Properties of Random Traffic in Nonblocking Telephone Connecting Networks

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Some of the properties of random traffic in nonblocking connecting networks are described and proved. Even though nonblocking networks are rare, they represent an important limiting case, approached as blocking is reduced by adding switches. For many purposes they provide a useful first approximation in the calculation of system parameters. The number of calls in progress is extensively studied in both equilibrium and transient regimes, and its properties are used to distinguish between the wide and strict senses of "nonblocking."

I. INTRODUCTION

In the continuing effort to understand the nature of congestion in telephone connecting networks, it is important to have a thorough knowledge of the special case of *no congestion*, exemplified by traffic in a nonblocking network. Such knowledge is useful not merely as a guide to theoretical investigations, but also in answering questions that are of immediate practical import in the design of networks with small congestion.

It is the purpose of this paper to describe some results concerning random traffic in *nonblocking* connecting networks; these results have important applications to traffic in networks that are not nonblocking. For although nonblocking networks are rare in present telephone practice, and are therefore of limited immediate interest to engineers, they form an important limiting case that is approached as the probability of blocking is reduced by the addition of links and switches to the network. Moreover, many parameters descriptive of the traffic can be calculated with ease for a nonblocking network, and only arduously or not at all for a network that has a nonzero probability of blocking. Hence for low blocking, certain results pertaining to the nonblocking case can be used to approximate those in the blocking case.

In other words, for many purposes the nonblocking case serves as a useful first approximation, as a guide for intuition and computation, in the general case. It is important not to misconstrue our claim. We are not making the banal and useless point that zero is a good first approximation to the probability of blocking when the blocking is small. We are making the point that if the blocking is small then various interesting parameters of the system, other than blocking, are very nearly related as they would be in the nonblocking case. This point has direct practical value.

The present work is, nevertheless, restricted to depicting the properties of nonblocking systems, and no attempt is made here to apply the results to systems with low blocking. Such applications are to appear in later papers, e.g., Ref. 1.

II. THEORETICAL MODEL

Let S be the set of permitted (i.e., physically meaningful) states of the one-sided connecting network ν (of T terminals) under study.† The set S is *partially ordered* by inclusion $\leqq$, where

$$x \leqq y$$

means that state x can be obtained from state y by removing zero or more calls. If x is a state, the notation $| x |$ will denote the number of calls in progress in state x, while if X is a set, $| X |$ will denote the number of elements of X. We also use, for a state x, the notations

A_x = set of states accessible from x by *adding* one call
B_x = set of states accessible from x by *removing* one call.

The following two probabilistic assumptions are made:

(*i*) Holding times of calls are mutually independent random variables, each with the negative exponential distribution of unit mean.

(*ii*) If u is an inlet idle in state x and $v \neq u$ is any outlet, there is a probability

$$\lambda h + o(h), \qquad \lambda > 0$$

that u attempt a call to v in the next interval of time of length h, as $h \to 0$.

The choice of unit mean for the holding times merely means that the mean holding time is being used as the unit of time, so that only the one parameter λ need be specified.

We can complete the description of the traffic model to be used by

† A given (network) graph can give rise to several *networks* ν depending on what states are permitted, i.e., belong to S.

indicating how routes for calls are chosen. For this purpose we introduce a routing matrix $R = (r_{xy})$, with these properties: For each $x \in S$ let Π_x be the partition of A_x induced by the equivalence relation of "having the same calls in progress"; then, for each $Y \in \Pi_x$, r_{xy} is a probability distribution over $y \in Y$; in all other cases $r_{xy} = 0$. As in Ref. 2, the interpretation of R is this: any $Y \in \Pi_x$ represents all the ways in which some call c not blocked in x could be completed when ν is in state x; for $y \in Y$, r_{xy} is the chance that if c is attempted in x, it will be routed through the network so as to take the system to state y. Evidently,

$$\begin{aligned}\sum_{y \in A_x} r_{xy} &= \text{number of calls each of which could actually be put up in state } x \\ &= s(x), \text{ ("successes" in } x)\end{aligned}$$

the second equality defining $s(\cdot)$ on S.

A Markov process x_t based on the preceding assumptions has been studied in previous work,[2] and is used here again as a mathematical description of an operating connecting network subject to random traffic.

