Some Extensions of Nyquist's Telegraph Transmission Theory

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(Manuscript received April 21, 1965)

The conditions necessary to achieve undistorted transmission of a pulse signal over a channel of finite bandwidth have been set down by Nyquist. These conditions are extended in this paper to eliminate the bandwidth restrictions. Conditions on the real and imaginary parts of the overall system characteristic which lead to the elimination of intersymbol amplitude and pulse width distortion are found. These generalized constraints do not depend on any sharp band limitation and permit one to find ideal conditions for band pass and gradual cutoff systems. The application of Nyquist's conditions usually amounts to equalizing the transmission characteristics in order to approximate an overall linear phase and some sort of symmetrical amplitude roll-off. This paper shows that the principles of channel shaping for distortionless transmission are a good deal more flexible than this. The application of this more general interpretation of Nyquist's theory is illustrated by several examples.

I. INTRODUCTION

Nyquist's classic paper[1] considered the conditions necessary for digital data transmission without intersymbol distortion, and these conditions have provided the guides for system design for many years. However, Nyquist treated the case in which no energy is transmitted at a frequency above twice the signaling speed, although he mentioned the general case in passing. As a consequence, his results cannot be applied directly to cases in which the amplitude characteristics extend beyond twice the signaling speed (gradual cutoff systems) or baseband systems without low-frequency components (bandpass). In addition, Nyquist's theory has been incompletely exploited in practice. The usual application of the principles of channel shaping amounts to equalizing the phase to make it linear across the band, and equalizing the amplitude to produce a symmetric roll-off characteristic. This procedure is valid and consistent with the theory, but is only a special application of the theory.

This paper extends the previous results by showing that it is not necessary to restrict the bandwidth to arrive at an efficient description of the amplitude and phase constraints for distortionless transmission. In other words, transmission systems without a sharp cutoff frequency are considered and constraints on the system characteristics are obtained. The removal of the bandwidth limitation means that one can easily find the constraints for gradual cutoff and bandpass systems.

In addition, the applications of the principles developed here are extended to give a good deal of flexibility in the design of transmission networks. In particular it is shown that distortionless transmission can be achieved under conditions of nonlinear phase and nonsymmetrical roll-off in amplitude, provided the proper relationships between these two quantities exist.

Fig. 1 illustrates the general baseband system to be examined. We can

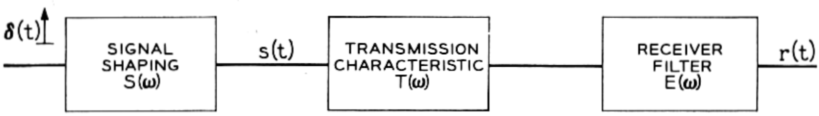

Fig. 1 — General digital transmission system.

assume, without loss of generality, that the information is contained in a random sequence of impulses at the input to the system. Thus a signal $s(t)$ having an amplitude of 0 or 1 is transmitted every T seconds. The system output, $r(t)$, with the Fourier transform

$$R(\omega) = S(\omega)T(\omega)E(\omega) \tag{1}$$

is used to decide whether $s(t)$ was transmitted with amplitude 0 or 1 at a particular time. The type of decision criterion used determines the constraints on $R(\omega)$. Decisions based upon pulse amplitude (usual PCM) at a fixed time and pulse width (telegraph) will be considered.

A sequence of input signals will, in general, produce a sequence of overlapping output pulses. To prevent intersymbol distortion at the output, either the pulse amplitude or the pulse width must be unaffected by the tails of adjacent signals. Fig. 2 illustrates the types of waveform which possess these characteristics. It should be noted that both waveforms require periodic zero crossings away from the main peak. These constraints on the time domain signals are translated into constraints in the frequency domain.*

* These constraints on channels are based on the preservation of periodic zero crossings in the output response. In the case where information is contained in the amplitude of a binary signal, this concept is straightforward. Complications arise, however, when information is associated with the pulse width (such as certain

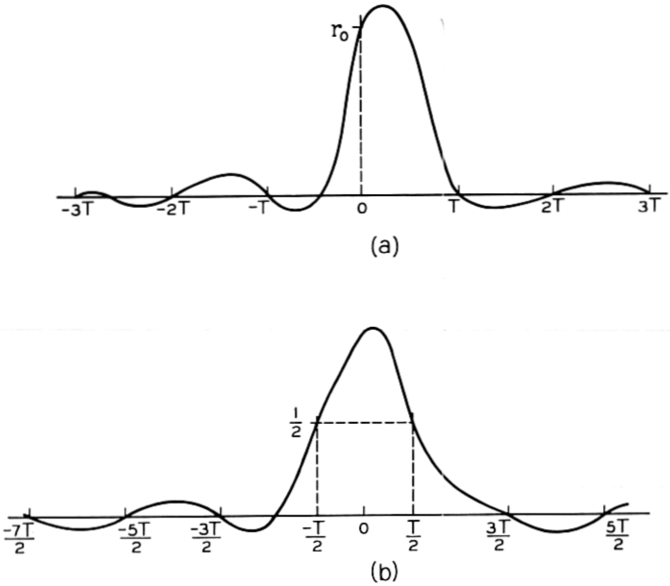

Fig. 2 — Undistorted system responses, (a) $r(t)$ with pulse amplitude undistorted by adjacent pulses; (b) $r(t)$ with pulse width undistorted by adjacent pulses.

