Some Results on the Theory of Physical Systems Governed by Nonlinear Functional Equations

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The main purpose of this paper is to present a reasonably complete picture of the results of the first phase of some recent research on the properties of solutions of nonlinear functional equations that frequently arise in the study of physical systems. We consider in detail the properties of a vector nonlinear Volterra integral equation of the second kind, and some conditions are presented for the norm boundedness of solutions of a functional equation of similar type defined on an abstract space.

More specifically, concerning the Volterra equation, conditions are presented under which the solutions (a) approach zero as $t \to \infty$, (b) approach zero exponentially as $t \to \infty$, (c) are uniformly bounded on $t \geqq 0$, (d) are square integrable on $[0, \infty)$, or (e) are ultimately periodic. On the basis of these results, it appears that an input-output stability theory of a large class of time-varying nonlinear systems of engineering interest is well within sight.

I. PRELIMINARY NOTATION AND DEFINITIONS

The set of real measurable N-vector-valued functions of the real variable t defined on $[0, \infty)$ is denoted by $\mathcal{K}_N(0, \infty)$ and the jth component of $f \in \mathcal{K}_N(0, \infty)$ is denoted by f_j.

The sets $\mathcal{L}_{\infty N}(0, \infty)$ and $\mathcal{L}_{2N}(0, \infty)$ are defined by

$$\mathcal{L}_{\infty N}(0, \infty) = \{f \mid f \in \mathcal{K}_N(0, \infty), \sup_{t \geqq 0} [f'(t)f(t)] < \infty\}$$

$$\mathcal{L}_{2N}(0, \infty) = \left\{f \mid f \in \mathcal{K}_N(0, \infty), \int_0^\infty f'(t)f(t)\, dt < \infty\right\},$$

in which $f'(t)$ denotes the transpose of $f(t)$. In order to be consistent with standard notation, we let $\mathcal{L}_2(0, \infty) = \mathcal{L}_{2N}(0, \infty)$ when $N = 1$.

Let $y \in (0, \infty)$ and define f_y by

$$f_y(t) = f(t) \quad \text{for} \quad t \in [0, y]$$
$$= 0 \quad \text{for} \quad t > y$$

for any $f \in \mathcal{H}_N(0, \infty)$, and let

$$\mathcal{E}_N = \{f \mid f \in \mathcal{H}_N(0, \infty), f_y \in \mathcal{L}_{2N}(0, \infty) \quad \text{for} \quad 0 < y < \infty\}.$$

With A an arbitrary real measurable $N \times N$ matrix-valued function of t with elements $\{a_{nm}\}$ defined on $[0, \infty)$, let $\mathcal{K}_{pN}$ $(p = 1,2)$ denote

$$\left\{A \mid \int_0^\infty |a_{nm}(t)|^p \, dt < \infty \qquad (n, m = 1,2, \cdots, N)\right\}.$$

For an arbitrary $f \in \mathcal{H}_N(0, \infty)$, let $\psi[f(t),t]$ denote

$$(\psi_1[f_1(t),t], \psi_2[f_2(t),t], \cdots, \psi_N[f_N(t),t])'$$

where $\psi_1(w,t), \psi_2(w,t), \cdots, \psi_N(w,t)$ are real-valued functions of the real variables w and t for $w \in (-\infty, \infty)$ and $t \in [0, \infty)$ such that

(*i*) $\psi_n(0,t) = 0$ for $t \in [0, \infty)$ and $n = 1,2, \cdots, N$

(*ii*) $\psi_n[w(t),t]$ $(n = 1,2, \cdots, N)$ is a measurable function of t whenever $w(t)$ is measurable.

Let α and β denote real numbers such that $\alpha \leqq \beta$. We shall say that $\psi[\cdot, \cdot] \in \Psi_0(\alpha,\beta)$ if and only if

$$\alpha \leqq \frac{\psi_n(w, t)}{w} \leqq \beta \qquad (n = 1,2, \cdots, N)$$

for $t \in [0, \infty)$ and all real $w \neq 0$; and we shall say that $\psi[\cdot, \cdot] \in \Psi(\alpha,\beta)$ if and only if

$$\alpha \leqq \frac{\psi_n(w_1, t) - \psi_n(w_2, t)}{w_1 - w_2} \leqq \beta \qquad (n = 1,2, \cdots, N)$$

for $t \in [0, \infty)$ and all real w_1, w_2 such that $w_1 \neq w_2$.

II. INTRODUCTION

In the study of physical systems containing time-varying nonlinear elements, attention is frequently focused on the properties of the equation

$$g(t) = f(t) + \int_0^t k(t - \tau)\psi[f(\tau), \tau] \, d\tau, \qquad t \geqq 0 \tag{1}$$

in which $g \in \mathcal{E}_N$, $f \in \mathcal{E}_N$, $k \in \mathcal{K}_{1N}$ and $\psi[\cdot, \cdot] \in \Psi_0(\alpha,\beta)$ for some α and β. For example, consider the multi-input multi-output nonlinear feedback system of Fig. 1 in which u, v, f, w, x, and y are assumed to denote elements of $\mathcal{E}_N$ with the input $u \;\varepsilon\; \mathcal{L}_{2N}(0,\infty)$. Let the block labeled ψ represent N memoryless time-varying nonlinear elements which introduce the constraint $w(t) = \psi[f(t),t]$ for $t \geqq 0$ with $\psi[\cdot\,,\cdot] \;\varepsilon\; \Psi_0(\alpha,\beta)$, and let the blocks labeled $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$, which represent linear time-invariant portions of the system, introduce the constraints

$$f(t) = \int_0^t k_1(t-\tau)v(\tau)\,d\tau + g_1(t)$$

$$x(t) = \int_0^t k_3(t-\tau)w(\tau)\,d\tau + g_3(t)$$

$$y(t) = \int_0^t k_2(t-\tau)x(\tau)\,d\tau + g_2(t)$$

for $t \geqq 0$, in which the impulse response matrices k_1, k_2, and k_3 are elements of $\mathcal{K}_{1N}$ and the initial condition functions g_1, g_2, and g_3 are elements of $\mathcal{L}_{2N}(0,\infty)$. Then

$$g(t) = f(t) + \int_0^t k(t-\tau)\psi[f(\tau),\tau]\,d\tau, \qquad t \geqq 0$$

where k, the inverse Fourier transform of

$$\int_0^\infty k_1(t)e^{-i\omega t}\,dt \int_0^\infty k_2(t)e^{-i\omega t}\,dt \int_0^\infty k_3(t)e^{-i\omega t}\,dt,$$

is an element of $\mathcal{K}_{1N}$, and g defined for $t \geqq 0$ by

$$g(t) = g_1(t) + \int_0^t k_1(t-\tau)u(\tau)\,d\tau - \int_0^t k_1(t-\tau)g_2(\tau)\,d\tau$$
$$- \int_0^t k_1(t-\tau)\int_0^\tau k_2(t-q)g_3(q)\,dq\,d\tau$$

is an element of $\mathcal{L}_{2N}(0,\infty)$.

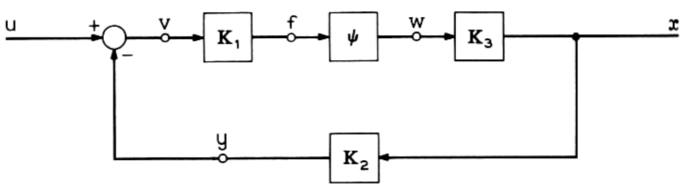

Fig. 1 — Nonlinear feedback system.

Equations of the form

$$\frac{d^n f}{dt^n} = g(t) + \int_0^t k(t - \tau)\psi[f(\tau), \tau]\, d\tau, \qquad t \geqq 0 \tag{2}$$

also arise in a natural way in the study of physical systems. For example, this type of equation with $n = 1$ is encountered in the theory of feedback control systems containing a motor in the forward path.

This paper is addressed primarily to the engineer interested in the mathematical aspects of nonlinear systems. Its main purpose is to present a reasonably complete picture of the results of the first phase of some recent research on the properties of solutions of (1) and of equations of similar type defined on an abstract space. The essentials of the material to be presented are drawn largely from Refs. 1 and 2.

With the exception of some observations in connection with the problem of determining lower bounds on the decay rate or upper bounds on the growth rate of solutions of (1), each of the results of Section III constitutes a set of sufficient conditions, in which a certain frequency-domain condition†, †† plays a central role, under which the solutions of (1) are stable in one of several significant senses. More specifically, conditions are presented under which: $f(t) \to 0$ (i.e., the zero vector) as $t \to \infty$, $g \; \varepsilon \; \mathfrak{L}_{2N}(0, \infty)$ implies $f \; \varepsilon \; \mathfrak{L}_{2N}(0, \infty)$, $g \; \varepsilon \; \mathfrak{L}_{\infty N}(0, \infty)$ implies $f \; \varepsilon \; \mathfrak{L}_{\infty N}(0, \infty)$, and g ultimately periodic with period T implies that f is ultimately periodic with period T. Conditions are also presented under which f depends continuously on g. On the basis of these results, it appears that an input-output stability theory of a large class of time-varying nonlinear systems is well within sight. In particular, the class is not restricted to lumped-parameter systems. An example is presented concerning the necessity of some of the conditions.

Section IV is devoted to the proof of a previously unpublished result concerning (2) with n an arbitrary nonnegative integer, and, for simplicity, f and g scalar functions (i.e., $N = 1$). A set of conditions is established under which $g \; \varepsilon \; \mathfrak{L}_2(0, \infty)$ implies that $f \; \varepsilon \; \mathfrak{L}_2(0, \infty)$.

In Section V, we consider some properties of equations defined on an

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† For some other results concerned with frequency-domain conditions for the stability of nonlinear or time-varying systems, see Refs. 3–6. In particular, Ref. 3 describes in detail the work of V. M. Popov.

