B.S.T.J. BRIEFS

Axis-Crossing Intervals of Rayleigh Processes

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I. INTRODUCTION

Let $R(t,a)$ denote the envelope of a stationary random process consisting of a sinusoidal signal of amplitude $\sqrt{2a}$ and frequency f_0 plus Gaussian noise of unit variance having a narrow-band power spectral density which is symmetrical about f_0. When $a = 0$ Rice[1] presented some theoretical results which are very useful for studying statistical properties of the axis-crossing intervals of $R(t,0)$. The axis-crossing points and the axis-crossing intervals of the Rayleigh process $R(t,a)$ are defined in Fig. 1. Some recent work concerning the axis-crossing in tervals of $R(t,a)$ was reported by Levin and Fomin[2], Goryainov,[3] and Rainal.[4,5] The purpose of this brief is to present some theoretical results when $a \geqq 0$. These results stem from a straightforward extension of Rice's analysis. The Rayleigh process $R(t,a)$ occurs at the output of a typical radio or radar receiver during the reception of a sinusoidal signal immersed in Gaussian noise.

II. THEORETICAL RESULTS

Using a notation consistent with Refs. 4 and 5 we define the following probability functions at an arbitrary level R of Fig. 1:

(1) $Q^-(\tau,R,a)d\tau$, the conditional probability that an upward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given a downward axis-crossing at t.

(2) $Q^+(\tau,R,a)d\tau$, the conditional probability that a downward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t.

(3) $[U(\tau,R,a) - Q(\tau,R,a)]d\tau$, the conditional probability that an upward axis-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward axis-crossing at t.

The reader should refer to Rice[1] for the definition of all notation which is not defined in this note. When $a \geqq 0$, Rice's (86) becomes:

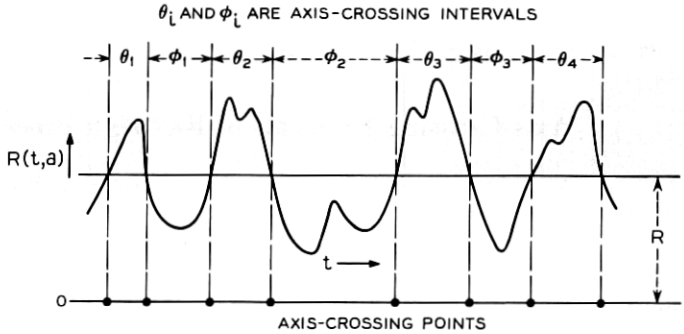

Fig. 1 — The level R defines the axis-crossing points and the axis-crossing intervals of the Rayleigh process $R(t,a)$.

$$
\begin{aligned}
Q^{-}(\tau,R,a) &= -\left(\frac{2\pi}{\beta}\right)^{\frac{1}{2}} R^{-1} e^{R^2/2}\, I_0^{-1}\,(RQ)\, e^{a} \\
&\quad \cdot \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_{-\infty}^{0} dR_1' \int_0^{\infty} dR_2' R_1' R_2' p(R,R_1',R_2',R,\theta_1,\theta_2)
\end{aligned} \tag{1}
$$

where: $I_0\,(x) = $ Bessel function of imaginary argument

$$
\begin{aligned}
p\,(R,R_1',R_2',R,\theta_1,\theta_2) &= \frac{R^2[M_{22}^{\;2} - M_{23}^{\;2}c^2]^{-\frac{1}{2}}}{8\pi^3} \\
&\quad \cdot \exp\left\{-\frac{1}{2M}\,[A\,(R_1'^2 + R_2'^2) + 2Ar R_1' R_2' + 2DR_1' + 2ER_2' + F]\right\}
\end{aligned}
$$

$$
A = [M_{22}^2 - M_{23}^2c^2]^{-1}[MM_{22}(1 - m^2)] \qquad Q = \sqrt{2a}
$$

$$
c = \cos\,(\theta_1 - \theta_2) \qquad s = \sin\,(\theta_1 - \theta_2) \qquad r = \frac{cM_{23}}{M_{22}}
$$

$$
\begin{aligned}
D = [M_{22}^2 - M_{23}^2c^2]^{-1}\{&R[M_{22} - M_{23}c][Mm'(c - m)] \\
&+ Q[M_{12} - M_{13}][M_{23}s(M_{22}\sin\theta_2 + M_{23}c\sin\theta_1) \\
&- \cos\theta_1(M_{22}^2 - M_{23}^2c^2)]\}
\end{aligned}
$$

$$
\begin{aligned}
E = [M_{22}^2 - M_{23}^2c^2]^{-1}\{&-R[M_{22} - M_{23}c][Mm'(c - m)] \\
&+ Q[M_{12} - M_{13}][M_{23}s(M_{22}\sin\theta_1 + M_{23}c\sin\theta_2) \\
&+ \cos\theta_2(M_{22}^2 - M_{23}^2c^2)]\}
\end{aligned}
$$

