by

$$\begin{aligned} c_n &= \int_0^1 G'(y) \sin 2\pi ny \sin [tG(y)]\, dy \\ &= \int_0^1 \sin [2\pi nL(x)] \sin tx\, dx, \\ d_n &= \int_0^1 G'(y) \cos 2\pi ny \sin [tG(y)]\, dy \\ &= \int_0^1 \cos [2\pi nL(x)] \sin tx\, dx, \end{aligned} \tag{20}$$

then integration by parts applied to (19) yields

$$\begin{aligned} a_n &= \frac{t}{\pi n} c_n , \\ b_n &= \frac{1 - \cos t}{\pi n} - \frac{t}{\pi n} d_n . \end{aligned} \tag{21}$$

The Bernoullian function

$$\rho(y) = \tfrac{1}{2} - \{y\}, \tag{22}$$

in which $\{y\}$ designates the *fractional part* of y, has the Fourier series

$$\rho(y) = \sum_{n=1}^{\infty} \frac{\sin 2\pi ny}{\pi n}, \tag{23}$$

hence, replacing a_n, b_n in (18) by their values in (21), one obtains

$$\begin{aligned} \cos [tG(y)] = H(t) &+ \frac{t}{\pi} \sum_{n=1}^{\infty} \frac{c_n}{n} \cos 2\pi ny \\ &- \frac{t}{\pi} \sum_{n=1}^{\infty} \frac{d_n}{n} \sin 2\pi ny + (1 - \cos t)\rho(y). \end{aligned} \tag{24}$$

By summation of the geometric series, one has

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^{N} \exp \left[i2\pi n \left(y + \frac{2j-1}{2N} \right) \right] &= e^{i2\pi ny} (-1)^{n/N}, \qquad N \mid n, \\ &= 0, \qquad N \nmid n, \end{aligned} \tag{25}$$

and hence, letting $n = Nk$ ($k > 0$ integral),

$$\frac{1}{N} \sum_{j=1}^{N} \cos 2\pi Nk \left(y + \frac{2j-1}{2N} \right) = (-1)^k \cos 2\pi Nky, \tag{26}$$

$$\frac{1}{N}\sum_{j=1}^{N} \sin 2\pi Nk\left(y + \frac{2j-1}{2N}\right) = (-1)^k \sin 2\pi Nky. \tag{27}$$

Equations (16), (24), (26), and (27) now yield

$$\begin{aligned} S_N(t,y) = H(t) &+ \frac{t}{\pi N}\sum_{k=1}^{\infty}(-1)^k \frac{c_{Nk}}{k}\cos 2\pi Nky \\ &- \frac{t}{\pi N}\sum_{k=1}^{\infty}(-1)^k \frac{d_{Nk}}{k}\sin 2\pi Nky \\ &+ (1-\cos t)\frac{1}{N}\sum_{j=1}^{N}\rho\left(y+\frac{2j-1}{2N}\right). \end{aligned} \tag{28}$$

The Fourier series for $\rho(y)$, (23), permits ready establishment of the identity

$$\sum_{j=1}^{N}\rho\left(y+\frac{2j-1}{2N}\right) = \rho\left(Ny+\frac{1}{2}\right), \tag{29}$$

hence

$$\begin{aligned} S_N(t,y) = H(t) &+ \frac{t}{\pi N}\sum_{k=1}^{\infty}(-1)^k \frac{c_{Nk}}{k}\cos 2\pi Nky \\ &- \frac{t}{\pi N}\sum_{k=1}^{\infty}(-1)^k \frac{d_{Nk}}{k}\sin 2\pi Nky \\ &+ \frac{1-\cos t}{N}\rho\left(Ny+\frac{1}{2}\right). \end{aligned} \tag{30}$$

Setting $y = 0$ in (30) yields the result of the lemma.