†† After this paper had been submitted for publication, the following related papers came to the writer's attention: B. N. Naumov and Ya. Z. Tsypkin, A Frequency Criterion for Absolute Process Stability in Nonlinear Automatic Control Systems, Automation and Remote Control, *25*, Jan. 1965; and V. A. Yakubovich, The Matrix-Inequality Method in the Theory of Stability of Nonlinear Control Systems: I. The Absolute Stability of Forced Vibrations, Automation and Remote Control, *25*, Feb. 1965.

abstract space. More specifically, the basic problem considered is a direct generalization of the problem of establishing the $\mathfrak{L}_{2N}(0, \infty)$-boundedness of solutions of nonlinear functional equations (i.e., a generalization of the central problem of Ref. 1). In particular, Theorem 8 is of immediate utility in obtaining results similar to those of Section III for other types of equations [e.g., the discrete analog of (1) which is of interest in the theory of sampled-data systems]. Section 5.2 is essentially a restatement of the key argument of Ref. 1 in a more abstract setting. In Section 5.3 some new results are proved.

In the Appendix we state some results concerning an integral equation similar to (1) that arises in the study of ordinary linear differential equations, and conditions are presented under which all solutions of a system of second-order equations with varying coefficients approach zero exponentially as $t \to \infty$.

III. RESULTS CONCERNING THE PROPERTIES OF (1)

3.1 *Further Notation and Definitions*

Let M denote an arbitrary matrix. We shall denote by M', M^*, and M^{-1}, respectively, the transpose, the complex-conjugate transpose, and the inverse of M. The positive square-root of the largest eigenvalue of M^*M is denoted by $\Lambda\{M\}$, and 1_N denotes the identity matrix of order N.

The norm of $f \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$ is denoted by $\| f \|$ and is defined by

$$\| f \| = \left(\int_0^\infty f'(t) f(t)\, dt \right)^{\frac{1}{2}}.$$

The symbol s denotes a scalar complex variable with $\sigma = \text{Re}\,[s]$ and $\omega = \text{Im}\,[s]$.

We shall say that k is an element of the set $\Phi(\alpha, \beta)$ if and only if $k \,\varepsilon\, \mathcal{K}_{1N}$ and, with

$$K(s) = \int_0^\infty k(t) e^{-st}\, dt \quad \text{for} \quad \sigma \geqq 0,$$

(*i*) $\det\,[1_N + \frac{1}{2}(\alpha + \beta)K(s)] \neq 0 \quad \text{for} \quad \sigma \geqq 0$

(*ii*) $\frac{1}{2}(\beta - \alpha) \sup\limits_{-\infty<\omega<\infty} \Lambda\{[1_N + \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1} K(i\omega)\} < 1.$

Comments: It can be shown that conditions (*i*) and (*ii*) above are satisfied if $\alpha \geqq 0$ and $[K(i\omega) + K(i\omega)^*]$ is nonnegative definite for all ω.

For $N = 1$, conditions (i) and (ii) above are met if $\beta > 0$ and one of the following three conditions is satisfied:

(i) $\alpha > 0$; and the locus of $K(i\omega)$ for $-\infty < \omega < \infty$ (a) lies outside the circle C_1 of radius $\frac{1}{2}(\alpha^{-1} - \beta^{-1})$ centered on the real axis of the complex plane at $[-\frac{1}{2}(\alpha^{-1} + \beta^{-1}),0]$, and (b) does not encircle C_1 (see Fig. 2)

(ii) $\alpha = 0$, and $\mathrm{Re}\,[K(i\omega)] > -\beta^{-1}$ for all real ω

(iii) $\alpha < 0$, and the locus of $K(i\omega)$ for $-\infty < \omega < \infty$ is contained within the circle C_2 of radius $\frac{1}{2}(\beta^{-1} - \alpha^{-1})$ centered on the real axis of the complex plane at $[-\frac{1}{2}(\alpha^{-1} + \beta^{-1}),0]$ (see Fig. 3).

Concerning the condition for $\alpha > 0$, if $\beta = \alpha$, then the circle C_1 degenerates to a point, and the criterion becomes the well-known Nyquist stability criterion.

3.2 *Results*

Our first theorem, which is proved in Ref. 1 as an application of an abstract result similar to Theorem 8 of Section V, is the key result of this section. It is of direct interest in the theory of stability of dynamical systems, and it plays an important role in the proof of each of the other theorems of this section.

Theorem 1: Let $k \,\varepsilon\, \Phi(\alpha,\beta)$, *let* $\psi[\cdot\,,\cdot] \,\varepsilon\, \Psi_0(\alpha,\beta)$, *and let*

$$g(t) = f(t) + \int_0^t k(t-\tau)\psi[f(\tau),\tau]\,d\tau, \qquad t \geqq 0$$

where $g \,\varepsilon\, \mathfrak{L}_{2N}(0,\infty)$ *and* $f \,\varepsilon\, \mathcal{E}_N$. *Then* $f \,\varepsilon\, \mathfrak{L}_{2N}(0,\infty)$, *and there exists a positive constant* ρ *which depends only on* k, α, *and* β *such that*

$$\| f \| \leqq \rho \, \| g \|.$$

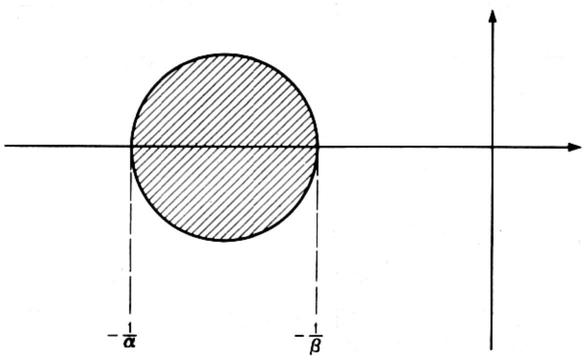

Fig. 2 — Location of the "critical circle" C_1 in the complex plane ($\alpha > 0$).

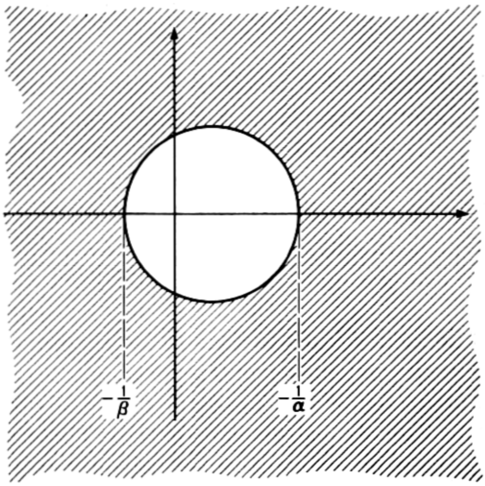

Fig. 3 — Location of the "critical circle" C_2 in the complex plane ($\alpha < 0$).

Corollary 1(a): Let $k \in \Phi(\alpha,\beta)$, *let* $\psi[\cdot\,,\cdot] \in \Psi(\alpha,\beta)$, *and let*

$$g_1(t) = f_1(t) + \int_0^t k(t-\tau)\psi[f_1(\tau),\tau]\,d\tau, \qquad t \geqq 0$$

$$g_2(t) = f_2(t) + \int_0^t k(t-\tau)\psi[f_2(t),\tau]\,d\tau, \qquad t \geqq 0$$

where $g_1, g_2, f_1, f_2 \in \mathcal{E}_N$ *and* $(g_1 - g_2) \in \mathfrak{L}_{2N}(0,\infty)$. *Then* $(f_1 - f_2) \in \mathfrak{L}_{2N}(0,\infty)$, *and there exists a positive constant* ρ *which depends only on* k, α, *and* β *such that*

$$\| f_1 - f_2 \| \leqq \rho \, \| g_1 - g_2 \|.$$

Proof of Corollary 1(a): Let $q_j(t)$ be defined on $[0,\infty)$ by

$$\begin{aligned} q_j(t) &= \frac{\psi_j[f_{1j}(t),t] - \psi_j[f_{2j}(t),t]}{f_{1j}(t) - f_{2j}(t)}, \qquad && t \in \{t \mid t \geqq 0, f_{1j}(t) \neq f_{2j}(t)\} \\ &= \tfrac{1}{2}(\alpha+\beta), && t \in \{t \mid t \geqq 0, f_{1j}(t) = f_{2j}(t)\} \end{aligned}$$

for $j = 1,2,\cdots,N$; and let $q(t)$ denote the diagonal matrix diag $[q_1(t),q_2(t),\cdots,q_N(t)]$. Then $\alpha \leqq q_j(t) \leqq \beta$ for $j = 1,2,\cdots,N$ and

$$\begin{aligned} g_1(t) - g_2(t) &= f_1(t) - f_2(t) \\ &\quad + \int_0^t k(t-\tau)q(\tau)[f_1(\tau) - f_2(\tau)]\,d\tau, \qquad t \geqq 0. \end{aligned}$$

The conclusion of the corollary follows from this equation and the theorem.

Remarks: A direct application of the Schwarz inequality and the Riemann-Lebesgue Lemma shows† that if the hypotheses of Theorem 1 are satisfied and $g(t) \to 0$ as $t \to \infty$ [i.e., $g_j(t) \to 0$ as $t \to \infty$ for $j = 1,2, \cdots, N$], then $f(t) \to 0$ as $t \to \infty$ provided that $k \in \mathcal{K}_{2N}$ [observe that $k \in \mathcal{K}_{2N}$ if $k \in \mathcal{K}_{1N}$ and the elements of k are uniformly bounded on $[0, \infty)$]. Similarly, if the hypotheses of Corollary 1(a) are met and $[g_1(t) - g_2(t)] \to 0$ as $t \to \infty$, then $[f_1(t) - f_2(t)] \to 0$ as $t \to \infty$ provided that $k \in \mathcal{K}_{2N}$.