We restrict attention entirely to the important case of "one-sided" networks in which all inlets are outlets.[2] Analogous results are valid for two-sided, and other, cases.

III. SUMMARY

The wide and strict senses of "nonblocking" are reviewed in Section IV, where it is also pointed out that for most of our purposes it will not be necessary to distinguish them. The equilibrium distribution of the number of calls in progress is calculated in Section V. The terms of the distribution are proportional to the (corresponding) terms of the Poisson distribution with parameter λ, the factors of proportionality indicating the "finite source effect" that is present.

In Section VI various relations among the moments of the distribution of calls in progress are explored. It is noted that the mean determines the variance, and that, as functions of λ, successive moments are related by a difference-differential equation, and can be obtained by logarithmic differentiation of the generating function of the number of assignments of k inlets to k outlets. An extremal property of the distribution of the number of calls in progress, closely related to the author's "thermodynamic" model[3] for telephone traffic, is studied in Section VII. In Section VIII it is shown that the number of calls in progress assumes a Poisson distribution in the limit as $\lambda \to 0$ and the number T of terminals becomes large, with λT^2 constant.

The remainder of the paper is concerned with the transient behavior of the process x_t representing network operation. The principal result of Section IX is that the past of the process (prior to 0) and the actual state at 0 are both irrelevant to the number of calls in progress at $t > 0$, if it is known how many calls are in progress at $t = 0$. It follows from this that the number $|x_t|$ of calls in progress at t is actually a Markov process, indeed, even a birth-and-death process. These results make it possible to calculate the covariance of $|x_t|$ in terms of $1 + [\frac{1}{2}T]$ characteristic values rather than the astronomical $|S|$ associated with x_t, and to give natural approximations (Sections X and XI). This covariance, it is to be recalled, is the essential ingredient in estimates of sampling error in traffic time-averages. In Section XII, finally, we conclude with characterizations of both the wide and the strict sense of "nonblocking" in terms of the stochastic properties of $|x_t|$.

IV. WIDE AND STRICT SENSES OF "NONBLOCKING"

In a previous paper[4] we have distinguished between a wide sense and a strict sense of the word "nonblocking," as follows: a network ν is nonblocking in the wide sense if there exists a routing matrix R which confines the trajectory of the operating system to nonblocking states, i.e., such that use of the rule R makes the system nonblocking; and ν is nonblocking in the strict sense if no call is ever blocked in *any* of its states. Topological equivalents of these properties were derived in the cited paper.

It is apparent that if ν is nonblocking in the wide sense, then for each rule R that makes ν nonblocking there exists another network ν' whose states are exactly those of ν that are accessible from the zero state under R, and ν' is nonblocking. For this reason most of our results can be (and are) stated for nonblocking networks without specifying whether the sense is wide or strict. The only excepted results are in Section XII, where the stochastic properties of $|x_t|$ are used to distinguish the wide sense of "nonblocking" from the strict.

V. THE NUMBER OF CALLS IN PROGRESS

The equation of statistical equilibrium for the stochastic process x_t is[2]

$$[\,|x| + \lambda s(x)]p_x = \sum_{y \in A_x} p_y + \lambda \sum_{y \in B_x} p_y r_{yx}\,, \qquad x \in S. \tag{1}$$

We let

$$p_k = \sum_{|x|=k} p_x\,, \qquad k = 0, 1, \cdots, \max_{x \in S} |x|\,,$$

be the probability that k calls are in progress. Our first result is the observation that the $\{p_k\}$ depend only on λ and T, if ν is nonblocking. Let α_x = number of idle inlet-outlet pairs of state x.

Theorem 1: Let ν be nonblocking. For $k = 1, \cdots, \max_{x \in S} | x | = [\frac{1}{2}T]$,

$$\begin{aligned} p_k &= p_0 \frac{\lambda^k}{k!} \prod_{j=0}^{k-1} \binom{T - 2j}{2} \\ &= p_0 \frac{(\frac{1}{2}\lambda)^k}{k!} \frac{T!}{(T - 2k)!}. \end{aligned} \tag{2}$$

Proof: We sum (1) over $| x | = k$. Since (with the third equality a definition)

$$s(x) = \alpha_x = \binom{T - 2 | x |}{2} = \alpha_{|x|}$$

if ν is nonblocking, we obtain

$$(k + \lambda\alpha_k)p_k = \sum_{|x|=k} \sum_{y \in A_x} p_y + \lambda \sum_{|x|=k} \sum_{y \in B_x} p_y r_{yx}.$$