II. SYSTEM CONSTRAINTS — UNDISTORTED AMPLITUDE TRANSMISSION

In this section decisions based upon pulse amplitude will be considered. From Fig. 2(a) the constraints on the output pulse may be written

$$r(kT) = r_k = r_0 \delta_{k0} . \tag{2}$$

These sample values may be written in terms of the Fourier transform

$$r(t) = \int_{-\infty}^{\infty} R(\omega) e^{j\omega t} \, d\omega \tag{3}$$

$$r_k = \int_{-\infty}^{\infty} R(\omega) e^{j\omega kT} \, d\omega \tag{4a}$$

$$= \sum_{n=-\infty}^{\infty} \int_{\frac{\pi}{T}(2n-1)}^{\frac{\pi}{T}(2n+1)} R(\omega) e^{j\omega kT} \, d\omega \tag{4b}$$

types of telegraph transmission and systems involving timing recovery). In such cases there may occur troublesome excursions of the signal in between those points which are preserved by the constraints. Unless special apparatus is used in the detection (or timing recovery) process errors will result. The analysis of this problem, which is inherent in the original Nyquist work as well as in the present study, is very complicated and beyond the scope of this paper.

$$= \sum_{n=-\infty}^{\infty} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} R\left(u + \frac{2n\pi}{T}\right) e^{jukT} \, du. \tag{5a}$$

Assuming that $\sum_n R[u + (2n\pi/T)]e^{jukT}$ is a uniformly convergent series, one obtains

$$r_k = \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right) e^{jukT} \, du. \tag{5b}$$

Notice that r_k is just the kth coefficient of an exponential Fourier series expansion of

$$\frac{2\pi}{T} \sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right) \qquad -\frac{\pi}{T} \leqq u \leqq \frac{\pi}{T}.$$

The requirement that $r_k = r_0\delta_{k0}$ implies that only the *zero*th coefficient of the expansion of

$$\frac{2\pi}{T} \sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right)$$

is not zero, and hence

$$\frac{2\pi}{T} \sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right) = r_0. \tag{6}$$

By using the amplitude and phase characteristics

$$R(\omega) = A(\omega)e^{j\alpha(\omega)} \tag{7}$$

one gets

$$\sum_{n=-\infty}^{\infty} A\left(u + \frac{2n\pi}{T}\right) \exp\left[j\alpha\left(u + \frac{2n\pi}{T}\right)\right] = \frac{r_0T}{2\pi}. \tag{8}$$

Separating (8) into real and imaginary parts one obtains

$$\sum_{n=-\infty}^{\infty} A\left(u + \frac{2n\pi}{T}\right) \cos\alpha\left(u + \frac{2n\pi}{T}\right) = \frac{r_0T}{2\pi} \tag{9a}$$

and

$$\sum_{n=-\infty}^{\infty} A\left(u + \frac{2n\pi}{T}\right) \sin\alpha\left(u + \frac{2n\pi}{T}\right) = 0 \tag{9b}$$

for $-\pi/T \leqq u \leqq \pi/T$. Because of symmetry conditions $[A(\omega) = A(-\omega), \alpha(\omega) = -\alpha(-\omega)]$ the interval $0 \leqq u \leqq \pi/T$ is sufficient.

Fig. 3 illustrates the constraints for a characteristic that is limited to

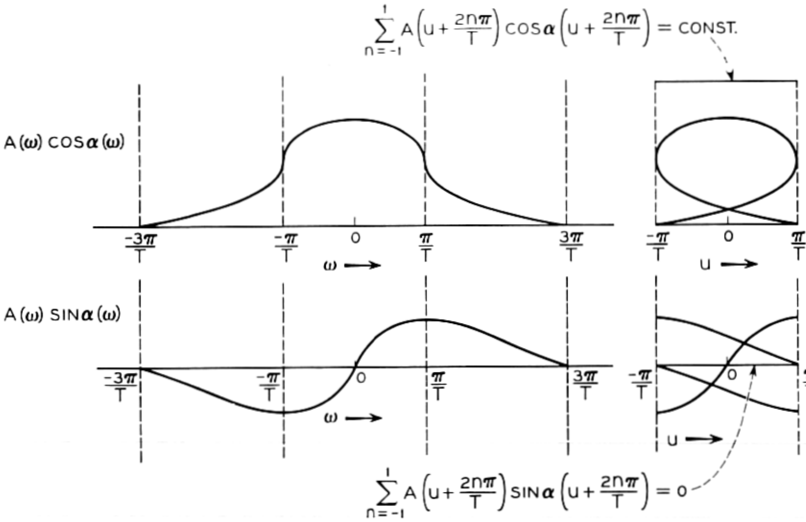

Fig. 3 — Constraints for no intersymbol amplitude distortion $[A(\omega) = 0 \mid \omega \mid \geqq 3\pi/T]$.

$\mid \omega \mid < 3\pi/T$. There is, however, no reason for this limitation other than for clarity in the diagram. The only restriction on the frequency characteristic is an asymptotic one. The condition that $\sum_n R[u + (2n\pi/T)] \cdot e^{jukT}$ be a uniformly convergent series is satisfied if $A(\omega) \to 1/\omega^q$, $q \geqq 2$, as $\omega \to \infty$. This is a more realistic restriction than forcing $A(\omega) = 0$ for large ω.

One may also note that the constraints are more general than Nyquist's symmetry conditions because of the elimination of the cutoff requirements. These symmetry conditions may be obtained by limiting $A(\omega)$ to the region $-2\pi/T < \omega < 2\pi/T$. From Fig. 4 it is easily seen that

$$A(u) \cos \alpha(u) + A[u - (2\pi/T)] \cos \alpha[u - (2\pi/T)] = \text{Const.} \quad (10a)$$

and

$$A(u) \sin \alpha(u) + A[u - (2\pi/T)] \sin \alpha[u - (2\pi/T)] = 0 \qquad \text{for } 0 \leqq u \leqq \pi/T \quad (10b)$$

which are Nyquist's conditions.

Consider, now, (9a) and (9b) and their ramifications. No longer is one confined to low-pass sharp cutoff systems. It is now possible to

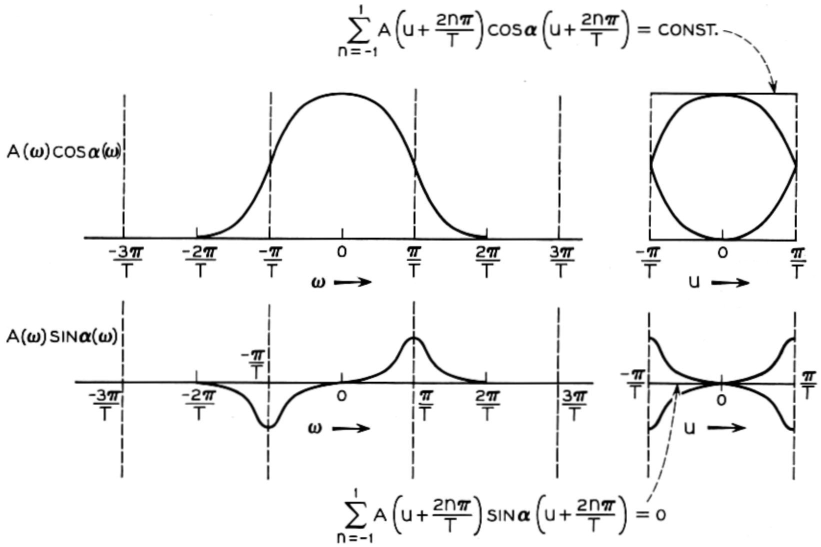

Fig. 4 — Constraints for no intersymbol amplitude distortion $[A(\omega) = 0 \mid \omega \mid \geqq 2\pi/T]$.

express compactly the conditions for distortionless transmission for bandpass or gradual cutoff systems as well. Fig. 3 shows a gradual cutoff system and Fig. 5 illustrates an acceptable bandpass characteristic.