Theorem 8 of Section V leads to a result for the integral equation (1) that is actually somewhat stronger than that stated as Theorem 1. If k, $\psi[\cdot\,,\cdot]$ and f are as defined in Theorem 1, and if $g \in \mathcal{E}_N$ satisfies the integral equation, then it can be shown that Theorem 8 implies the existence of a positive constant ρ which depends only on k, α, and β such that $\| f_y \| \leqq \rho \| g_y \|$ for all $y > 0$.

Theorem 2: Let

$$g(t) = f(t) + \int_0^t k(t-\tau)\psi[f(\tau),\tau]\,d\tau, \qquad t \geqq 0 \tag{3}$$

in which $\psi[\cdot\,,\cdot] \in \Psi_0(\alpha,\beta)$, $f \in \mathcal{E}_N$, and there exists a real constant c_1 such that

(*i*) $ge^{c_1 t} \in \mathcal{L}_{2N}(0,\infty)$

(*ii*) $ke^{c_1 t} \in \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$

(*iii*) $ke^{c_1 t} \in \Phi(\alpha,\beta)$.

Then there exists a positive constant c_2 such that

$$|f_j(t)| \leqq |g_j(t)| + c_2 e^{-c_1 t}, \qquad t \geqq 0$$

for $j = 1, 2, \cdots, N$.

Corollary 2(a): Let

$$g(t) = f(t) + \int_0^t k(t-\tau)\psi[f(\tau),\tau]\,d\tau, \qquad t \geqq 0$$

in which $\psi[\cdot\,,\cdot] \in \Psi_0(\alpha,\beta)$, $k \in \Phi(\alpha,\beta)$, $f \in \mathcal{E}_N$, and there exists a positive constant c_1 such that

(*i*) $ge^{c_1 t} \in \mathcal{L}_{2N}(0,\infty)$

(*ii*) $ke^{c_1 t} \in \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$.

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† See the proof of Theorem 6 of Ref. 1.

Then there exist positive constants c_2 and c_3 such that

$$|f_j(t)| \leqq |g_j(t)| + c_2 e^{-c_3 t}, \qquad t \geqq 0.$$

Proof of Theorem 2: From the fact that f and g satisfy (3), we have

$$e^{c_1 t} g(t) = \hat{f}(t) + \int_0^t e^{c_1(t-\tau)} k(t-\tau) e^{c_1 \tau} \psi[e^{-c_1 \tau} \hat{f}(\tau), \tau]\, d\tau, \qquad t \geqq 0 \quad (4)$$

in which $\hat{f}(t) = f(t) e^{c_1 t}$. Since

$$\alpha \leqq \frac{e^{c_1 t} \psi_n[e^{-c_1 t} x, t]}{x} \leqq \beta \qquad (n = 1, 2, \cdots, N)$$

for all real $x \neq 0$ and $t \geqq 0$, it follows from Theorem 1 that $\hat{f} \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$. Thus $e^{c_1 t} \psi[f(\cdot), \cdot] \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$, and by the Schwarz inequality, there exists a positive constant c_2 such that the modulus of the jth component of

$$\int_0^t e^{c_1(t-\tau)} k(t-\tau) e^{c_1 \tau} \psi[f(t), \tau]\, d\tau$$

does not exceed c_2 for $t \geqq 0$ and $j = 1, 2, \cdots, N$. Thus, using (4),

$$|f_j(t)| \leqq |g_j(t)| + c_2 e^{-c_1 t}, \qquad t \geqq 0$$

for $j = 1, 2, \cdots, N$.

Proof of Corollary 2(a): Let

$$K(i\omega - \rho) = \int_0^\infty k(t) e^{-(i\omega - \rho)t}\, dt$$

for $\rho \leqq c_1$ and $-\infty < \omega < \infty$. It clearly suffices to prove that there exists a positive constant $c_4 \leqq c_1$ such that $k e^{\rho t} \,\varepsilon\, \Phi(\alpha,\beta)$ for $0 \leqq \rho \leqq c_4$. The existence of such a constant follows easily from the fact that each element of $[K(s) - K(s - \rho)]$ approaches zero uniformly in $\sigma \geqq 0$ as $\rho \to 0+$. The details are omitted.

For some results related to Theorem 2 and Corollary 2(a), see the Appendix.

The following theorem is proved in Ref. 2 with the aid of Theorem 1.

Theorem 3: Let $k \,\varepsilon\, \Phi(\alpha,\beta)$ with $t^p k \,\varepsilon\, \mathfrak{K}_{1N} \cap \mathfrak{K}_{2N}$ for $p = 0,1,2$. Let $\psi[\cdot\, , \cdot] \,\varepsilon\, \Psi_0(\alpha,\beta)$, and let

$$g(t) = f(t) + \int_0^t k(t-\tau) \psi[f(\tau), \tau]\, d\tau, \qquad t \geqq 0$$

where $g \in \mathfrak{L}_{\infty N}(0,\infty)$ *and* $f \in \mathcal{E}_N$. *Then* $f \in \mathfrak{L}_{\infty N}(0,\infty)$, *there exists a positive constant* ρ *which depends only on* k, α, *and* β *such that*

$$\max_j \sup_{t \geqq 0} |f_j(t)| \leqq \rho \max_j \sup_{t \geqq 0} |g_j(t)|,$$

and $f_j(t) \to 0$ *as* $t \to \infty$ *for* $j = 1,2, \cdots, N$ *whenever* $g_j(t) \to 0$ *as* $t \to \infty$ *for* $j = 1,2, \cdots, N$.

Corollary 3(a): Let $k \in \Phi(\alpha,\beta)$ *with* $t^p k \in \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$ *for* $p = 0,1,2$. *Let* $\psi[\cdot\, ,\cdot] \in \Psi(\alpha,\beta)$, *and let*

$$g_1(t) = f_1(t) + \int_0^t k(t-\tau)\psi[f_1(\tau),\tau]\, d\tau, \qquad t \geqq 0$$

$$g_2(t) = f_2(t) + \int_0^t k(t-\tau)\psi[f_2(\tau),\tau]\, d\tau, \qquad t \geqq 0$$

where $g_1, g_2, f_1, f_2 \in \mathcal{E}_N$ *and* $(g_1 - g_2) \in \mathfrak{L}_{\infty N}(0,\infty)$. *Then* $(f_1 - f_2) \in \mathfrak{L}_{\infty N}(0,\infty)$, *there exists a positive constant* ρ *which depends only on* k, α, *and* β *such that*

$$\max_j \sup_{t \geqq 0} |f_{1j}(t) - f_{2j}(t)| \leqq \rho \max_j \sup_{t \geqq 0} |g_{1j}(t) - g_{2j}(t)|,$$

and $[f_{1j}(t) - f_{2j}(t)] \to 0$ *as* $t \to \infty$ *for* $j = 1,2, \cdots, N$ *whenever* $[g_{1j}(t) - g_{2j}(t)] \to 0$ *as* $t \to \infty$ *for* $j = 1,2, \cdots, N$.

Comments: If the hypotheses of Theorem 3 are altered to the extent that the integrability condition on $t^p k$ is replaced with the assumption that there exists a positive constant c_1 such that $e^{c_1 t} k \in \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$, then it is possible to give a considerably simpler proof (than that of Ref. 2) of the fact that $f \in \mathfrak{L}_{\infty N}(0,\infty)$. Specifically, under the new assumptions, it can be easily verified that for any positive constant $c_2 < c_1$, there exists a positive constant c_3 such that the modulus of the jth component of

$$\int_0^y k(y-\tau)\psi[f(\tau),\tau]d\tau$$

does not exceed $c_3 e^{-c_2 y} \| e^{c_2 t} f_y \|$ for all $y > 0$ and $j = 1, 2, \cdots, N$. By arguments very similar to those of the proofs of Theorem 2 and its corollary, it can be shown that there exist positive constants ρ and c_4 such that $c_4 < c_1$ and

$$\| e^{c_4 t} f_y \| \leqq \rho \| e^{c_4 t} g_y \|$$

for all $y > 0$. Since $g \varepsilon \mathfrak{L}_{\infty N}(0, \infty)$, and for $y > 0$

$$f(y) = g(y) - \int_0^y k(y - \tau)\psi[f(\tau),\tau]d\tau$$

and

$$\| e^{c_4 t} g_y \| \leqq \left(\frac{N}{2c_4}\right)^{\frac{1}{2}} e^{c_4 y} \max_j \sup_{t \geqq 0} | g_j(t) | ,$$

it follows that $f \varepsilon \mathfrak{L}_{\infty N}(0, \infty)$. This type of approach, when coupled with the techniques of Section V, can be used to establish the $\mathfrak{L}_{\infty N}(0, \infty)$-boundedness of solutions of more general functional equations.

For results similar to Theorems 1 and 3 concerning the discrete analog of (1), see Ref. 2.

Definition: Let T be a real positive constant, and let

$$\mathfrak{D} = \{f \mid f \varepsilon \mathfrak{L}_{\infty N}(-\infty, \infty), \quad f(t) = f(t + T) \quad \text{for} \quad -\infty < t < \infty\}$$

where $\mathfrak{L}_{\infty N}(-\infty, \infty)$ is the natural extension of the space $\mathfrak{L}_{\infty N}(0, \infty)$ to N-vector-valued functions defined on the entire real line.

Theorem 4: Let $k \varepsilon \Phi(\alpha,\beta)$ *with* $t^p k \varepsilon \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$ *for* $p = 0,1,2$. *Let* $g_1 \varepsilon \mathfrak{D}$, $g_2 \varepsilon \mathfrak{L}_{\infty N}(0, \infty)$, $g_2(t) \to 0$ *as* $t \to \infty$, *and* $\psi[\cdot , \cdot] \varepsilon \Psi(\alpha,\beta)$ *with* $\psi_n(w,t) = \psi_n(w,t + T)$ *for all real* w *and* $t \geqq 0$. *Let* $f \varepsilon \mathcal{E}_N$ *satisfy*

$$g_1(t) + g_2(t) = f(t) + \int_0^t k(t - \tau)\psi[f(\tau), \tau]\, d\tau, \qquad t \geqq 0.$$

Then $\mathfrak{D}$ *contains an element* $\hat{f}$, *which does not depend on* g_2, *such that* $[f(t) - \hat{f}(t)] \to 0$ *as* $t \to \infty$.