Lemma 2: $r \geqq 2$, *integral*, $W^{(r)}(x) \geqq \varepsilon_r > 0$ *or*

$$W^{(r)}(x) \leqq -\varepsilon_r < 0 \quad \textit{for} \quad a \leqq x \leqq b$$

$$\Rightarrow \left|\int_a^b \cos W(x)\, dx\right| \leqq r2^{(r+1)/2}\varepsilon_r^{-(1/r)}.$$

Proof: It is clear that only the inequality $W^{(r)}(x) \geqq \varepsilon_r > 0$ need be considered. The case $r = 2$ will be considered first. The function $W'(x)$ is monotonic increasing, hence it vanishes at most once in $[a,b]$, say at $x = c$, then

$$\int_a^b \cos W(x)\, dx = \int_a^c \cos W(x)\, dx + \int_c^b \cos W(x)\, dx. \tag{31}$$

Let $0 \leqq \delta \leqq b - c$ be chosen, then

$$\int_c^b \cos W(x)\, dx = \int_c^{c+\delta} \cos W(x)\, dx + \int_{c+\delta}^b \cos W(x)\, dx, \quad (32)$$

and hence

$$\left| \int_c^b \cos W(x)\, dx \right| \leqq \delta + \left| \int_{c+\delta}^b \cos W(x)\, dx \right|. \quad (33)$$

One has

$$\begin{aligned} \int_{c+\delta}^b \cos W(x)\, dx &= \int_{c+\delta}^b \frac{1}{W'(x)}\, d \sin W(x) \\ &= \frac{1}{W'(c+\delta)} \int_{c+\delta}^{\xi} d \sin W(x), \end{aligned} \quad (34)$$

in which the second mean-value theorem was used, and hence

$$\left| \int_{c+\delta}^b \cos W(x)\, dx \right| \leqq \frac{2}{W'(c+\delta)}. \quad (35)$$

Since

$$W'(c+\delta) = \int_c^{c+\delta} W''(x)\, dx \geqq \delta \varepsilon_2, \quad (36)$$

one obtains, from (33),

$$\left| \int_c^b \cos W(x)\, dx \right| \leqq \delta + \frac{2}{\delta \varepsilon_2}. \quad (37)$$

The choice

$$\delta = \sqrt{2}\, \varepsilon_2^{-\frac{1}{2}} \quad (38)$$

yields

$$\left| \int_c^b \cos W(x)\, dx \right| \leqq 2\sqrt{2}\, \varepsilon_2^{-\frac{1}{2}}. \quad (39)$$

The value of δ in (38) may exceed $b - c$, however, in this case the inequality of (39) is certainly correct since the integral always admits the estimate $b - c$.

Similarly choose $0 \leqq \delta \leqq c - a$, then

$$\int_a^c \cos W(x)\, dx = \int_a^{c-\delta} \cos W(x)\, dx + \int_{c-\delta}^c \cos W(x)\, dx, \quad (40)$$

and hence

$$\left| \int_a^c \cos W(x)\, dx \right| \leqq \left| \int_a^{c-\delta} \cos W(x)\, dx \right| + \delta. \tag{41}$$

One has

$$\begin{aligned} \int_a^{c-\delta} \cos W(x)\, dx &= \int_a^{c-\delta} \frac{1}{W'(x)}\, d \sin W(x) \\ &= \frac{1}{W'(c-\delta)} \int_\xi^{c-\delta} d \sin W(x), \end{aligned} \tag{42}$$

and hence

$$\left| \int_a^{c-\delta} \cos W(x)\, dx \right| \leqq -\frac{2}{W'(c-\delta)}. \tag{43}$$

Since

$$-W'(c-\delta) = \int_{c-\delta}^{c} W''(x)\, dx \geqq \delta\varepsilon_2 . \tag{44}$$

one obtains from (41)

$$\left| \int_a^c \cos W(x)\, dx \right| \leqq \delta + \frac{2}{\delta\varepsilon_2}. \tag{45}$$

Hence

$$\left| \int_a^c \cos W(x)\, dx \right| \leqq 2\sqrt{2}\, \varepsilon_2^{-\frac{1}{2}}, \tag{46}$$

and, from (31),

$$\left| \int_a^b \cos W(x)\, dx \right| \leqq 4\sqrt{2}\, \varepsilon_2^{-\frac{1}{2}}. \tag{47}$$

The lemma is thus established for $r = 2$.