Note that (9a) and (9b) represent constraints on the real and imaginary parts of the characteristics and not upon the amplitude and phase. In general, these equations imply nothing about conditions on the amplitude and phase individually (the exception being the bandlimited case $[A(\omega) = 0, \mid \omega \mid > \pi/T]$ where $A(\omega) = K$ and $\alpha(\omega) = 0$ are the conditions). Constraints on $A(\omega)$ are imposed only if $\alpha(\omega)$ is arbitrarily chosen or vice versa. The usual application of Nyquist's results (linear phase and symmetric roll-off) is just such an arbitrary choice.

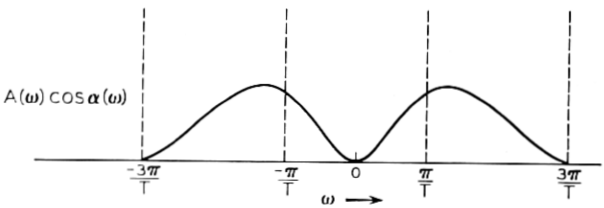

Fig. 5 — Distortionless bandpass characteristic.

teristics may be such that the required $A(\omega)$ in one of the intervals may approach infinity (if $\alpha_j(u) = \alpha_i(u) + n\pi$). With phase equalization, for $\alpha_i(u)$ or $\alpha_j(u)$ to be real phase angles it is necessary that

$$\{A_i(u) + A_j(u)\}^2 \geqq F^2(u) + G^2(u) \geqq \{A_i(u) - A_j(u)\}^2 \qquad -(\pi/T) \leqq u \leqq \pi/T. \tag{15}$$

This condition determines the intervals, if any, in which phase equalization may be applied. It may happen that, because of a poor choice of transmission speed or poor characteristics outside the i and j intervals, this type of equalization cannot be used. In most practical cases, however, the transmission rate can be judiciously chosen, and phase equalization is theoretically possible. It might be pointed out that (15) or its generalization (where phase equalization is allowed over the entire spectrum) can be used to determine the maximum rate for a fixed amplitude characteristic. The application and some ramifications of (15) are illustrated in Appendix A for the Nyquist problem of (10).

As a specific example of some of the concepts outlined, consider the usual Nyquist problem ($A(\omega) = 0$ for $\omega > 2\pi/T$) given in (10a) and (10b). One can obtain the constraints on either $A(\omega)$ or $\alpha(\omega)$ by letting

$$F(u) = K \tag{16a}$$

$$G(u) = 0 \tag{16b}$$

$$A_i(u) = A(u), \qquad \alpha_i(u) = \alpha(u) \tag{16c}$$

$$A_j(u) = A[u - (2\pi/T)], \qquad \alpha_j(u) = \alpha[u - (2\pi/T)] \tag{16d}$$

in (14a–d). The resulting equations become

$$A(u) = \frac{K \sin \alpha\left(u - \frac{2\pi}{T}\right)}{\sin\left[\alpha\left(u - \frac{2\pi}{T}\right) - \alpha(u)\right]} \tag{17a}$$

$$A\left(u - \frac{2\pi}{T}\right) = \frac{-K \sin \alpha(u)}{\sin\left[\alpha\left(u - \frac{2\pi}{T}\right) - \alpha(u)\right]}, \tag{17b}$$

$$\alpha(u) = \cos^{-1} \frac{K^2 + A^2(u) - A^2\left(u - \frac{2\pi}{T}\right)}{2KA(u)} \tag{17c}$$

and

$$\alpha\left(u - \frac{2\pi}{T}\right) = \cos^{-1} \frac{K^2 + A^2\left(u - \frac{2\pi}{T}\right) - A^2(u)}{2KA\left(u - \frac{2\pi}{T}\right)} \tag{17d}$$

$$\text{for } 0 \leqq u \leqq \frac{\pi}{T}.$$

Equations (17a–d) form a relationship which must be satisfied for ideal transmission. In general, $\alpha(\omega)$ need not be linear and $A(\omega)$ need not have the usual symmetrical roll-off. All that is required is that the phase and amplitude satisfy the equations.

For an unequalized channel, with known $A(\omega)$ and $\alpha(\omega)$, this can be accomplished by either leaving the phase unchanged and computing the matching amplitude from (17a–b) or by leaving the amplitude unchanged and computing the matching phase by (17c–d). It is apparent that this gives a good deal more freedom and flexibility to one confronted with the task of equalizing a channel. Some examples of the use of equations will now be considered.

2.1 *Examples*

The amplitude characteristic $A(\omega)$ of a channel with some kind of resonant peaking is shown in Fig. 6(a) together with the minimum phase characteristic associated with $A(\omega)$. Since these channel characteristics do not satisfy ideal transmission conditions, the impulse response of the channel will be distorted. This is indicated in Fig. 6(b) in which the zero crossings of the response do not coincide with the sampling points. As stated before, there are several ways of equalizing the channel. Phase equalization may be achieved by substituting the value of $A(\omega)$ into (17c–d) and obtaining the matching phase. This is shown in Fig. 7(a) (with the original minimum phase shown dashed for comparison). The resulting impulse response, shown in Fig. 7(b), is seen to have zero crossings which are properly spaced, thus satisfying the condition for undistorted transmission.