If, in addition to the hypotheses stated above, there exist positive constants c_1, c_2, *and* c_3 *such that*

$$e^{c_1 t} k \varepsilon \mathcal{K}_{1N} \cap \mathcal{K}_{2N}\, ; \qquad \int_t^\infty | k_{mn}(x) | \, dx \leqq c_2 e^{-c_3 t}, \qquad t \geqq 0$$

for $m,n = 1,2, \cdots , N$; *and*

$$| g_{2j}(t)| \leqq c_2 e^{-c_3 t}, \qquad t \geqq 0$$

for $j = 1,2, \cdots , N$; *then there exist positive constants* c_4 *and* c_5 *such that*

$$|f_j(t) - \hat{f}_j(t)| \leqq c_4 e^{-c_5 t}, \qquad t \geqq 0$$

for $j = 1,2, \cdots , N$.

Proof of Theorem 4: Assume that the hypotheses of the first part of Theorem 4 are satisfied. Let $\psi_n(w,t)$ be defined for $t < 0$ by the condition that $\psi_n(w,t) = \psi_n(w,t + T)$ for *all* real t, all real w, and $n = 1,2, \cdots, N$. We need the following result, which is easily provable (see the proofs of Lemmas 4 and 5 of Ref. 8) with the aid of Theorem 4 of Ref. 7 and the remarks relating to its proof.

Lemma: The set $\mathfrak{D}$ contains a unique element $\hat{f}$ such that

$$g_1(t) = \hat{f}(t) + \int_{-\infty}^{t} k(t - \tau)\psi[\hat{f}(\tau), \tau]\, d\tau, \qquad -\infty < t < \infty.$$

Thus we have

$$g_1(t) - \int_{-\infty}^{0} k(t - \tau)\psi[\hat{f}(\tau), \tau]\, d\tau = \hat{f}(t) + \int_{0}^{t} k(t - \tau)\psi[\hat{f}(\tau), \tau]\, d\tau, \qquad t \geqq 0$$

$$g_1(t) + g_2(t) = f(t) + \int_{0}^{t} k(t - \tau)\psi[f(\tau), \tau]\, d\tau, \qquad t \geqq 0.$$

Since

$$g_2(t) + \int_{-\infty}^{0} k(t - \tau)\psi[\hat{f}(\tau), \tau]\, d\tau \to 0 \quad \text{as} \quad t \to \infty,$$

by Corollary 3(a), $[f(t) - \hat{f}(t)] \to 0$ as $t \to \infty$.

The second part of the theorem follows at once from Corollary 2(a) and the fact that here

$$g_2(t) + \int_{-\infty}^{0} k(t - \tau)\psi[\hat{f}(\tau), \tau]\, d\tau = [f(t) - \hat{f}(t)] + \int_{0}^{t} k(t - \tau)\{\psi[f(\tau), \tau] - \psi[\hat{f}(\tau), \tau]\}\, d\tau, \qquad t \geqq 0$$

with $\psi[\cdot\,,\cdot] \in \Psi(\alpha,\beta)$.

Comments: A result similar to the first part of Theorem 4 is proved in Ref. 8. There it is assumed that $g_2 \in \mathfrak{L}_{2N}(0, \infty)$.

Under the additional assumptions that $g_1(t)$ is a *constant* N-vector, and that $\psi_n(w,t)$ is independent of t for $n = 1,2, \cdots, N$, it can be shown that $\hat{f}(t)$ of the lemma is a *constant* N-vector, and hence that $f(t)$ of Theorem 4 approaches a limit as $t \to \infty$.

It is a simple matter to construct examples involving f's not contained in $\mathfrak{L}_{\infty N}(0, \infty)$ which illustrate that the conclusion of the first part of Theorem 4 can be false if k does not belong to $\Phi(\alpha,\beta)$ (with the understanding that the remaining hypotheses are satisfied). The following example shows that the conclusion can be false in some relatively simple situations in which $f \,\varepsilon\, \mathfrak{L}_{\infty N}(0, \infty)$, if k does not belong to $\Phi(\alpha,\beta)$.

Let $N = 1$, and for $t \geqq 0$ let

$$\begin{aligned} \psi(w,t) &= w, & -\infty < w \leqq 1 \\ &= (w)^{\frac{1}{2}}, & 1 \leqq w \leqq 9 \\ &= \tfrac{1}{6}w + \tfrac{3}{2}, & w \geqq 9. \end{aligned}$$

Let $t^p k \,\varepsilon\, \mathcal{K}_{11} \cap \mathcal{K}_{21}$ for $p = 0, 1, 2$; and let $K(0) = 0$ and $K(i) = -4$. Here $\psi[\cdot,\cdot] \,\varepsilon\, \Psi(\frac{1}{6}, 1)$ and, since $K(i)$ is a point on the real-axis diameter of the disk of Fig. 2 when $\alpha = \frac{1}{6}$ and $\beta = 1$, it is clear that k does not belong to $\Phi(\frac{1}{6}, 1)$.

For $t \geq 0$, let

$$g_1(t) = \tfrac{9}{2} - \tfrac{1}{2}\cos 2t$$

$$\begin{aligned} g_{2a}(t) = e^{-t} + \int_0^t k(t-\tau)\{\psi[\tfrac{9}{2} + 4\sin\tau - \tfrac{1}{2}\cos 2\tau + e^{-\tau},0] \\ - \psi[\tfrac{9}{2} + 4\sin\tau - \tfrac{1}{2}\cos 2\tau,0]\}d\tau \\ - \int_{-\infty}^0 k(t-\tau)\{\psi[\tfrac{9}{2} + 4\sin\tau - \tfrac{1}{2}\cos 2\tau,0]\}d\tau. \end{aligned}$$

(Observe that $g_{2a}(t)$ is uniformly bounded on $[0, \infty)$ and that $g_{2a}(t) \to 0$ as $t \to \infty$.) Then, using the identity $(2 + \sin t) = (\frac{9}{2} + 4\sin t - \frac{1}{2}\cos 2t)^{\frac{1}{2}}$ which is valid for all real t, it can be verified that $f_1(t) = \frac{9}{2} + 4\sin t - \frac{1}{2}\cos 2t + e^{-t}$ satisfies

$$g_1(t) + g_{2a}(t) = f_1(t) + \int_0^t k(t-\tau)\psi[f_1(\tau),0]d\tau, \qquad t \geqq 0.$$

Note that although f_1 is ultimately periodic, it contains a component of one half the frequency of g_1.

At this point it is convenient to comment on the necessity of the hypotheses of Corollaries 1(a) and 3(a). Let g_1, f_1, k, and ψ be as defined in the preceding two paragraphs, and assume that

$$\int_t^\infty |k(\tau)|\, d\tau \,\varepsilon\, \mathfrak{L}_2(0, \infty).$$

For $t \geqq 0$, let

$$g_{2b}(t) = e^{-t} + \int_0^t k(t-\tau)\{\psi[\tfrac{9}{2} - 4\sin\tau - \tfrac{1}{2}\cos 2\tau + e^{-\tau},0] - \psi[\tfrac{9}{2} - 4\sin\tau - \tfrac{1}{2}\cos 2\tau,0]\}d\tau - \int_{-\infty}^0 k(t-\tau)\psi[\tfrac{9}{2} - 4\sin\tau - \tfrac{1}{2}\cos 2\tau,0]d\tau.$$

Then, using the identity mentioned above, it can be verified that $f_2(t) = \frac{9}{2} - 4\sin t - \frac{1}{2}\cos 2t + e^{-t}$ satisfies

$$g_1(t) + g_{2b}(t) = f_2(t) + \int_0^t k(t-\tau)\psi[f_2(\tau),0]d\tau, \qquad t \geqq 0.$$

Although $[g_{2a}(t) - g_{2b}(t)]$ approaches zero at infinity and belongs to both $\mathcal{L}_{\infty 1}(0,\infty)$ and $\mathcal{L}_2(0,\infty)$, it is obvious that $[f_1(t) - f_2(t)]$ (i.e., $8\sin t$) does not approach zero at infinity and does not belong to $\mathcal{L}_2(0,\infty)$.

IV. SUFFICIENT CONDITIONS FOR THE $\mathcal{L}_2$-BOUNDEDNESS OF THE SOLUTIONS OF (2)

Let f^n denote df^n/dt^n for $n = 1,2,\cdots$, and let $f^0 = f$.