Induction will now be employed. The lemma is assumed true for $r = k \geqq 2$. Since $W^{(k+1)}(x) > 0$, $W^{(k)}(x)$ is monotonic increasing, and hence vanishes at most once in $[a,b]$, say at $x = c$. Choose $0 \leqq \delta \leqq b - c$, then

$$\int_c^b \cos W(x)\, dx = \int_c^{c+\delta} \cos W(x)\, dx + \int_{c+\delta}^b \cos W(x)\, dx, \tag{48}$$

and hence

$$\left| \int_c^b \cos W(x)\, dx \right| \leqq \delta + \left| \int_{c+\delta}^b \cos W(x)\, dx \right|. \tag{49}$$

The inductive hypothesis states

$$\left| \int_{c+\delta}^{b} \cos W(x)\, dx \right| \leqq k2^{(k+1)/2} W^{(k)}(c+\delta)^{-(1/k)}, \tag{50}$$

hence

$$\left| \int_{c}^{b} \cos W(x)\, dx \right| \leqq \delta + k2^{(k+1)/2} W^{(k)}(c+\delta)^{-(1/k)}. \tag{51}$$

Since

$$W^{(k)}(c+\delta) = \int_{c}^{c+\delta} W^{(k+1)}(x)\, dx \geqq \delta\varepsilon_{k+1}, \tag{52}$$

one has

$$\left| \int_{c}^{b} \cos W(x)\, dx \right| \leqq \delta + k2^{(k+1)/2} \delta^{-(1/k)} {\varepsilon_{k+1}}^{-(1/k)}. \tag{53}$$

The choice

$$\delta = 2^{k/2} {\varepsilon_{k+1}}^{-(1/k+1)} \tag{54}$$

yields

$$\left| \int_{c}^{b} \cos W(x)\, dx \right| \leqq (k+1)\, 2^{k/2} {\varepsilon_{k+1}}^{-(1/k+1)}. \tag{55}$$

The inequality of (55) remains correct even for $\delta > b - c$.

Similarly, choose $0 \leqq \delta \leqq c - a$, then

$$\int_{a}^{c} \cos W(x)\, dx = \int_{a}^{c-\delta} \cos W(x)\, dx + \int_{c-\delta}^{c} \cos W(x)\, dx, \tag{56}$$

and hence

$$\left| \int_{a}^{c} \cos W(x)\, dx \right| \leqq \left| \int_{a}^{c-\delta} \cos W(x)\, dx \right| + \delta. \tag{57}$$

The inductive hypothesis yields

$$\left| \int_{a}^{c} \cos W(x)\, dx \right| \leqq \delta + k2^{(k+1)/2} \left[-W^{(k)}(c-\delta) \right]^{-(1/k)}. \tag{58}$$

Since

$$-W^{(k)}(c-\delta) = \int_{c-\delta}^{c} W^{(k+1)}(x)\, dx \geqq \delta\varepsilon_{k+1}, \tag{59}$$

one has

$$\left| \int_a^c \cos W(x)\, dx \right| \leqq \delta + k 2^{(k+1)/2} \delta^{-(1/k)} \varepsilon_{k+1}{}^{-(1/k)}. \tag{60}$$

Thus

$$\left| \int_a^c \cos W(x)\, dx \right| \leqq (k+1)\, 2^{k/2}\, \varepsilon_{k+1}{}^{-(1/k+1)}, \tag{61}$$

and hence

$$\left| \int_a^b \cos W(x)\, dx \right| \leqq (k+1)\, 2^{(k+2)/2}\, \varepsilon_{k+1}{}^{-(1/k+1)}. \tag{62}$$

The lemma is now established.

Theorem 1 provides an estimate of $R_N(t)$.

Theorem 1: $r \geqq 2$, *integral,* $L^{(r)}(x) \geqq \varepsilon_r > 0$ *or*

$$L^{(r)}(x) \leqq -\varepsilon_r < 0 \quad \textit{for} \quad 0 \leqq x \leqq 1$$

$$\Rightarrow |R_N(t)| \leqq r 2^{(r+1)/2-(1/r)} \pi^{-1-(1/r)} \zeta(1 + (1/r)) \varepsilon_r{}^{-(1/r)} |t| N^{-1-(1/r)}.$$

Proof: One has, from Lemma 1,

$$c_n = \int_0^1 \sin\,[2\pi n L(x)] \sin tx\, dx, \tag{63}$$

and hence

$$c_n = \frac{1}{2} \int_0^1 \cos\,[2\pi n L(x) - tx]\, dx - \frac{1}{2} \int_0^1 \cos\,[2\pi n L(x) + tx]\, dx. \tag{64}$$