It can be seen that equalizing a channel by means of (17c,d) offers considerable reduction in complexity over the method which requires a flat delay and symmetrically shaped amplitude. For example, in the above illustration it was necessary to alter only one of the characteristics instead of both. It was not required that the phase be linear but only that its shape be altered in a prescribed manner. An important practical factor stems from the fact that the delay for the equalized channel is not

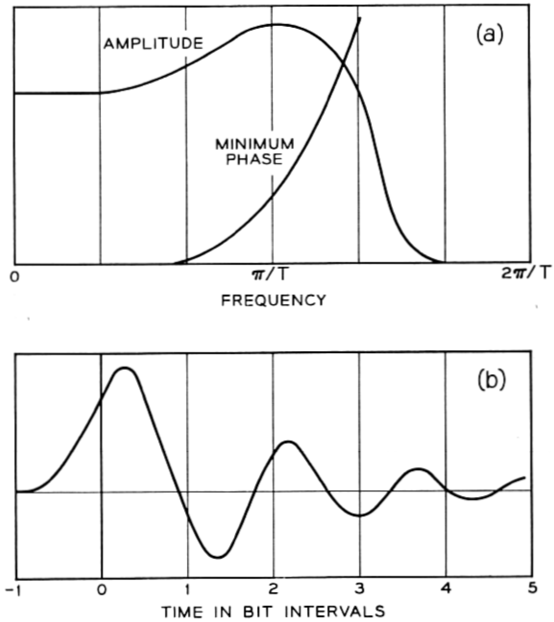

Fig. 6 — Initial system response, (a) transmission frequency characteristics; (b) impulse response.

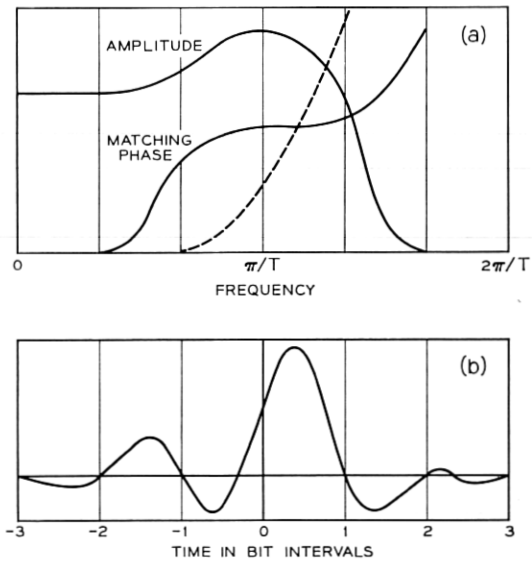

Fig. 7 — System response with phase correction, (a) transmission frequency characteristics; (b) impulse response.

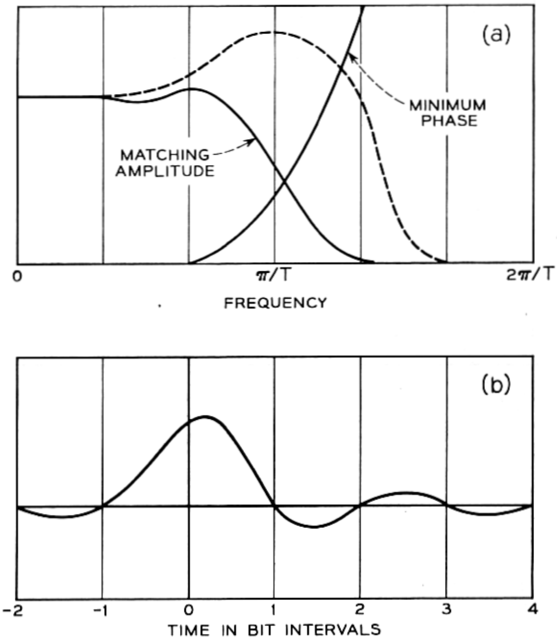

Fig. 8 — System response with amplitude correction, (a) transmission frequency characteristics; (b) impulse response.

flat. While it is usually thought desirable to have a channel with a flat delay, it is apparent that in this case linear phase across the band would degrade rather than improve transmission.

A second method of equalizing the channel of Fig. 6 is obtained when the equalized amplitude characteristic is obtained from the original minimum phase by (17a–b). The resulting $A(\omega)$ is shown in Fig. 8(a) together with the impulse response for the equalized channel in Fig. 8(b).

III. SYSTEM CONSTRAINTS — PULSE WIDTH UNDISTORTED

If the pulse width is to be undistorted by adjacent pulses, $r(t)$ must satisfy the conditions

$$\begin{aligned} r_k &= r\left(\frac{2k-1}{2}\,T\right) = 0 \qquad k \neq 0, 1 \\ r_0 &= r_1 = \frac{1}{2}. \end{aligned} \tag{18}$$

Again, writing these sample values in terms of the Fourier transform, one obtains (3)

$$r(t) = \int_{-\infty}^{\infty} R(\omega) e^{j\omega t}\, d\omega$$
$$r_k = \int_{-\infty}^{\infty} R(\omega) e^{-j\omega(T/2)}\, e^{j\omega kT}\, d\omega \tag{19a}$$

$$= \sum_{n=-\infty}^{\infty} \int_{\frac{\pi}{T}(2n-1)}^{\frac{\pi}{T}(2n+1)} R(\omega) e^{-j\omega(T/2)}\, e^{j\omega kT}\, dw \tag{19b}$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} R\left(u + \frac{2n\pi}{T}\right) \exp\left[-j\left(u + \frac{2n\pi}{T}\right)\frac{T}{2}\right] e^{jkTu}\, du. \tag{20a}$$

Assuming that $\sum_n R[u + (2n\pi/T)]e^{-jn\pi}e^{-ju(T/2)}e^{jukT}$ is a uniformly convergent series one obtains

$$r_k = \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left\{ \sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right) e^{-jn\pi} \right\} e^{-ju(T/2)}\, e^{jukT}\, du \tag{20b}$$

$$= \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left\{ \sum_{n=-\infty}^{\infty} (-1)^n R\left(u + \frac{2n\pi}{T}\right) \right\} e^{-ju(T/2)}\, e^{jukT}\, du. \tag{20c}$$