Theorem 5: Let $\psi[\cdot\,,\cdot] \in \Psi_0(\alpha,\beta)$ *with* $N = 1$, $g \in \mathcal{L}_2(0,\infty)$, $k \in \mathcal{K}_{11}$, *and, with* n *a nonnegative integer,*

$$f^n(t) = g(t) + a\psi[f(t),t] + \int_0^t k(t-\tau)\psi[f(\tau),\tau]\,d\tau, \qquad t \geqq 0$$

where $f \in \mathcal{H}_1(0,\infty)$, $f^n \in \mathcal{E}_1$, *and* a *is a real constant. Suppose that, with*

$$K(s) = a + \int_0^\infty k(t)e^{-st}\,dt \quad for \quad \sigma \geqq 0,$$

(*i*) $s^n - \frac{1}{2}(\alpha+\beta)K(s) \neq 0$ for $\sigma \geqq 0$ $(s^0 = 1)$, *and*

$$1 - \tfrac{1}{2}(\alpha+\beta)a \neq 0 \quad \text{if} \quad n = 0$$

(*ii*) $\frac{1}{2}(\beta-\alpha)\sup_\omega |[(i\omega)^n - \frac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}K(i\omega)| < 1.$

Then

$f \in \mathcal{L}_2(0,\infty)$; *and* $\|f\| \leqq (1-\rho_0)^{-1}\rho_1\|g\| + (1-\rho_0)^{-1}\rho_2$, *where*

$$\rho_0 = \tfrac{1}{2}(\beta-\alpha)\sup_\omega |[(i\omega)^n - \tfrac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}K(i\omega)|$$

$$\rho_1 = \sup_\omega |[(i\omega)^n - \tfrac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}| \qquad [(i\omega)^0 = 1]$$

and

$$\rho_2 = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|[(i\omega)^n - \tfrac{1}{2}(\alpha+\beta)K(i\omega)]^{-1} \cdot \sum_{j=0}^{(n-1)} (i\omega)^j f^{(n-1-j)}(0)\right|^2 d\omega\right]^{\frac{1}{2}}$$

for $n > 0$; *for* $n = 0$, $\rho_2 = 0$.

Proof: Assume that the hypotheses of the theorem are satisfied, and let y be an arbitrary positive number. From the fact that

$$f^n(t) = g(t) + a\psi[f(t), t] + \int_0^t k(t-\tau)\psi[f(\tau), \tau]\, d\tau, \qquad t \geqq 0$$

we have for $-\infty < \omega < \infty$

$$\int_0^y e^{-i\omega t} f^n(t)\, dt = \int_0^y e^{-i\omega t} g(t)\, dt + a\int_0^y e^{-i\omega t}\psi[f_y(t), t]\, dt + \int_0^y e^{-i\omega t}\int_0^t k(t-\tau)\psi[f_y(\tau), \tau]\, d\tau$$

in which

$$\begin{aligned} f_y(t) &= f(t) \quad \text{for} \quad t \,\varepsilon\, [0,y] \\ &= 0 \qquad \text{for} \quad t > y. \end{aligned}$$

For $-\infty < \omega < \infty$, let

$$F_y = \int_0^y e^{-i\omega t} f_y(t)\, dt$$

$$G_y = \int_0^y e^{-i\omega t} g_y(t)\, dt$$

$$H_y = \int_0^y e^{-i\omega t}\psi[f_y(t), t]\, dt$$

$$X = a\int_y^\infty e^{-i\omega t}\psi[f_y(t), t]\, dt + \int_y^\infty e^{-i\omega t}\int_0^t k(t-\tau)\psi[f_y(\tau), \tau]\, d\tau$$

$$U = \sum_{j=0}^{(n-1)} (i\omega)^j f^{(n-1-j)}(y), \qquad n > 0$$

$$V = \sum_{j=0}^{(n-1)} (i\omega)^j f^{(n-1-j)}(0), \qquad n > 0$$

$$U = V = 0,\ n = 0.$$

Then, using the fact that

$$\int_0^y e^{-i\omega t} f^n(t)\, dt = (i\omega)^n F_y + e^{-i\omega y} U - V$$

for $n \geqq 0$ and $-\infty < \omega < \infty$, we have

$$(i\omega)^n F_y + e^{-i\omega y} U - V = G_y + K(i\omega) H_y - X$$

for $n \geqq 0$ and $-\infty < \omega < \infty$. It follows that

$$\begin{aligned} F_y + [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}[X + e^{-i\omega y}U] \\ = [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)[H_y - \tfrac{1}{2}(\alpha + \beta)F_y] \quad (5) \\ + [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}[G_y + V] \end{aligned}$$

for $n \geqq 0$ and $-\infty < \omega < \infty$.

Observe that $[(i\omega)^n - \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}$ is uniformly bounded for $\omega \, \varepsilon \, (-\infty, \infty)$, and that $[(i\omega)^n - \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}U$ is square integrable on the ω-set $(-\infty, \infty)$. Thus, since $\psi[f_y(t), t] \, \varepsilon \, \mathfrak{L}_2(0, \infty)$,

$$[(i\omega)^n - (\alpha + \beta)K(i\omega)]^{-1}[X + e^{-i\omega y}U]$$

is the Fourier transform of a square-integrable function $\hat{f}_y$. Further, since both

$$[s^n - \tfrac{1}{2}(\alpha + \beta)K(s)]^{-1} \quad \text{and} \quad [s^n - \tfrac{1}{2}(\alpha + \beta)K(s)]^{-1} \sum_{j=0}^{(n-1)} s^j f^{(n-1-j)}(y)$$

are analytic and uniformly bounded for $\sigma > 0$, it follows[9,10] that $\hat{f}_y(t) = 0$ almost everywhere on $(-\infty, y)$.

Using (5), Parseval's identity, and Minkowski's inequality,

$$\begin{aligned} &\left(\int_0^y | f_y(t) |^2 \, dt\right)^{\frac{1}{2}} \\ &\leqq \left(\int_0^\infty | f_y(t) + \hat{f}_y(t) |^2 \, dt\right)^{\frac{1}{2}} \\ &\leqq \left(\frac{1}{2\pi} \int_{-\infty}^\infty | [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1} K(i\omega) \right. \\ &\qquad \left. \cdot [H_y - \tfrac{1}{2}(\alpha + \beta)F_y] |^2 \, d\omega\right)^{\frac{1}{2}} \\ &+ \left(\frac{1}{2\pi} \int_{-\infty}^\infty | [(i\omega)^n - (\alpha + \beta)K(i\omega)]^{-1}[G_y + V] |^2 \, d\omega\right)^{\frac{1}{2}} \end{aligned}$$

$$\leqq \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} | [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega) \cdot [H_y - \tfrac{1}{2}(\alpha + \beta)F_y] |^2 \, d\omega\right)^{\frac{1}{2}}$$

$$+ \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} | [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}G_y |^2 \, d\omega\right)^{\frac{1}{2}}$$

$$+ \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \left| [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1} \cdot \sum_{j=0}^{(n-1)} (i\omega)^j f^{(n-1-j)}(0) \right|^2 d\omega\right)^{\frac{1}{2}}$$

with the understanding that the last integral vanishes if $n = 0$.

Thus

$$\left(\int_0^y | f_y(t) |^2 \, dt\right)^{\frac{1}{2}} \leqq \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} | [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega) \cdot [H_y - \tfrac{1}{2}(\alpha + \beta)F_y] |^2 \, d\omega\right)^{\frac{1}{2}} + \sup_{\omega} | [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1} | \left(\int_0^{\infty} | g(t) |^2 \, dt\right)^{\frac{1}{2}} + \rho_2$$

where

$$\rho_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} | [(i\omega)^n - \tfrac{1}{2}(\alpha + \beta)K(i\omega)]^{-1} \cdot \sum_{j=0}^{(n-1)} (i\omega)^j f^{(n-1-j)}(0) |^2 \, d\omega\right)^{\frac{1}{2}}$$

if $n > 0$ and $\rho_2 = 0$ if $n = 0$.

Since $| x^{-1}\psi(x,t) - \frac{1}{2}(\alpha + \beta)| \leqq \frac{1}{2}(\beta - \alpha)$ for all real $x \neq 0$ and $t \geqq 0$, we have

$$\left(\int_{-\infty}^{\infty} | H_y(i\omega) - \tfrac{1}{2}(\alpha - \beta)F_y(i\omega) |^2 \, d\omega\right)^{\frac{1}{2}} = \left(2\pi \int_0^y | \psi[f_y(t), t] - \tfrac{1}{2}(\alpha + \beta)f_y(t) |^2 \, dt\right)^{\frac{1}{2}} \leqq \tfrac{1}{2}(\beta - \alpha) \left(2\pi \int_0^y | f_y(t) |^2 \, dt\right)^{\frac{1}{2}}.$$

It follows that

$$\left(\int_0^y |f_y(t)|^2\,dt\right)^{\frac{1}{2}} \leqq (1-\rho_0)^{-1} \sup_\omega |[(i\omega)^n - \tfrac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}| \times \left(\int_0^\infty |g(t)|^2\,dt\right)^{\frac{1}{2}} + (1-\rho_0)^{-1}\rho_2 \tag{6}$$

in which

$$\rho_0 = \tfrac{1}{2}(\beta-\alpha) \sup_\omega |[(i\omega)^n - \tfrac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}K(i\omega)|$$

(recall that $\rho_0 < 1$ by assumption). Thus, since ρ_2 and

$$\sup_\omega |[(i\omega)^n - \tfrac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}|$$

are finite, and (6) is valid for all $y > 0$, it follows that $f \varepsilon \mathcal{L}_2(0,\infty)$, and that

$$\|f\| \leqq (1-\rho_0)^{-1} \sup_\omega |[(i\omega)^n - \tfrac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}|\cdot\|g\| + (1-\rho_0)^{-1}\rho_2 .$$

Comment: With regard to the hypothesis concerning f'' of Theorem 5, observe that if $f \varepsilon \mathcal{E}_1$, and if f'' exists and satisfies the integral equation, with g, a, k, and $\psi[\cdot\,,\cdot]$ as defined, then $f'' \varepsilon \mathcal{E}_1$.

V. SOME RESULTS ON THE PROPERTIES OF EQUATIONS DEFINED ON AN ABSTRACT SPACE

5.1 *Definitions and Notation*

Let $\mathcal{K}$ denote an abstract linear space that contains a normed linear space $\mathcal{L}$ with norm $\|\cdot\|$.

Let Ω denote a set of real numbers, and let P_y denote a linear mapping of $\mathcal{K}$ into $\mathcal{L}$ for each $y \varepsilon \Omega$.

Let $g \varepsilon \mathcal{L}$ if and only if $g \varepsilon \mathcal{K}$ and $\|P_y g\|$ is uniformly bounded on Ω.