Lemma 2 applied to the integrals of (64) yields

$$|c_n| \leqq r 2^{(r+1)/2-(1/r)} \pi^{-(1/r)} \varepsilon_r{}^{-(1/r)} n^{-(1/r)}. \tag{65}$$

The infinite series for $R_N(t)$ in Lemma 1 may now be estimated. Using (65), one obtains

$$|R_N(t)| \leqq r 2^{(r+1)/2-(1/r)} \pi^{-1-(1/r)} \varepsilon_r{}^{-(1/r)} |t| N^{-1-(1/r)} \sum_{k=1}^{\infty} \frac{1}{k^{1+(1/r)}}. \tag{66}$$

Since the series of (66) is $\zeta(1 + (1/r))$, the inequality of the theorem follows.

An example of the above analysis is provided by the choice

$$H(t) = J_o(t), \tag{67}$$

that is the Bessel function of first kind and order zero. For this case

$$F(x) = \frac{2}{\pi\sqrt{1 - x^2}}, \tag{68}$$

and hence

$$L(x) = \frac{2}{\pi} \sin^{-1} x. \tag{69}$$

Thus

$$x = \sin \frac{\pi}{2} y, \tag{70}$$

and

$$x_j = \sin \frac{\pi}{2} \frac{2j - 1}{2N}. \tag{71}$$

The function $L(x)$ satisfies

$$L'''(x) \geqq 1 = \varepsilon_3, \tag{72}$$

and hence, after numerical simplification, the error is estimated by

$$| R_N(t) | < 8 \, | t | \, N^{-(4/3)}. \tag{73}$$

Another example is given by

$$H(t) = (\sin \tfrac{1}{2}t/\tfrac{1}{2}t)^2. \tag{74}$$

One has

$$F(x) = 2 - 2x, \tag{75}$$

$$L(x) = 2x - x^2, \tag{76}$$

and

$$G(y) = 1 - \sqrt{1 - y}. \tag{77}$$

Thus

$$x_j = 1 - \sqrt{1 - (2j - 1)/2N}. \tag{78}$$

Since

$$L'' = -2 = -\varepsilon_2, \tag{79}$$

the error estimate obeys

$$| R_N(t) | < 1.3 \, | t | \, N^{-(3/2)}. \tag{80}$$

If $F(x)$ has high order of contact at the endpoints zero and one, then $H(t)$ will decrease rapidly with increasing t, and hence the sidelobes will be small. In particular, let

$$F^{(j)}(0) = 0, \qquad F^{(j)}(1) = 0, \qquad 0 \leqq j \leqq k \tag{81}$$

then, integration by parts applied to (3) yields

$$H(t) = -(-1)^{k/2} \frac{1}{t^{k+1}} \int_0^1 F^{(k+1)}(x) \sin tx \, dx, \qquad k \text{ even}, \tag{82}$$

and

$$H(t) = -(-1)^{(k-1)/2} \frac{1}{t^{k+1}} \int_0^1 F^{(k+1)}(x) \cos tx \, dx, \qquad k \text{ odd}. \tag{83}$$

If $F^{(k+1)}(x)$ is of bounded variation, then

$$| H(t) | \leqq \frac{V}{t^{k+2}}, \tag{84}$$

in which V is the total variation of $F^{(k+1)}(x)$. Equation (84) shows the rapid decay of the sidelobes.

An example of this type of tapered design is given by

$$F_k(x) = (2k + 1) \binom{2k}{k} [x(1 - x)]^k \tag{85}$$

which has order of contact $k - 1$ and for which

$$| H(t) | \leqq V/t^k. \tag{86}$$

In this case, V is the total variation of $F_k^{(k)}(x)$. Since

$$L^{(2k+1)}(x) = -(2k + 1)! \binom{2k}{k} = -\varepsilon_{2k+1} < 0 \tag{87}$$

one has, from Theorem 1,

$$| R_N(t) | \leqq E_k \, | t | \, N^{-1-(1/2k+1)}, \tag{88}$$

in which E_k is the constant determined by the theorem.