The value of r_k is the kth coefficient of an exponential Fourier series expansion of

$$(2\pi/T) \sum_{n=-\infty}^{\infty} (-1)^n R[u + (2n\pi/T)] e^{-ju(T/2)}.$$

From (20c) it is seen that the expansion is

$$\sum_{n=-\infty}^{\infty} (-1)^n R[u + (2n\pi/T)] e^{-ju(T/2)} = (T/2\pi) \sum_k r_k{}^{-jukT}. \tag{21}$$

Letting

$$G_R(u) + jG_I(u) = \left\{ \sum_{n=-\infty}^{\infty} (-1)^n R[u + (2n\pi/T)] \right\} e^{-ju(T/2)} \tag{22}$$

and using the conditions $r_0 = r_1 = \frac{1}{2}$ and $r_k = 0,\ k \neq 0{,}1$ one gets

$$G_R(u) + jG_I(u) = (T/2\pi)[\tfrac{1}{2} + \tfrac{1}{2}e^{-juT}]. \tag{23}$$

Separating the real and imaginary parts of the equation yields

$$G_R(u) = (T/4\pi)(1 + \cos uT) \tag{24a}$$

and

$$G_I(u) = (T/4\pi)(-\sin uT) \qquad \text{for } -(\pi/T) \leqq u \leqq \pi/T. \tag{24b}$$

Letting

$$R_R(u) = \operatorname{Re}\{\sum_n (-1)^n R[u + (2n\pi/T)]\} \tag{25a}$$

and

$$R_I(u) = \operatorname{Im}\{\sum_n (-1)^n R[u + (2n\pi/T)]\} \qquad \text{for } -(\pi/T) \leqq u \leqq \pi/T \tag{25b}$$

one gets

$$\begin{aligned} G_R(u) &= R_R(u) \cos (uT/2) + R_I(u) \sin (uT/2) \\ &= (T/4\pi)(1 + \cos uT) \end{aligned} \tag{26a}$$

and

$$\begin{aligned} -G_I(u) &= R_R(u) \sin (uT/2) - R_I(u) \cos (uT/2) \\ &= (T/4\pi) \sin uT \qquad \text{for } -(\pi/T) \leqq u \leqq \pi/T. \end{aligned} \tag{26b}$$

Solving these two equations, the constraints for no intersymbol interference become

$$R_R(u) = \operatorname{Re}\{\sum_n (-1)^n R[u + (2n\pi/T)]\} = (\mathrm{T}/2\pi) \cos (uT/2) \tag{27a}$$

and

$$R_I(u) = \operatorname{Im}\{\sum_n (-1)^n R[u + (2n\pi/T)]\} = 0 \qquad \text{for } -(\pi/T) \leqq u \leqq \pi/T. \tag{27b}$$

Finally, writing

$$R(\omega) = A(\omega) e^{j\alpha(\omega)}$$

one obtains

$$\sum_n (-1)^n A[u + (2n\pi/T)] \cos \alpha[u + (2n\pi/T)] = (T/2\pi) \cos (uT/2) \tag{28a}$$

and

$$\sum_n (-1)^n A[u + (2n\pi/T)] \sin \alpha[u + (2n\pi/T)] = 0 \qquad \text{for } -(\pi/T) \leqq u \leqq \pi/T. \tag{28b}$$

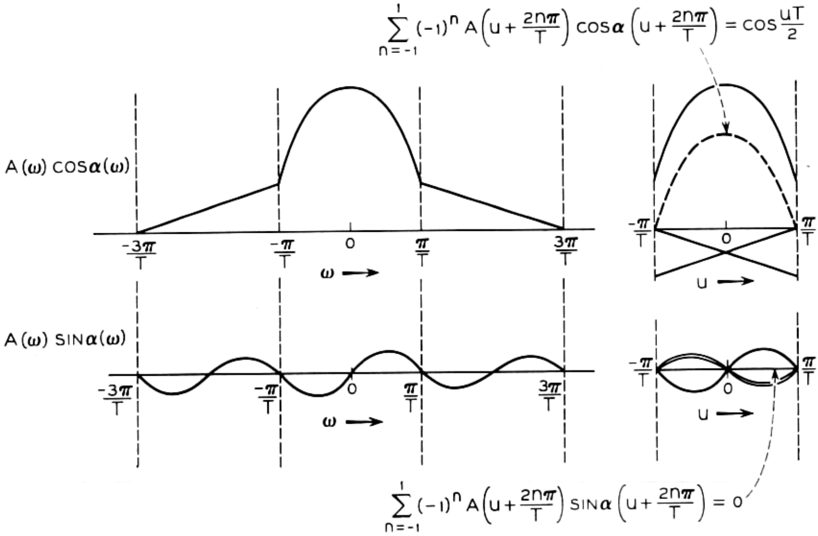

Fig. 9 — Constraints for no intersymbol pulse width distortion $[A(\omega) = 0 \mid \omega \mid \geqq 3\pi/T]$.

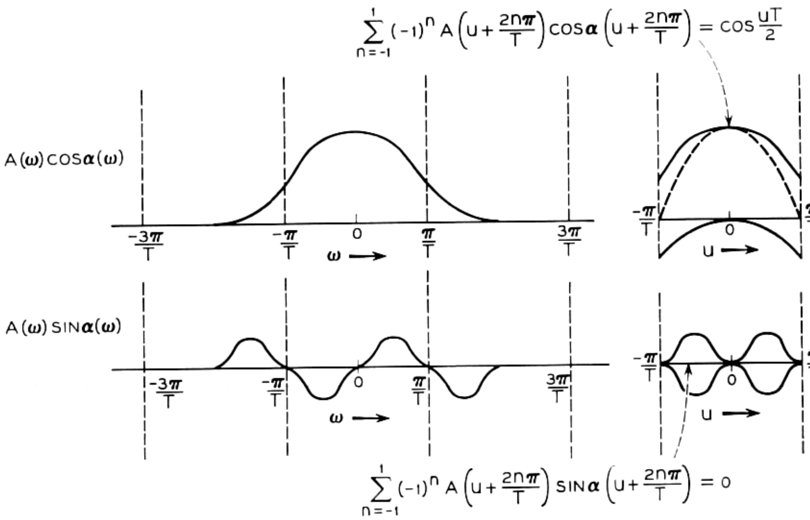

Fig. 10 — Constraints for no intersymbol pulse width distortion $[A(\omega) = 0 \mid \omega \mid \geqq 2\pi/T]$.