The norm of a linear transformation A defined on $\mathcal{L}$ is denoted by $\|A\|$, and I denotes the identity operator on $\mathcal{K}$.

We shall say that a (not necessarily linear) operator T is an element of the set Θ if and only if T maps $\mathcal{K}$ into itself and $P_yT = P_yTP_y$ on $\mathcal{K}$ for $y \varepsilon \Omega$.†

† As a concrete example, we note that the development of Ref. 1 is concerned with the complex-valued-function version of the case in which $\mathcal{L} = \mathcal{L}_{2N}(0,\infty)$, $\mathcal{K} = \mathcal{E}_N$, $\Omega = [0,\infty)$, and P_y is defined by: $(P_yf)(t) = f(t)$ for $t \varepsilon [0,y]$ and $(P_yf)(t) = 0$ for $t > y$ where f is an arbitrary element of $\mathcal{E}_N$. With that definition of P_y, an operator belongs to Θ only if it is "causal."

As a matter of convenience, we shall let g_y denote $P_y g$ for $g \varepsilon \mathcal{K}$ and $y \varepsilon \Omega$. Thus, for example, if $f \varepsilon \mathcal{K}$ and $T \varepsilon \Theta$, then $(Tf_y)_y$ denotes $P_y T P_y f$ for $y \varepsilon \Omega$.

5.2 *Results of the Type Presented in Ref. 1*

Our first two observations, which are stated as Theorems 6 and 7, are instructive.

Theorem 6: Let $T \varepsilon \Theta$; $f_1, f_2 \varepsilon \mathcal{K}$; and

$$g_1 = f_1 + Tf_1$$
$$g_2 = f_2 + Tf_2 .$$

Suppose that there exists a positive constant $k < 1$ such that

$$\| (Tf_{1y})_y - (Tf_{2y})_y \| \leqq k \| f_{1y} - f_{2y} \|, \qquad y \varepsilon \Omega.$$

Then

(*i*) $\| f_{1y} - f_{2y} \| \leqq (1 - k)^{-1} \| g_{1y} - g_{2y} \|, \qquad y \varepsilon \Omega$
(*ii*) $(f_1 - f_2) \varepsilon \mathfrak{L}$ *provided that* $(g_1 - g_2) \varepsilon \mathfrak{L}$.

Proof of Theorem 6: For $y \varepsilon \Omega$,

$$\begin{aligned} f_{1y} - f_{2y} &= g_{1y} - g_{2y} - P_y[Tf_1 - Tf_2] \\ &= g_{1y} - g_{2y} - P_y[Tf_{1y} - Tf_{2y}]. \end{aligned}$$

Thus

$$\begin{aligned} \| f_{1y} - f_{2y} \| &\leqq \| g_{1y} - g_{2y} \| + \| (Tf_{1y})_y - (Tf_{2y})_y \| \\ &\leqq \| g_{1y} - g_{2y} \| + k \| f_{1y} - f_{2y} \| \end{aligned}$$

for $y \varepsilon \Omega$. Hence

$$\| f_{1y} - f_{2y} \| \leqq (1 - k)^{-1} \| g_{1y} - g_{2y} \|, \qquad y \varepsilon \Omega.$$

If $(g_1 - g_2) \varepsilon \mathfrak{L}$, then

$$\sup_{\Omega} \| f_{1y} - f_{2y} \| \leqq (1 - k)^{-1} \sup_{\Omega} \| g_{1y} - g_{2y} \| < \infty,$$

and hence $(f_1 - f_2) \varepsilon \mathfrak{L}$. This proves the theorem. A similar argument establishes

Theorem 7: Let $T \varepsilon \Theta$; $f \varepsilon \mathcal{K}$; and

$$g = f + Tf.$$

Suppose that there exists a positive constant $k < 1$ *such that*

$$\| (Tf_y)_y \| \leqq k \| f_y \|, \qquad y \in \Omega.$$

Then

(*i*) $\| f_y \| \leqq (1 - k)^{-1} \| g_y \|, \qquad y \in \Omega$

(*ii*) $f \in \mathfrak{L}$ *provided that* $g \in \mathfrak{L}$.

Our next result is quite useful. For example, it leads to Theorem 1.

Theorem 8: Let P_y *be such that* $\| P_y h \| \leqq \| h \|$ *and* $P_y P_y h = P_y h$ *for* $h \in \mathfrak{L}$ *and* $y \in \Omega$. *Let* $L,N \in \Theta$, *with* L *linear and* L *mapping* $\mathfrak{L}$ *into itself. Let* $f \in \mathcal{K}$, *and*

$$g = f + LNf.$$

Suppose that there exists a scalar λ (λ *real if* $\mathfrak{L}$ *is a real space*) *such that*

(*i*) $(I + \lambda L)^{-1}$ *exists on* $\mathfrak{L}$, *and*

$$P_y(I + \lambda L)^{-1} = P_y(I + \lambda L)^{-1}P_y$$

on $\mathfrak{L}$ *for* $y \in \Omega$

(*ii*) $\| (I + \lambda L)^{-1} L \| < \infty$

(*iii*) *there exists a positive constant* k_λ *with the property that*

$$\| (Nf_y)_y - \lambda f_y \| \leqq k_\lambda \| f_y \| \quad \textit{for} \quad y \in \Omega$$

(*iv*) $\| (I + \lambda L)^{-1} L \| k_\lambda < 1.$

Then

$$\| f_y \| \leqq (1 - r)^{-1} \| P_y(I + \lambda L)^{-1} g_y \|, \qquad y \in \Omega.$$

in which

$$r = \| (I + \lambda L)^{-1} L \| k_\lambda .$$

Corollary 8(a): Suppose that the hypotheses of Theorem 8 are satisfied and that $g \in \mathfrak{L}$. *Then* $f \in \mathfrak{L}$.

Proof of Theorem 8: Let $y \in \Omega$. From the fact that $g = f + LNf$, we clearly have

$$g_y = f_y + P_y LNf.$$

Since L and N belong to Θ,

$$\begin{aligned} g_y &= f_y + P_y L P_y N f \\ &= f_y + P_y L P_y N f_y . \end{aligned}$$

Thus

$$g_y = P_y(I + \lambda L)f_y + P_yLP_y(Nf_y - \lambda f_y).$$

Since $(I + \lambda L)^{-1}$ exists on $\mathfrak{L}$ and $P_y(I + \lambda L)^{-1} = P_y(I + \lambda L)^{-1}P_y$, we have

$$P_y(I + \lambda L)^{-1}P_y(I + \lambda L)f_y = f_y,$$

and hence

$$f_y = -P_y(I + \lambda L)^{-1}P_yLP_y(Nf_y - \lambda f_y) + P_y(I + \lambda L)^{-1}g_y .$$

Using the fact that

$$\| P_y(I + \lambda L)^{-1}P_yL \| = \| P_y(I + \lambda L)^{-1}L \|$$
$$\leqq \| P_y \| \cdot \| (I + \lambda L)^{-1}L \| \leqq \| (I + \lambda L)^{-1}L \|,$$

it follows that

$$\| f_y \| \leqq \| P_y(I + \lambda L)^{-1}P_yL \| \cdot \| (Nf_y)_y - \lambda f_y \| + \| P_y(I + \lambda L)^{-1}g_y \|$$
$$\leqq \| (I + \lambda L)^{-1}L \| k_\lambda \| f_y \| + \| P_y(I + \lambda L)^{-1}g_y \|.$$

Therefore

$$\| f_y \| \leqq (1 - r)^{-1} \| P_y(I + \lambda L)^{-1}g_y \|, \qquad y \,\varepsilon\, \Omega$$

in which

$$r = \| (I + \lambda L)^{-1}L \| k_\lambda .$$

This proves the theorem.

Proof of Corollary 8(a): Assume that the hypotheses of the corollary are satisfied. Then

$$\sup_{y \varepsilon \Omega} \| f_y \| \leqq (1 - r)^{-1} \sup_{y \varepsilon \Omega} \| P_y(I + \lambda L)^{-1}g_y \|$$
$$\leqq (1 - r)^{-1} \sup_{y \varepsilon \Omega} \| P_y(I + \lambda L)^{-1}g \| < \infty$$

and hence $f \,\varepsilon\, \mathfrak{L}$.

Arguments similar to those of the proof of Theorem 8 and its corollary establish the following theorem and corollary.

Theorem 9: Let P_y be such that $\| P_yh \| \leqq \| h \|$ and $P_yP_yh = P_yh$ for $h \,\varepsilon\, \mathfrak{L}$ and $y \,\varepsilon\, \Omega$. Let $L,N \,\varepsilon\, \Theta$; with L linear and L mapping $\mathfrak{L}$ into itself. Let f_1, $f_2 \,\varepsilon\, \mathfrak{K}$; and

$$g_1 = f_1 + LNf_1$$
$$g_2 = f_2 + LNf_2 .$$

Suppose that there exists a scalar λ (λ *real if* $\mathfrak{L}$ *is a real space*) *such that*

(*i*) $(I + \lambda L)^{-1}$ *exists on* $\mathfrak{L}$, *and* $P_y(I + \lambda L)^{-1} = P_y(I + \lambda L)^{-1}P_y$ *on* $\mathfrak{L}$ *for* $y \varepsilon \Omega$

(*ii*) $\| (I + \lambda L)^{-1} L \| < \infty$

(*iii*) *there exists a positive constant* k_λ *with the property that*

$$\| (Nf_{1y})_y - (Nf_{2y})_y - \lambda(f_{1y} - f_{2y}) \| \leqq k_\lambda \| f_{1y} - f_{2y} \|$$

for $y \varepsilon \Omega$

(*iv*) $\| (I + \lambda L)^{-1} L \| k_\lambda < 1$.