$$
\begin{aligned}
F = [M_{22}^2 - M_{23}^2c^2]^{-1}\{&2[Q^2 - QR(\cos\theta_1 + \cos\theta_2)] \\
&\cdot [M_{11} + M_{14}][M_{22}^2 - M_{23}^2c^2]
\end{aligned}
$$

$$- 2M_{13}QRs[M_{12} - M_{13}][M_{22} - M_{23}c][\sin\theta_1 - \sin\theta_2]$$
$$- M_{22}Q^2[M_{12} - M_{13}]^2[\sin^2\theta_1 + \sin^2\theta_2]$$
$$- 2M_{23}Q^2c[M_{12} - M_{13}]^2 \sin\theta_1 \sin\theta_2$$
$$+ 2R^2[M_{22} - M_{23}c][(M_{22} + M_{23}c)(M_{11} + M_{14}c) - M_{13}^2s^2]\}.$$

Equation (1) can be put in a form analogous to Rice's (97) and (55):

$$Q^-(\tau,R,a) = \frac{e^a R M_{22} e^{R^2/2} I_0^{-1}(RQ)}{2\pi\sqrt{2\pi\beta}\,(1 - m^2)^2} \int_0^{2\pi}\int_0^{2\pi} e^{-(G/2M)} J(r,h,k)\, d\theta_1 d\theta_2 \quad (2)$$

where:

$$J(r,h,k) \equiv \frac{1}{2\pi s_1}\int_h^\infty dx \int_k^\infty dy\,(x-h)(y-k)e^z$$

$$z = -\frac{x^2 + y^2 - 2rxy}{2(1 - r^2)}; \qquad h = -a_1\left[\frac{1 - m^2}{M_{22}}\right]^{\frac{1}{2}};$$

$$k = a_2\left[\frac{1 - m^2}{M_{22}}\right]^{\frac{1}{2}}$$

$$a_1 = A^{-1}[1 - r^2]^{-1}[D - rE] \qquad a_2 = A^{-1}[1 - r^2]^{-1}[E - rD]$$

$$G = A^{-1}[1 - r^2]^{-1}[2rDE - D^2 - E^2] + F; \qquad s_1 = \sqrt{1 - r^2}.$$

We also find that:

$$J(r,h,k) = \frac{s_1}{2\pi}\exp\left[-\frac{(k^2 - 2rhk + h^2)}{2(1 - r^2)}\right] - \frac{he^{-k^2/2}}{2\sqrt{2\pi}} \cdot\left[1 - P\left(\frac{h - rk}{s_1}\right)\right] - \frac{ke^{-h^2/2}}{2\sqrt{2\pi}}\left[1 - P\left(\frac{k - rh}{s_1}\right)\right] + (hk + r)K(r,h,k) \quad (3)$$

where:

$$P(x) = \frac{2}{\sqrt{2\pi}}\int_0^x e^{-t^2/2}\,dt$$

$$K(r,h,k) \equiv \text{Karl}^6 \text{ Pearson's}\left(\frac{d}{N}\right) = \frac{1}{2\pi s_1}\int_h^\infty dx \int_k^\infty dy e^z.$$

For a recent table of $K(r,h,k)$ see Ref. 7. For a recent discussion of $K(r,h,k)$ see the recent work of Gupta.[8] In these latter two references $K(r,h,k)$ is denoted by $L(h,k,r)$.

$Q^+(\tau,R,a)$ is obtained from (1) by changing the signs of the ∞'s in the limits of integration. We find that $Q^+(\tau,R,a)$ is equal to the right-hand side of (2) with h,k replaced by $-h,-k$.

$[U(\tau,R,a) - Q(\tau,R,a)]$ is obtained from (1) by changing the lower limit of integration of R_1' to $+\infty$. We find that:

$$U(\tau,R,a) - Q(\tau,R,a) = \frac{e^a RM_{22}e^{R^2/2}\, I_0^{-1}(RQ)}{2\pi\sqrt{2\pi\beta}\,(1 - m^2)^2} \cdot \int_0^{2\pi}\int_0^{2\pi} e^{-(G/2M)} J_1(r,h,k)\, d\theta_1 d\theta_2 \tag{4}$$

where:

$$J_1(r,h,k) \equiv \frac{1}{2\pi s_1}\int_h^{-\infty} dx \int_k^{\infty} dy\,(x-h)(y-k)e^z.$$

We find that $J(r,h,k)$ and $J_1(r,h,k)$ are related by:

$$J_1(r,h,k) = J(r,h,k) + \frac{h}{\sqrt{2\pi}} e^{-k^2/2} - \frac{(hk+r)}{2}[1 - P(k)]. \tag{5}$$

Equations (4) and (5) are the generalizations of (64) and (35) of Ref. 5.