In the first sum on the right, each p_y gets counted $(k + 1)$ times, because if $| y | = (k + 1)$, then $y \in A_x$ for exactly $(k + 1)$ values of x. Thus this sum has the value

$$(k + 1) \sum_{|y|=(k+1)} p_y = (k + 1)p_{k+1} .$$

The second sum is

$$\sum_{|x|=k} \sum_{|y|=k-1} p_y r_{yx} = \sum_{|y|=k-1} p_y \sum_{|x|=k} r_{yx} .$$

However, by the definition of the routing matrix R,

$$\begin{aligned} \sum_{|x|=k} r_{yx} &= \sum_{x \in A_y} r_{yx} \\ &= s(y) \\ &= \alpha_{|y|} , \end{aligned}$$

because ν is nonblocking. Hence the second sum is

$$p_{k-1}\alpha_{k-1} ,$$

and we have shown that

$$(k + \lambda\alpha_k)p_k = (k + 1)p_{k+1} + \lambda\alpha_{k-1}p_{k-1} ,$$

with the convention $p_k = 0$ if $k < 0$ or $k > [\frac{1}{2}T]$. Thus

$$kp_k = \lambda \alpha_{k-1} p_{k-1} \qquad k = 1, \cdots, [\tfrac{1}{2}T].$$

By iteration, the theorem follows.

We remark that the probability p_0 that no calls are in progress, determined from the normalization

$$\sum_{k=0}^{[\frac{1}{2}T]} p_k = 1,$$

is just

$$p_0 = \frac{1}{1 + \sum_{k=1}^{[\frac{1}{2}T]} \frac{(\frac{1}{2}\lambda)^k}{k!} \frac{T!}{(T-2k)!}}. \tag{3}$$

VI. MOMENTS OF THE NUMBER OF CALLS IN PROGRESS

From the formulas (2) and (3) giving the distribution of the number of calls in progress, any moment of the distribution of calls in progress can be calculated in principle. More important, though, are the several systematic relationships that obtain among the moments and the parameters λ and T of the system. To these we now turn our attention.

We use the abbreviations

$$a_k = \begin{cases} \dfrac{T!}{2^k k!(T-2k)!} & k = 0, \cdots, [\tfrac{1}{2}T], \\ 0, & k > [\tfrac{1}{2}T] \end{cases}$$

$$m_i = \sum_{x \in S} |x|^i p_x \qquad i = 1, 2, \cdots,$$

$$= i\text{th moment of } \{p_k\},$$

$$\Phi(\lambda) = \sum_{k \geq 0} \lambda^k a_k,$$

$$\sigma^2 = m_2 - m_1^2 = \text{variance of calls in progress}$$

and $m_1 = m$.

First, it has been shown[2] that whether ν is nonblocking or not, a stochastic process x_t based on our assumptions has the property that the probability Pr {bl} of blocking, the mean m and variance σ^2 of the number of calls in progress, and the parameters λ and T, are all related by the formula, for one-sided networks ν,

$$1 - \Pr\{\text{bl}\} = \frac{1}{\lambda} \frac{2m}{(T-2m)^2 - T + 2m + 4\sigma^2}.$$

(A similar, but different, formula obtains for two-sided ν.) It follows that when ν is nonblocking, the mean and variance of calls in progress are related by

$$(T - 2m)^2 - T + 2m + 4\sigma^2 = 2m/\lambda, \tag{4}$$

and thus determine each other uniquely when λ and T are specified. This means that for a nonblocking ν the important parameters m and σ^2 cannot assume just any values, but must lie on the curve defined by (4).