Then

$$\| f_{1y} - f_{2y} \| \leqq (1 - r)^{-1} \| P_y(I + \lambda L)^{-1}(g_{1y} - g_{2y}) \|, \qquad y \varepsilon \Omega$$

in which

$$r = \| (I + \lambda L)^{-1} L \| k_\lambda .$$

Corollary 9(a): Suppose that the hypotheses of Theorem 9 are satisfied and that $(g_1 - g_2) \varepsilon \mathfrak{L}$. *Then* $(f_1 - f_2) \varepsilon \mathfrak{L}$.

5.3 *Results for the Case in Which* $\mathfrak{L}$ *is an Inner-Product Space*

In this section we employ the definitions and notation of Section 5.1. It is further assumed here that $\mathfrak{L}$ is an inner-product space, with inner product $\langle \cdot , \cdot \rangle$. The norm of $f \varepsilon \mathfrak{L}$ is $\| f \| = \langle f, f \rangle^{\frac{1}{2}}$.

It is also assumed throughout this section that A and B are elements of Θ.

Lemma 1: Let $f \varepsilon \mathcal{K}$, $h = Bf$, *and* $g = f + Ah$. *Then*

$$| Re \langle (Bf_y)_y , f_y \rangle + Re \langle (Ah_y)_y , h_y \rangle | \leqq \| g_y \| \cdot \| h_y \|$$

for $y \varepsilon \Omega$.

Proof: Since A and B belong to Θ, we have

$$h_y = (Bf_y)_y \quad \text{and} \quad g_y = f_y + (Ah_y)_y \quad \text{for} \quad y \varepsilon \Omega.$$

Thus, for $y \varepsilon \Omega$,

$$\begin{aligned} \text{Re} \langle (Bf_y)_y , f_y \rangle + \text{Re} \langle (Ah_y)_y , h_y \rangle &= \text{Re} \langle f_y , (Bf_y)_y \rangle + \text{Re} \langle (Ah_y)_y , h_y \rangle \\ &= \text{Re} \langle f_y + (Ah_y)_y , h_y \rangle = \text{Re} \langle g_y , h_y \rangle \end{aligned}$$

and so, by the Schwarz inequality,

$$| \text{Re} \langle (Bf_y)_y , f_y \rangle + \text{Re} \langle (Ah_y)_y , h_y \rangle | \leqq \| g_y \| \cdot \| h_y \|$$

for $y \varepsilon \Omega$.

Remarks: An application of the Schwarz inequality similar to that of the proof of Lemma 1 shows that if $g = f + Af$ with $f \varepsilon \mathcal{K}$ and if there exists a constant $\delta > 0$ such that

$$| \, \| f_y \|^2 + \langle (Af_y)_y , f_y \rangle \, | \geqq \delta \, \| f_y \|^2, \qquad y \varepsilon \Omega$$

then $\| f_y \| \leqq \delta^{-1} \| g_y \|$ for $y \varepsilon \Omega$ and thus then $f \varepsilon \mathfrak{L}$ provided that $g \varepsilon \mathfrak{L}$. Observe that for $\mathfrak{L}$ an inner-product space, this is a stronger result than that of Theorem 7.

Theorem 10: Let $g \varepsilon \mathfrak{L}$, $f \varepsilon \mathcal{K}$, and $g = f + ABf$. Let

$$Re \langle (Aq_y)_y , q_y \rangle \geqq 0 \quad \text{and} \quad Re \langle (Bq_y)_y , q_y \rangle \geqq 0$$

for $q \varepsilon \mathcal{K}$ and $y \varepsilon \Omega$. Then $f \varepsilon \mathfrak{L}$ provided that at least one of the following conditions is satisfied:

(i) A maps $\mathfrak{L}$ into $\mathfrak{L}$, and there exist constants $\alpha > 0$ and $p > 1$ such that $Re \langle (Aq_y)_y , q_y \rangle \geqq \alpha \| q_y \|^p$ for $q \varepsilon \mathcal{K}$ and $y \varepsilon \Omega$

(ii) there exist constants $k > 0$, $\alpha > 0$, and $p > 1$ such that $\| (Bq_y)_y \| \leqq k \|q_y \|$ and $Re \langle (Bq_y)_y , q_y \rangle \geqq \alpha \| q_y \|^p$ for $q \varepsilon \mathcal{K}$ and $y \varepsilon \Omega$

(iii) there exist constants $k_1 > 0$, $k_2 > 0$, and $p > 1$ such that $Re \langle (Aq_y)_y , q_y \rangle \geqq k_1 \| (Aq_y)_y \|^p$ and $\| (Bq_y)_y \| \leqq k_2 \| q_y \|$ for $q \varepsilon \mathcal{K}$ and $y \varepsilon \Omega$

(iv) A maps $\mathfrak{L}$ into $\mathfrak{L}$, and there exist constants $k > 0$ and $p > 1$ such that $Re \langle (Bq_y)_y , q_y \rangle \geqq k \| (Bq_y)_y \|^p$ for $q \varepsilon \mathcal{K}$ and $y \varepsilon \Omega$

Proof of Theorem 10: Let $h = Bf$. Suppose that condition (i) is met Then, using Lemma 1,

$$\alpha \| h_y \|^p \leqq \text{Re} \langle (Ah_y)_y , h_y \rangle \leqq \| g_y \| \cdot \| h_y \|, \qquad y \varepsilon \Omega$$

from which

$$\| h_y \| \leqq (\alpha^{-1} \| g_y \|)^{(p-1)^{-1}}, \qquad y \varepsilon \Omega.$$

Thus

$$\sup_{\Omega} \| h_y \| \leqq (\alpha^{-1})^{(p-1)^{-1}} (\sup_{\Omega} \| g_y \|)^{(p-1)^{-1}} < \infty$$

and hence $h \varepsilon \mathfrak{L}$. Since A is assumed to map $\mathfrak{L}$ into itself, $Ah \varepsilon \mathfrak{L}$ and since $\mathfrak{L}$ is a linear space, $f = (g - Ah) \varepsilon \mathfrak{L}$.

Suppose that condition (ii) is satisfied. Then, using Lemma 1,

$$\alpha \| f_y \|^p \leqq \text{Re} \langle (Bf_y)_y , f_y \rangle \leqq \| g_y \| \cdot \| h_y \| \leqq k \| g_y \| \cdot \| f_y \|$$

for $y \varepsilon \Omega$, and so

$$\| f_y \| \leqq (k/\alpha)^{(p-1)^{-1}} (\| g_y \|)^{(p-1)^{-1}}, \qquad y \varepsilon \Omega.$$

Since $\sup_{\Omega} \| g_y \| < \infty, f \varepsilon \mathcal{L}$.

Assume now that condition (*iii*) is met. Then, using the lemma,

$$k_1 \| (Ah_y)_y \|^p \leqq \operatorname{Re} \langle (Ah_y)_y , h_y \rangle \leqq \| g_y \| \cdot \| h_y \|$$

for $y \varepsilon \Omega$. Thus

$$k_1 \| g_y - f_y \|^p \leqq k_2 \| g_y \| \cdot \| f_y \|, \qquad y \varepsilon \Omega.$$

It follows that $\sup_{\Omega} \| f_y \| < \infty$.

Finally, assume that condition (*iv*) is satisfied. Then, using the lemma,

$$k \| h_y \|^p \leqq \operatorname{Re} \langle (Bf_y)_y , f_y \rangle \leqq \| g_y \| \cdot \| h_y \|$$

for $y \varepsilon \Omega$. Therefore

$$\| h_y \| \leqq (k^{-1})^{(p-1)^{-1}} (\| g_y \|)^{(p-1)^{-1}},$$

from which it is clear that $h \varepsilon \mathcal{L}$. Since A is assumed to map $\mathcal{L}$ into $\mathcal{L}$, and $\mathcal{L}$ is a linear space, $f \varepsilon \mathcal{L}$.

Remarks: The requirement of conditions (*i*) and (*iv*) that A map $\mathcal{L}$ into itself can be replaced with the condition that

$$\sup_{\Omega} \operatorname{Re} \langle (Bq_y)_y , q_y \rangle = +\infty \quad \text{for} \quad q \varepsilon (\mathcal{K} - \mathcal{L})$$

which (through Lemma 1) implies that $f \varepsilon \mathcal{L}$ whenever $h \varepsilon \mathcal{L}$ and

$$\operatorname{Re} \langle (Aq_y)_y , q_y \rangle \geqq 0 \quad \text{and} \quad \operatorname{Re} \langle (Bq_y)_y , q_y \rangle \geqq 0$$

for $q \varepsilon \mathcal{K}$ and $y \varepsilon \Omega$.

It is possible to extend Theorem 10 in many other directions. For example, an argument essentially the same as that used to establish the sufficiency of condition (*iii*) of Theorem 10 shows that if $g \varepsilon \mathcal{L}, f \varepsilon \mathcal{K}$, and $g = f + ABf$, then $f \varepsilon \mathcal{L}$ provided that (a) $\operatorname{Re} \langle (Bq_y)_y , q_y \rangle \geqq 0$ for $q \varepsilon \mathcal{K}$ and $y \varepsilon \Omega$ and (b) there exist constants $k_1 > 0$, $k_2 > 0$, $k_3 > 0$, and $p > 1$ such that

$$\operatorname{Re} \langle (Aq_y)_y , q_y \rangle \geqq k_1 \| (Aq_y)_y \|^p - k_3 \| (Aq_y)_y \| - k_3$$

and $\| (Bq_y)_y \| \leqq k_2 \| q_y \|$ for $q \varepsilon \mathcal{K}$ and $y \varepsilon \Omega$.

For some earlier material related to Theorem 10, see Ref. 1, Theorem 3, and the observations of Ref. 7 concerning the relation between passivity and conditions for certain nonlinear operators to be contraction mappings.

The following lemma and theorem can be proved with essentially the same arguments used in the proof of Lemma 1 and Theorem 10.