Again, there is only an asymptotic bandwidth restriction on these constraints. Fig. 9 illustrates a satisfactory characteristic with $A(\omega) = 0, |\omega| \geqq 3\pi/T$ for clarity. With $A(\omega) = 0, |\omega| \geqq 2\pi/T$ the conditions become the familiar Nyquist results shown in Fig. 10.

The general statements of Section II about the implications of (9a–b) can be applied here to (28a–b). The specific results of Section II can be obtained by replacing

$$A[u + (2n\pi/T)] \text{ by } (-1)^n A[u + (2nT/\pi)]$$

and K by $K \cos (uT/2)$. For the specific case of the usual bandwidth limitation $[A(\omega) = 0, |\omega| \geqq 2\pi/T]$ one gets from (17a–d) the constraints

$$A(u) = \frac{K \cos \frac{uT}{2} \sin \alpha \left(u - \frac{2\pi}{T}\right)}{\sin \left[\alpha \left(u - \frac{2\pi}{T}\right) - \alpha(u)\right]}, \tag{29a}$$

$$A\left(u - \frac{2\pi}{T}\right) = \frac{K \cos \frac{uT}{2} \sin \alpha(u)}{\sin \left[\alpha \left(u - \frac{2\pi}{T}\right) - \alpha(u)\right]}, \tag{29b}$$

$$\alpha(u) = \cos^{-1} \left[\frac{K^2 \cos^2 \frac{uT}{2} + A^2(u) - A^2\left(u - \frac{2\pi}{T}\right)}{2KA(u) \cos \frac{uT}{2}}\right] \tag{29c}$$

and

$$\alpha\left(u - \frac{2\pi}{T}\right) = \cos^{-1} \left[\frac{K^2 \cos^2 \frac{uT}{2} + A^2\left(u - \frac{2\pi}{T}\right) - A^2(u)}{-2KA\left(u - \frac{2\pi}{T}\right) \cos \frac{uT}{2}}\right] \tag{29d}$$

$$0 \leqq u \leqq \frac{\pi}{T}.$$

IV. CONCLUDING REMARKS

This paper has extended Nyquist's work on transmission theory to eliminate bandwidth restrictions. The extension is important for a full understanding of data systems. In the past, incomplete results have been obtained from the imposition of arbitrary band limitations. For example, one paper[2] stated that only one waveform jointly satisfies the two criteria discussed here (pulse height and pulse width preservation). In Appendix B this is shown to be false in general but true if $A(\omega) = 0, |\omega| \geqq 2\pi/T$.

Although distortionless transmission has been the main consideration, it is possible, with the approach used in the paper, to obtain an estimate of system quality when the conditions of ideal transmission are not met. In Appendix C, a measure of the distortion (for systems which base decisions on pulse amplitude) is derived in terms of the frequency domain characteristics.

The discussion makes clear that the constraints are not obtained on the phase and amplitude characteristics individually, but only the real and imaginary parts of the transfer characteristics. Specific constraints on the amplitude and phase are the result of arbitrary design choices. Equalization requirements are thus less stringent than usually assumed. It is seen that equalization is only necessary over intervals of π/T or $2\pi/T$ (subject to the conditions discussed) and not over the entire band. Further, it may only be necessary to compensate either the amplitude or the phase but not both.

APPENDIX A

Realizability Conditions for Phase Equalization

In Section II, the question of equalizer realizability was briefly considered. This question is closely related to the choice of transmission rate which is of sufficient importance to discuss further at this point. Thus, it is possible to illustrate the realizability conditions for phase equalization by considering a transmission system with variable phase equalizer and determining the maximum signaling speed. By assuming that the system has a continuous sharp cutoff amplitude characteristic $[A(\omega) = 0, \omega \geqq \omega_c]$ and that it is desirable that the signaling speed $(\omega_s = \pi/T)$ be

$$\omega_c/2 \leqq \pi/T \leqq \omega_c \tag{30}$$

one has the usual Nyquist problem. Under these assumptions, the conditions for phase equalization (15) become

$$\{A(u) + A[u - (2\pi/T)]\}^2 \geqq K^2 \geqq \{A(u) - A[u - (2\pi/T)]\}^2 \tag{31a}$$

or

$$\begin{aligned} A(u) + A[u - (2\pi/T)] \\ \geqq K \geqq A(u) - A[u - (2\pi/T)] \geqq -K \end{aligned} \tag{31b}$$

or

$$A_+(u) \geqq K \geqq A_-(u) \geqq -K \text{ for } 0 \leqq u \leqq \pi/T. \tag{31c}$$

The more general condition would be used if the above assumptions are removed, but this example illustrates the concepts adequately. By using condition (30) and the fact that

$$A(\omega) = 0, \qquad \omega \geqq \omega_c$$

one obtains

$$A(u) + A[u - (2\pi/T)] \geqq A(0) \geqq A(u) - A[u - (2\pi/T)] \geqq -A(0) \tag{32}$$

and

$$A(\pi/T) \geqq \tfrac{1}{2}A(0), \tag{33}$$

and these must be satisfied for phase equalization.

By examining the amplitude characteristics graphically, it is easier to study some of the other implications of the equations. As an example, consider the problem of finding the maximum signaling speed for the amplitude characteristic shown in Fig. 11(a). From the previous results, it is known that the maximum speed lies between $\omega_c/2$ and $z(A(z) = A(0)/2)$. Fig. 11(b) shows $A_+(u)$ and $A_-(u)$ for

$$\omega_c/2 < \pi/T < z$$

and Fig. 11(c) shows the same curves for

$$\pi/T = \omega_c/2.$$

Notice that $A_+(u) \not> A(0)$ for all u in Fig. 11(b), and phase equalization cannot yield distortionless transmission. For $\pi/T = \omega_c/2$ the network can be phase equalized. Notice also that $\omega_c/2$ is the maximum signaling speed for distortionless transmission with phase equalization. In other words, any amplitude characteristic which is strictly decreasing $[A(\omega + \delta) < A(\omega)]$ cannot have undistorted signaling above $\omega_c/2$. Because $A'(0^+) \neq 0$, a slightly higher signaling speed would mean that both $A_+(u)$ and $A_-(u)$ would be identical and have a slope different from zero at $u \approx 0$. This situation would not satisfy (32).