Second, it is intuitively obvious that, for many networks ν, $m = m(\lambda)$ should be an increasing function of λ. The rationale for this claim is, of course, that if the calling rate per idle pair λ increases, the network will carry a greater (equilibrium) load. For nonblocking networks ν, the claim is a consequence of

Theorem 2: For nonblocking ν, and $i = 1, 2, \cdots$,

$$\frac{d}{d\lambda} m_i = \frac{1}{\lambda}(m_{i+1} - m_i m_1).$$

Proof: We have

$$m_i = \frac{\sum_{k>0} k^i \lambda^k a_k}{\Phi(\lambda)}$$

$$\frac{d}{d\lambda} m_i = \frac{(\sum_{k>0} k^{i+1}\lambda^{k-1} a_k)\Phi(\lambda) - (\sum_{k>0} k_i \lambda^k a_k)(\sum_{k>0} k\lambda^{k-1} a_k)}{\Phi^2(\lambda)}$$

$$= \frac{1}{\lambda}(m_{i+1} - m_i m_1).$$

In particular

$$\frac{dm}{d\lambda} = \frac{\sigma^2(\lambda)}{\lambda} \tag{5}$$

and so m is a strictly increasing function of λ.

Corollary 1: The mean number m of calls in progress as a function of λ satisfies the differential equation

$$\frac{dm}{d\lambda} = \frac{m}{2\lambda^2} - \frac{(T - 2m)^2 - T + 2m}{4\lambda}$$

with the initial conditions $m(0) = 0$, $m'(0) = \binom{T}{2}$.

Proof: We substitute (5) in (4) with Pr {bl} = 0. The initial conditions follow from

$$m(\lambda) = \lambda \binom{T}{2} + o(\lambda), \quad \text{as} \quad \lambda \to 0.$$

It can be verified that Theorem 2 can be rephrased as saying that all the moments of $\{p_k\}$ can be obtained from the logarithmic derivatives of the generating function $\Phi(\cdot)$ of the numbers $\{a_k\}$. Thus for example

$$m = m_1 = \lambda \frac{d}{d\lambda} \log \Phi,$$

$$\sigma^2 = \lambda^2 \frac{d^2}{d\lambda^2} \log \Phi + \lambda^2 \frac{d}{d\lambda} \log \Phi.$$

Indeed, it now becomes apparent that $\{p_k\}$ has the same relationship to the function $\Phi(\cdot)$ as the distribution of calls in progress in the "thermodynamic" model of Ref. 3 had to the generating function of the number of ways of having k calls in progress. It will turn out in the next section that $\Phi(\cdot)$ is actually the generating function of the number of *assignments* of k inlets to k outlets, without reference to how many states of ν, if any, actually realize a given assignment.

VII. AN EXTREMAL PROPERTY OF THE DISTRIBUTION OF CALLS IN PROGRESS

With X the set of T terminals of the network ν, let us consider the set A of all fixed-point free maps of X into itself, together with all submaps thereof. The physical significance of A is that it consists of all the possible "assignments" of k inlets to k outlets with $0 \leqq k \leqq [\frac{1}{2}T]$. The fixed-point free restriction reflects the physically realistic circumstance that no customer will request connection to himself. It is readily seen that the set A of assignments is partially ordered by inclusion, and in fact forms a semilattice. Also there is a natural map of S onto A, the map $\gamma(\cdot)$ of Ref. 4, which takes every state of ν into the assignment it realizes. It can be seen that $\gamma(\cdot)$ preserves order and intersections, so that $\gamma(\cdot)$ is a semilattice homomorphism of S onto A.

Let us now pose the problem of finding a probability distribution $\{p_a, a \in A\}$ which maximizes the entropy functional

$$H(p) = -\sum_{a \in A} p_a \log p_a$$

subject to the condition that

$$\sum_{a \in A} |a| p_a = m,$$

where m is a given positive number with $0 < m < [\frac{1}{2}T]$, and $| a |$, the norm of a, is the number of inlets mapped into outlets by a, i.e., the number of "intended calls in progress" called for by the assignment a. It follows from Lemma 1 of Ref. 3 that this maximum is achieved by

$$p_a = \frac{\lambda^{|a|}}{\sum_{a \epsilon A} \lambda^{|a|}}$$

i.e., the "canonical" distribution of thermodynamics, with $| \cdot |$ playing the role of energy (see Ref. 3), and $\lambda > 0$ determined uniquely by

$$m = \lambda \frac{d}{d\lambda} \log \sum_{a \epsilon A} \lambda^{|a|}.$$

It follows that the probability assigned by $\{p_a , a \epsilon A\}$ to the set of assignments with k "intended calls in progress" is just

$$\frac{\lambda^k \sum_{|a|=k} 1}{\sum_{x \epsilon A} \lambda^{|a|}} = p_k ,$$

since there are exactly

$$\sum_{|a|=k} 1 = \frac{T!}{k!2^k(T - 2k)!} \qquad 0 \leqq k \leqq [\tfrac{1}{2}T]$$

fixed-point free maps of k elements out of a set of T into k others from the set, so that $\Phi(\lambda) = \sum_{a \epsilon A} \lambda^{|a|}$.