Lemma 2: Let $f_1, f_2 \varepsilon \mathcal{K}$; $h_1 = Bf_1$, $h_2 = Bf_2$; *and*

$$g_1 = f_1 + Ah_1$$
$$g_2 = f_2 + Ah_2$$

Then

$$| Re \langle (Bf_{1y} - Bf_{2y})_y, f_{1y} - f_{2y} \rangle + Re \langle (Ah_{1y} - Ah_{2y})_y, h_{1y} - h_{2y} \rangle | \leqq \| g_{1y} - g_{2y} \| \cdot \| h_{1y} - h_{2y} \|$$

for $y \varepsilon \Omega$.

Theorem 11: Let $f_1, f_2 \varepsilon \mathcal{K}$; *and let*

$$g_1 = f_1 + ABf_1$$
$$g_2 = f_2 + ABf_2$$

with $(g_1 - g_2) \varepsilon \mathfrak{L}$. *Let*

$$Re \langle (Aq_{1y} - Aq_{2y})_y, q_{1y} - q_{2y} \rangle \geqq 0$$

and

$$Re \langle (Bq_{1y} - Bq_{2y})_y, q_{1y} - q_{2y} \rangle \geqq 0$$

for $q_1, q_2 \varepsilon \mathcal{K}$ *and* $y \varepsilon \Omega$. *Then* $(f_1 - f_2) \varepsilon \mathcal{K}$ *provided that at least one of the following conditions is satisfied.*

(*i*) $(Aq_1 - Aq_2) \varepsilon \mathfrak{L}$ *whenever* $q_1, q_2 \varepsilon \mathcal{K}$ *and* $(q_1 - q_2) \varepsilon \mathfrak{L}$; *and there exist constants* $\alpha > 0$ *and* $p > 1$ *such that* $Re \langle (Aq_{1y} - Aq_{2y})_y, q_{1y} - q_{2y} \rangle \geqq \alpha \| q_{1y} - q_{2y} \|^p$ *for* $q_1, q_2 \varepsilon \mathcal{K}$ *and* $y \varepsilon \Omega$.

(*ii*) *there exist constants* $k > 0$, $\alpha > 0$, *and* $p > 1$ *such that*

$$\| (Bq_{1y} - Bq_{2y})_y \| \leqq k \| q_{1y} - q_{2y} \|$$

and $Re \langle (Bq_{1y} - Bq_{2y})_y, q_{1y} - q_{2y} \rangle \geqq \alpha \| q_{1y} - q_{2y} \|^p$ *for* $q_1, q_2 \varepsilon \mathcal{K}$ *and* $y \varepsilon \Omega$.

(*iii*) *there exist constants* $k_1 > 0$, $k_2 > 0$, *and* $p > 1$ *such that* $Re \langle (Aq_{1y} - Aq_{2y})_y, q_{1y} - q_{2y} \rangle \geqq k_1 \| (Aq_{1y} - Aq_{2y})_y \|^p$ *and*

$$\| (Bq_{1y} - Bq_{2y})_y \| \leqq k_2 \| q_{1y} - q_{2y} \|$$

for $q \varepsilon \mathcal{K}$ *and* $y \varepsilon \Omega$.

(*iv*) $(Aq_1 - Aq_2) \varepsilon \mathfrak{L}$ *whenever* $q_1, q_2 \varepsilon \mathcal{K}$ *and* $(q_1 - q_2) \varepsilon \mathfrak{L}$; *and there exist constants* $k > 0$ *and* $p > 1$ *such that* $Re \langle (Bq_{1y} - Bq_{2y}), q_{1y} - q_{2y} \rangle \geqq k \| (Bq_{1y} - Bq_{2y})_y \|^p$ *for* $q_1, q_2 \varepsilon \mathcal{K}$ *and* $y \varepsilon \Omega$.

Results similar to Theorems 10 and 11 can be established for the equation

$$g = Af + Bf.$$

In particular, we can very easily prove

Lemma 3: Let $f \in \mathcal{K}$, *and* $g = Af + Bf$. *Then*

$$| Re \langle (Bf_y)_y , f_y \rangle + Re \langle (Af_y)_y , f_y \rangle | \leqq \| g_y \| \cdot \| f_y \|$$

for $y \in \Omega$.

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APPENDIX

Some Results Related to Theorem 2 and Corollary 2(a)

Let $\underline{\mathcal{K}}_{1N}$, $\underline{\mathcal{K}}_{2N}$, $\underline{\mathcal{E}}_N$ and $\underline{\mathcal{L}}_{2N}(0,\infty)$ denote the natural complex extensions of the real sets $\mathcal{K}_{1N}$, $\mathcal{K}_{2N}$, $\mathcal{E}_N$ and $\mathcal{L}_{2N}(0,\infty)$, respectively.

Using arguments very similar to those of the proofs of Theorem 2, Corollary 2(a), and the lemma of Ref. 11, it is not difficult to prove the following theorem and corollary.

Theorem 12: Let $Q(\cdot)$ *denote a complex measurable* $N \times N$ *matrix-valued function of* t *defined on* $[0,\infty)$, *and let the elements of* $Q(t)$ *be uniformly bounded on* $[0,\infty)$. *Let*

$$g(t) = f(t) + \int_0^t k(t-\tau)Q(\tau)f(\tau)\,d\tau, \qquad t \geqq 0 \tag{1}$$

in which $f \in \underline{\mathcal{E}}_N$, *and there exists a real constant* c_1 *such that*

(*i*) $ge^{c_1 t} \in \underline{\mathcal{L}}_{2N}(0,\infty)$
(*ii*) $ke^{c_1 t} \in \underline{\mathcal{K}}_{1N} \cap \underline{\mathcal{K}}_{2N}$
(*iii*) *with*

$$K(i\omega - c_1) = \int_0^\infty k(t)e^{(c_1 - i\omega)t}\,dt \quad \text{for} \quad -\infty < \omega < \infty,$$

$$\sup_{t \geqq 0} \Lambda\{Q(t)\} \sup_{-\infty<\omega<\infty} \Lambda\{K(i\omega - c_1)\} < 1.$$

Then there exists a positive constant c_2 *such that*

$$|f_j(t)| \leqq |g_j(t)| + c_2 e^{-c_1 t}, \qquad t \geqq 0$$

for $j = 1,2,\cdots,N$.

Corollary 12(a): Let $Q(\cdot)$ *denote a complex measurable* $N \times N$ *matrix-*

valued function of t defined on $[0, \infty)$. *and let the elements of* $Q(t)$ *be uniformly bounded on* $[0, \infty)$. *Let*

$$g(t) = f(t) + \int_0^t k(t - \tau)Q(\tau)f(\tau)\, d\tau, \qquad t \geqq 0$$

in which $f \varepsilon \underline{\mathcal{E}}_N$, *and there exists a positive constant* c_1 *such that*

(*i*) $ge^{c_1 t} \varepsilon \underline{\mathfrak{L}}_{2N}(0, \infty)$
(*ii*) $ke^{c_1 t} \varepsilon \underline{\mathcal{K}}_{1N} \cap \underline{\mathcal{K}}_{2N}$.

With

$$K(i\omega) = \int_0^\infty k(t)e^{-i\omega t}\, dt \quad \mathit{for} \quad -\infty < \omega < \infty,$$

let

$$\sup_{t \geqq 0} \Lambda\{Q(t)\} \sup_{-\infty<\omega<\infty} \Lambda\{K(i\omega)\} < 1.$$

Then there exist positive constants c_2 and c_3 such that

$$|f_j(t)| \leqq |g_j(t)| + c_2 e^{-c_3 t}, \qquad t \geqq 0$$

for $j = 1,2, \cdots, N$.

With the aid of Corollary 12(a) and arguments very similar to those used to establish the corollary of Ref. 11, it is a simple matter to prove the following result concerned with conditions under which all solutions of a well known type of linear differential equation approach zero *exponentially* at infinity.†

Theorem 13: Let A denote a constant positive-definite $N \times N$ *Hermitian matrix. Let* $B(t)$ *denote an* $N \times N$ *positive-definite Hermitian-matrix-valued function of t for* $t \geqq 0$, *and let the elements of* $B(t)$ *be measurable and uniformly bounded for* $t \geqq 0$. *Let f be a complex N-vector valued function of t defined and twice differentiable on* $[0, \infty)$ *such that*

$$\frac{d^2 f(t)}{dt^2} + A\frac{df(t)}{dt} + B(t)f(t) = g(t), \qquad t \geqq 0$$

with

$$ge^{c_1 t} \varepsilon \underline{\mathfrak{L}}_{2N}(0, \infty)$$

for some positive constant c_1.

† Theorem 13 can be obtained also from the corollary of Ref. 11 by using it to show that there exists a positive constant δ such that, under the conditions of Theorem 13, $f(t)$ can be written as $h(t)e^{-\delta t}$ for $t \geqq 0$ with $h(t) \to 0$ as $t \to \infty$. That is, the corollary can be applied to the differential equation in h for δ sufficiently small.

Let $\underline{\lambda}\{A\}$, $\underline{\lambda}\{B(t)\}$, and $\bar{\lambda}\{B(t)\}$ denote, respectively, the smallest eigenvalue of A, the smallest eigenvalue of $B(t)$, and the largest eigenvalue of $B(t)$. Suppose that

$$\inf_{t \geqq 0} \underline{\lambda}\{B(t)\} > 0$$

and that

$$\underline{\lambda}\{A\} > (\sup_{t \geqq 0} \bar{\lambda}\{B(t)\})^{\frac{1}{2}} - (\inf_{t \geqq 0} \underline{\lambda}\{B(t)\})^{\frac{1}{2}}.$$

Then there exist positive constants c_2 and c_3 such that

$$|f_j(t)| \leqq c_2 e^{-c_3 t}, \qquad t \geqq 0$$

for $j = 1,2, \cdots, N$.

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