The function

$$H(t) = \sin t/t \tag{89}$$

corresponds to

$$F(x) = 1, \qquad L(x) = x, \qquad G(y) = y. \tag{90}$$

The distribution of radiators is

$$x_j = (2j - 1)/2N \tag{91}$$

and therefore is uniform. The approximability of this function is poor compared to the previous examples. Theorem 1 does not cover this case since $L''(x) = 0$; however, the Fourier coefficients c_n (20) may be explicitly evaluated, and the final determination of $R_N(t)$ obtained from Lemma 1. The result is

$$R_N(t) = \frac{t}{\pi N} \sum_{k=-\infty}^{\infty}{}' \frac{\sin (t + \pi k)}{k(t + 2\pi N k)}, \tag{92}$$

in which the prime shows the absence of the term $k = 0$. Evaluation of the integral

$$\int_0^1 \rho\left(Nx + \frac{1}{2}\right) \sin tx\, dx, \tag{93}$$

using (23), shows that

$$R_N(t) = -\frac{t}{N}\int_0^1 \rho\left(Nx + \frac{1}{2}\right) \sin tx\, dx. \tag{94}$$

Since

$$|\rho(Nx + \tfrac{1}{2})| \leqq \tfrac{1}{2}, \qquad |\sin tx| \leqq 1, \tag{95}$$

one has

$$|R_N(t)| \leqq \tfrac{1}{2}|t|N^{-1}. \tag{96}$$

III. TCHEBYCHEFF QUADRATURE METHOD

Let $\varphi(x) \varepsilon C^{(M)}[-1,1]$, then the Tchebycheff quadrature formula[1] is

$$\int_{-1}^1 K(x)\varphi(x)\, dx \cong \frac{1}{M}\sum_{j=1}^{M} \varphi(\alpha_j), \qquad \int_{-1}^1 K(x)\, dx = 1. \tag{97}$$

The fundamental points α_j are determined by the conditions

$$M\int_{-1}^1 x^\nu K(x)\, dx = \sum_{j=1}^{M} \alpha_j^\nu = b_\nu, \qquad 0 \leqq \nu \leqq M. \tag{98}$$

Define the polynomial $\omega(z)$ by the polynomial portion of the Laurent expansion of

$$\exp\left(M\int_{-1}^1 K(x) \ln (z - x)\, dx\right) = z^M \exp\left(-\frac{b_1}{z} - \frac{b_2}{2z^2} - \frac{b_3}{3z^3} - \cdots\right) \tag{99}$$

about the origin,[2] then the zeros of $\omega(z)$ are the required numbers $\alpha_1, \cdots, \alpha_M$. This procedure yields an approximation which, by (98), is exact if $\varphi(x)$ is a polynomial of degree not exceeding M. To obtain an approximation to $H(t)$ of the required form (1), one may set

$$\varphi(x) = \cos tx, \qquad K(x) = \tfrac{1}{2}F(x), \qquad M = 2N; \tag{100}$$

the points $x_1, \cdots, x_N$ are now chosen as those α_j which are positive. Equations (3) and (97) yield the required result.

Define the error, $R_N{}^T$, of Tchebycheff quadrature by

$$R_M{}^T = \int_{-1}^{1} K(x)\varphi(x)\, dx - \frac{1}{M}\sum_{j=1}^{M}\varphi(\alpha_j), \tag{101}$$

then Theorem 2 provides an estimate.

Theorem 2: The real numbers $\alpha_1, \cdots, \alpha_M$ are determined as the zeroes of the polynomial $\omega(x)$ defined in (99)

$$\Rightarrow \exists -1 < \xi < 1 \ni R_M{}^T = \frac{1}{M!}\int_{-1}^{1} K(x)\omega(x)\varphi^{(M)}(\xi)\, dx.$$

Proof: It will be shown that Tchebycheff quadrature is an instance of Newton-Cotes quadrature.