III. STATISTICAL DEPENDENCE OF AXIS-CROSSING INTERVALS

By expanding $m(\tau)$ as:

$$m(\tau) = 1 - \frac{\beta}{2}\tau^2 + \frac{b_3|\tau^3|}{3!} + \frac{b_4\tau^4}{4!} + \frac{b_5|\tau^5|}{5!} + \frac{b_6\tau^6}{6!} + \frac{b_7|\tau^7|}{7!} + o(\tau^7) \tag{6}$$

we find that as $\tau \to 0$ from the right:

$$M_{11} = 2\beta b_3\tau - (b_3^2 - \beta b_4 + \beta^3)\tau^2 + o(\tau^2) \tag{7}$$

$$M_{12} = \beta b_3\tau^2 - \tfrac{1}{2}(b_3^2 - \beta b_4 + \beta^3)\tau^3 + o(\tau^3) \tag{8}$$

$$M_{13} = \beta b_3\tau^2 - \tfrac{1}{2}(b_3^2 - \beta b_4 + \beta^3)\tau^3 + o(\tau^3) \tag{9}$$

$$M_{14} = -2\beta b_3\tau + (b_3^2 - \beta b_4 + \beta^3)\tau^2 + o(\tau^2) \tag{10}$$

$$M_{22} = \tfrac{2}{3}\beta b_3\tau^3 + \tfrac{1}{4}(\beta b_4 - b_3^2 - \beta^3)\tau^4 + o(\tau^4) \tag{11}$$

$$M_{23} = \tfrac{1}{3}\beta b_3\tau^3 + \tfrac{1}{12}(3\beta b_4 - b_3^2 - 3\beta^3)\tau^4 + o(\tau^4). \tag{12}$$

When $b_3 \neq 0$ we find that:

$$M_{12} - M_{13} = -\tfrac{1}{6}\beta^2 b_3\tau^4 + o(\tau^4) \tag{13}$$

$$M_{11} + M_{14} = \tfrac{1}{6}\beta b_3^2\tau^4 + o(\tau^4) \tag{14}$$

$$M_{22} - M_{23} = \tfrac{1}{3}\beta b_3\tau^3 + o(\tau^3) \tag{15}$$

$$M = \tfrac{1}{3}\beta b_3^2\tau^4 + o(\tau^4). \tag{16}$$

When $b_3 = 0$ and $b_5 \neq 0$ we find that:

$$M_{12} - M_{13} = \tfrac{1}{60}\beta^2 b_5\tau^6 + o(\tau^6) \tag{17}$$

$$M_{11} + M_{14} = -\tfrac{1}{120}\beta b_4 b_5\tau^7 + o(\tau^7) \tag{18}$$

$$M_{22} - M_{23} = -\tfrac{1}{30}\beta b_5\tau^5 + o(\tau^5) \tag{19}$$

$$M = \frac{\beta b_5}{60}(\beta^2 - b_4)\tau^7 + o(\tau^7). \tag{20}$$

When $b_3 = b_5 = 0$ we find that:

$$M_{12} - M_{13} = \frac{5\beta}{720}(\beta b_6 + b_4^2)\tau^7 + o(\tau^7) \tag{21}$$

$$M_{11} + M_{14} = \frac{-5b_4}{1440}(\beta b_6 + b_4^2)\tau^8 + o(\tau^8) \tag{22}$$

$$M_{22} - M_{23} = -\tfrac{1}{72}(\beta b_6 + b_4^2)\tau^6 + o(\tau^6) \tag{23}$$

$$M = \tfrac{1}{144}(\beta^2 - b_4)(\beta b_6 + b_4^2)\tau^8 + o(\tau^8). \tag{24}$$

As $\tau \to 0$ we see that the terms of the quantities D, E, and F which involve the sine wave amplitude Q are of higher order in τ than the terms which do not involve Q. This behavior as $\tau \to 0$ is consistent with a result reported by Levin and Fomin[2]. Thus, a theorem presented in Ref. 5 also applies to the Rayleigh process $R(t,a)$. That is: If $R(t,a)$ is a Rayleigh process, defined in paragraph one, having a finite expected number of axis-crossing points per unit time at any level R, then two successive axis-crossing intervals at that level R are statistically dependent.

The theorem implies that the successive axis-crossing points of the Rayleigh process $R(t,a)$ at any level R do not form a Markov point process.

IV. ACKNOWLEDGMENT

It gives me great pleasure to acknowledge stimulating discussions with S. O. Rice.

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Errata

A Note on a Signal Recovery Problem, by I. W. Sandberg, B.S.T.J., *43,* November 1964, pp. 3065–3067.

On page 3066, replace $|f_2(t)|^{\frac{1}{2}}$ by $|f_2(t)|^2$, and replace $\tilde{\psi}[w] = \tilde{\psi}[w] - w$ by $\tilde{\psi}[w] = \psi[w] - w$. On page 3067, replace max (c_1, c_2) by $(c_1^2 + c_2^2)^{\frac{1}{2}}$.