The above observation can be generalized by noting that the upper signaling speed is limited by $\frac{1}{2}(\omega_c + \omega_l)$ where ω_l = lowest frequency at which $A'(\omega) \neq 0$. Fig. 12 illustrates this feature for a peaked amplitude response. Here

$$\pi/T = \tfrac{1}{2}(\omega_c + \omega_l)$$

and a slightly higher speed would again mean that $A_+(u) = A_-(u)$ at some point with $A_+{}'(u) = A_-{}'(u) \neq 0$.

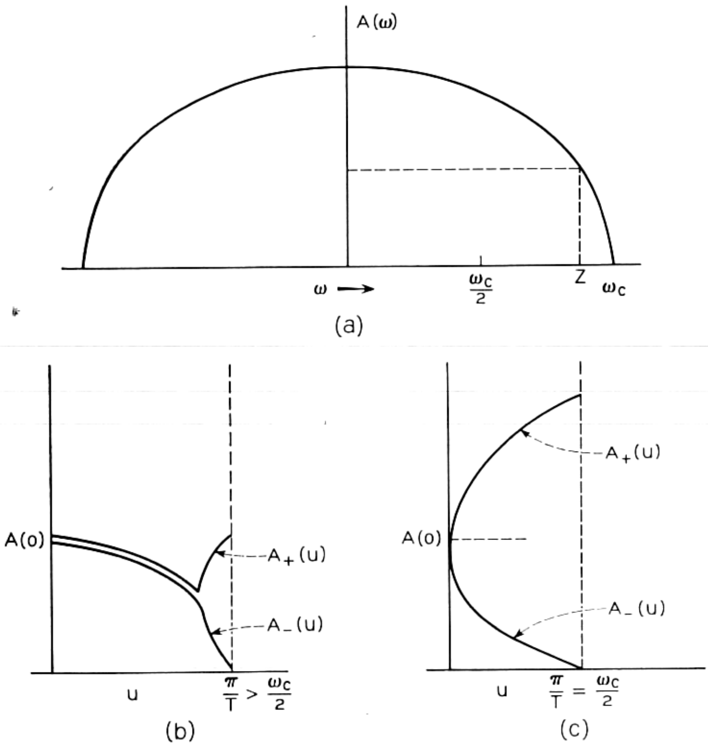

Fig. 11 — Maximum signaling speed when $A'(0^+) \neq 0$.

To show that $\frac{1}{2}(\omega_c + \omega_l)$ is only a limit and not the true maximum speed, consider the example in Fig. 13. It is apparent that the frequency $\frac{1}{2}(\omega_c + \omega_l)$ is too high and thus the true maximum is ω_l.

It is difficult to sum up in words all of the considerations in deciding whether equalization is possible or, equivalently, what is the highest signaling speed at which it is possible. Equation (32) contains all of the required information, and this section was intended to give some idea of its use.

APPENDIX B

Combination of the Two Cases

In the previous analysis, two types of undistorted transmission were treated independently. It will now be determined under what conditions these two cases can be realized simultaneously. The equations which

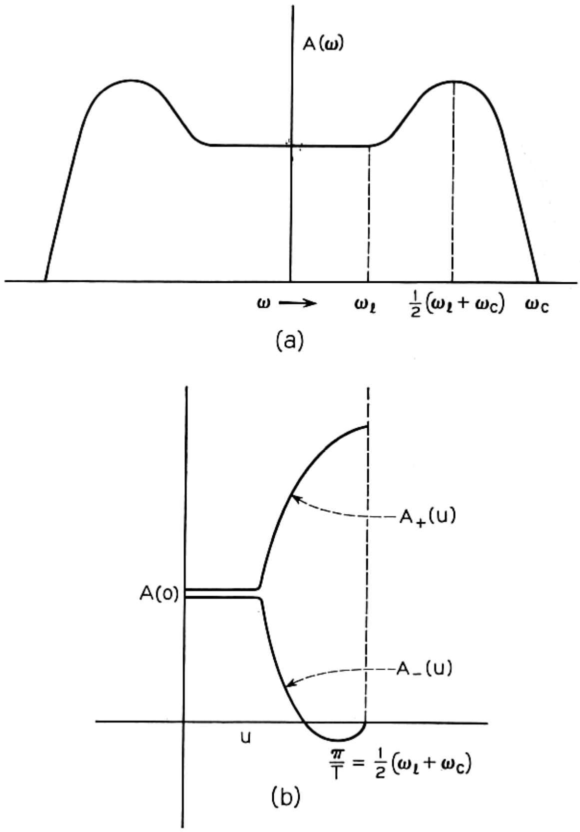

Fig. 12 — Maximum signaling speed for peaked amplitude response.

must be satisfied are:

equation (9a)

$$\sum_n A[u + (2n\pi/T)] \cos \alpha[u + (2n\pi/T)] = K,$$

equation (9b)

$$\sum_n A[u + (2n\pi/T)] \sin \alpha[u + (2n\pi/T)] = 0,$$

equation (28a)

$$\sum_n (-1)^n A[u + (2n\pi/T)] \cos \alpha[u + (2n\pi/T)] = K \cos (uT/2),$$

and equation (28b)

$$\sum_{n} (-1)^n A[u + (2n\pi/T)] \sin \alpha[u + (2n\pi/T)] = 0 \quad \text{for } -(\pi/T) \leqq u \leqq \pi/T.$$

The simultaneous solutions to these equations are

$$\sum_{n \text{ odd}} A[u + (2n\pi/T)] \sin \alpha[u + (2n\pi/T)] = 0, \tag{34}$$

$$\sum_{n \text{ even}} A[u + (2n\pi/T)] \sin \alpha[u + (2n\pi/T)] = 0, \tag{35}$$

$$\sum_{n \text{ even}} A[u + (2n\pi/T)] \cos \alpha[u + (2n\pi/T)] = \tfrac{1}{2}K\{1 + \cos (uT/2)\} \tag{36}$$

and

$$\sum_{n \text{ odd}} A[u + (2n\pi/T)] \cos \alpha[u + (2n\pi/T)] = \tfrac{1}{2}K\{1 - \cos (uT/2)\} \quad \text{for } -(\pi/T) \leqq u \leqq \pi/T. \tag{37}$$