Define

$$\ell_j(x) = \frac{\omega(x)}{(x-\alpha_j)\omega'(\alpha_j)}, \tag{102}$$

then the Lagrange interpolation formula is

$$\varphi(x) = \sum_{j=1}^{M}\varphi(\alpha_j)\, \ell_j(x) + \frac{\varphi^{(M)}(\xi)}{M!}\,\omega(x), \tag{103}$$

in which ξ satisfies

$$\min(x, \alpha_1, \cdots, \alpha_M) < \xi < \max(x, \alpha_1, \cdots, \alpha_M). \tag{104}$$

The coefficients of the Newton-Cotes quadrature formula are given by

$$c_j = \int_{-1}^{1} K(x)\ell_j(x)\, dx \tag{105}$$

and hence, one has

$$\int_{-1}^{1} K(x)\varphi(x)\, dx = \sum_{j=1}^{M} c_j\varphi(\alpha_j) + \frac{1}{M!}\int_{-1}^{1} K(x)\omega(x)\varphi^{(M)}(\xi)\, dx. \tag{106}$$

Since Tchebycheff quadrature is exact when $\varphi(x)$ is a polynomial of

Thus, secondary lobes may be produced of strength nearly equal to the main beam. These are called *grating lobes*. Their existence is the subject of Theorem 5.

Theorem 5: There always exist grating lobes.

Proof: Dirichlet's theorem[4] on simultaneous approximation states:

Given $x_1, \cdots, x_N$, a positive integer q, and a positive integer τ_o, there exists a number τ in the range

$$\tau_o \leqq \tau \leqq \tau_o q^N, \tag{114}$$

and integers $p_1, \cdots, p_N$, such that

$$| \tau x_j - p_j | \leqq 1/q, \qquad 1 \leqq j \leqq N. \tag{115}$$

Accordingly, choose $\tau_o = 1$ and $t = 2\pi\tau$, then

$$tx_j = 2\pi\tau x_j = 2\pi p_j + (2\pi/q)\theta, \qquad | \theta | \leqq 1, \tag{116}$$

and

$$\cos tx_j = \cos \frac{2\pi}{q}\theta > 1 - (2\pi^2/q^2). \tag{117}$$

Thus

$$S_N(t) = \sum_{j=1}^{N} A_j \cos tx_j > 1 - \frac{2\pi^2}{q^2}. \tag{118}$$

Since q may be chosen arbitrarily large, the theorem is proved.

An inspection of all the error formulae of this paper shows the common feature that they increase with increasing $| t |$ and ultimately become trivial. For large $| t |$, $H(t)$ is small, hence, since $S_N(t)$, by Theorem 5, must ultimately become large, the error estimates must also become large. It follows that the grating lobe cannot occur until $R_N(t)$, $R_N^T(t)$, or $R_N^G(t)$ are at least one. The error estimates, therefore, provide a lower bound for the value of $| t |$ at which a grating lobe can occur. This is especially important in those designs where it is desired to eliminate the grating lobe from the scan sector. The methods of synthesis presented in this paper provide different estimates of location of the first grating lobe. Let T designate that location, then, for

$$H(t) = J_o(t), \qquad T > \tfrac{1}{8}N^{4/3}, \tag{119}$$

$$H(t) = (\sin \tfrac{1}{2}t/\tfrac{1}{2}t)^2, \qquad T > .77N^{3/2}, \tag{120}$$

$$H(t) = \sin t/t, \qquad T > 2N. \tag{121}$$

The above are the estimates obtained from the Riemannian sum method. The Tchebycheff and Gaussian quadrature methods do not yield estimates of grating lobe location nearly as advantageous as the Riemannian sum method. Thus, for

$$\begin{aligned} H(t) &= J_o(t), \qquad T > 4N, \\ H(t) &= \sin t/t, \qquad T > 4N. \end{aligned} \tag{122}$$

These results were obtained by rough approximations to the factorials in (112) and (113), however, they serve to show the difference between the Riemannian sum, and the Tchebycheff and Gaussian quadrature methods. Nonetheless, the last two methods may show a much smaller estimate of error for small $|t|$ than the Riemannian sum method.

REFERENCES

1. Tomlinson Fort, *Finite Differences*, Oxford University Press, 1948.
2. Milne-Thomson, L. M., *The Calculus of Finite Differences*, Macmillan and Co., 1933.
3. Sansone, G., *Orthogonal Functions*, Interscience, 1959.
4. Titchmarsh, E. C., *The Theory of the Riemann Zeta-Function*, Oxford University Press, 1951.