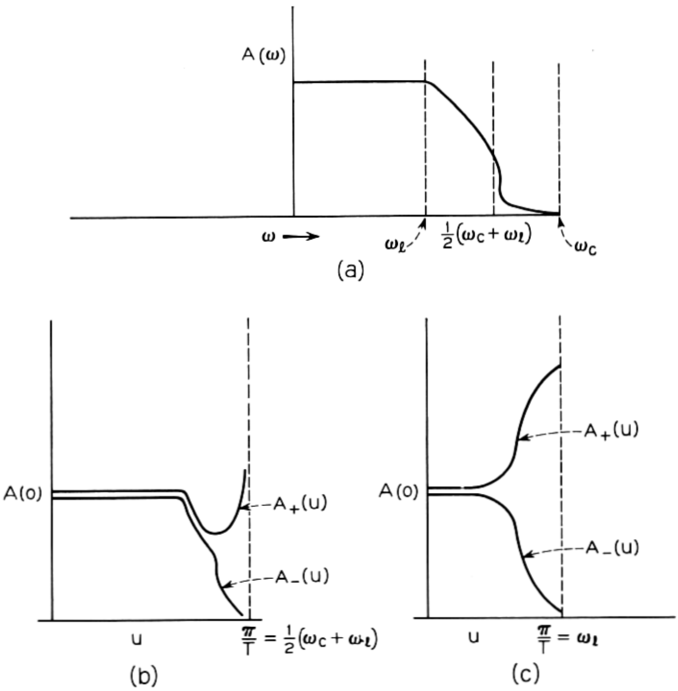

Fig. 13 — Maximum signaling speed equal to ω_l .

In general there will be many possible solutions to these four equations. For the particular case $A(\omega) = 0$, $|\omega| \geqq 2\pi/T$ each of the above summations reduces to one term and

$$\alpha(u) = 0, \tag{38}$$

$$\alpha[u - (2\pi/T)] = 0, \tag{39}$$

$$A(u) = (K/2)\{1 + \cos(uT/2)\}, \tag{40}$$

and

$$A[u - (2\pi/T)] = (K/2)\{1 - \cos(uT/2)\} \qquad \text{for } 0 \leqq u \leqq \pi/T. \tag{41}$$

Taken together, (40) and (41) define the amplitude characteristic across the band as the familiar[2] full cosine roll-off, which may be written by a single expression

$$A(\omega) = (K/2) + (K/2)\cos(\omega T/2) \qquad -(2\pi/T) \leqq \omega \leqq (2\pi/T). \tag{42}$$

This amplitude characteristic is shown in Fig. 14(a). The corresponding

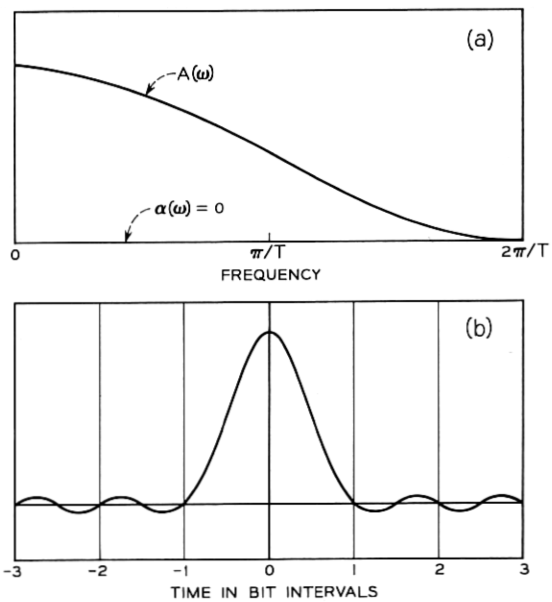

Fig. 14 — System response satisfying both critria, (a) transmission frequency characteristics; (b) impulse response.

impulse response in Fig. 14(b) satisfies both types of undistorted transmission, as expected.

APPENDIX C

A Distortion Measure

It is possible to use the results of the paper to obtain an estimate of system quality when the conditions of ideal transmission are not met. The variance of the intersymbol distortion distribution.

$$\sum_{k\neq 0} r_k^2$$

can be shown to provide an indication of transmission quality (for undistorted amplitude transmission). Since

$$r_k = \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \sum_n R\left(u + \frac{2n\pi}{T}\right) e^{jukT} du \text{ (equation 5b)}$$

one could write

$$\sum_n R\left(u + \frac{2n\pi}{T}\right) = \frac{T}{2\pi} \sum_k r_k\, e^{-jukT} \tag{43a}$$

or

$$\sum_n R\left(u + \frac{2n\pi}{T}\right) - \frac{r_0 T}{2\pi} = \frac{T}{2\pi} \sum_{k\neq 0} r_k\, e^{-jukT} \tag{43b}$$

and

$$\left[\sum_n R\left(u + \frac{2n\pi}{T}\right) - \frac{r_0 T}{2\pi}\right]^* = \frac{T}{2\pi} \sum_{k\neq 0} r_k\, e^{jukT} \tag{43c}$$

$$\text{for } -\frac{\pi}{T} \leqq u \leqq \frac{\pi}{T}.$$

Multiplying (43b) and (43c) and integrating, one obtains

$$\frac{T}{2\pi} \sum_{k\neq 0} r_k^2 = \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left[\sum_n R\left(u + \frac{2n\pi}{T}\right) - \frac{r_0 T}{2\pi}\right]^* \cdot \left[\sum_n R\left(u + \frac{2n\pi}{T}\right) - \frac{r_0 T}{2\pi}\right] du \tag{44a}$$

or

$$\sum_{k\neq 0} r_k^2 = \frac{2\pi}{T}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left\{\left[\sum_n A\left(u+\frac{2n\pi}{T}\right)\right.\right.$$

$$\left.\cos\,\alpha\left(u+\frac{2n\pi}{T}\right)-\frac{r_0T}{2\pi}\right]^2+\left[\sum_n A\left(u+\frac{2n\pi}{T}\right)\right. \tag{44b}$$

$$\left.\left.\cdot\sin\alpha\left(u+\frac{2n\pi}{T}\right)\right]^2\right\}du \qquad \text{for } -\frac{\pi}{T}\leqq u\leqq\frac{\pi}{T}.$$

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