Thus the distribution $\{p_k\}$ of the number of calls in progress in a nonblocking network arises naturally from maximizing the entropy functional for a probability distribution over the set A of assignments subject to a given average value for $| a |$, and then calculating the probability of the set of assignments of k calls.

In a similar way, it can be shown that $\{p_k\}$ maximizes the entropy functional $-\sum_k p_k \log p_k$, subject to

$$m = \sum kp_k ,$$

over all distributions having the form $b_k a_k$.

VIII. A POISSON LIMIT THEOREM

It is intuitively reasonable to expect that a nonblocking network with a very large number T of inlets (= outlets, here) and a very small calling rate λ per idle inlet pair will behave roughly like Palm's "infinite trunk" model for telephone traffic.[5] In particular, if λ becomes small and T becomes large in the right way, the distribution of the number of calls in progress in equilibrium should become Poisson. That this occurs is the content of

Theorem 3: Let a be a positive number, and let $\lambda \to 0$ and $T \to \infty$ in such a way that

$$a = \lambda T^2/2.$$

Then

$$p_k \to e^{-a}(a^k/k!), \qquad k = 0, 1, 2, \cdots.$$

Proof: We have

$$\begin{aligned} p_k/p_0 &= \frac{\left(\frac{\lambda T^2}{2}\right)^k}{k!}\left(1 - \frac{1}{T}\right)\cdots\left(1 - \frac{2k-1}{T}\right) \\ &\to \frac{a^k}{k!}. \end{aligned}$$

Since

$$p_0^{-1} = 1 + \sum_{k=1}^{[\frac{1}{2}T]} p_k/p_0,$$

the result follows.

The reason why λT^2, and not, e.g., λT, must be of the order of the average carried load, is that λ is the calling rate per pair of idle inlets (= outlets, here), so that if all are idle, this calling rate is just

$$\lambda \binom{T}{2},$$

omitting attempts by a customer to himself. Indeed, the load carried by one customer's line is

$$q = (2m/T) = \lambda T((1-q)^2 - T^{-1}(1-q) + T^{-2}4\sigma^2)).$$

It is easily seen that q and $T^{-2}\sigma^2$ are bounded independently of λ and T, so that

$$q \sim \lambda T \to 0$$

in the limit taken.

IX. TIME-DEPENDENT BEHAVIOR OF THE NUMBER OF CALLS IN PROGRESS

So far all our results have concerned only the equilibrium behavior of the process x_t representing the operation of a nonblocking connecting network. We now turn to the transient or time-dependent behavior.

The matrix $Q = (q_{xy})$ of transition rates of x_t is given by

$$q_{xy} = \begin{cases} 1, & x \in A_y \\ \lambda r_{xy}, & x \in B_y \\ -|x| - \lambda s(x), & x = y \\ 0 & \text{otherwise.} \end{cases}$$

The matrices $P(t) = (p_{xy}(t))$, t real, of transition probabilities, i.e., such that

$$p_{xy}(t) = \Pr\{x_t = y \mid x_0 = x\},$$

satisfy the Kolmogorov equations

$$P'(t) = QP(t) = P(t)Q, \qquad P(0) = I.$$

We let

$$p_{ij}(t) = \Pr\{|x_t| = j \mid |x_0| = i\}$$
$$p_{xj}(t) = \Pr\{|x_t| = j \mid x_0 = x\}.$$

Intuitively, if ν is nonblocking and $|x_t| = j$, then the (conditional) probabilities of the possible changes in the number of calls in progress in the next interval of time of length h are

$$jh + o(h), \qquad \text{for a hangup,}$$
$$\lambda\binom{T-2j}{2} + o(h), \qquad \text{for a new call,}$$

as $h \to 0$. Indeed, one expects that these evaluations remain true even if information about x_s for $s < t$ is added to what is known at time t, for the reason that only the fact that $|x_t| = j$ is relevant to what happens to $|x_s|$ for $s > t$. In other words it is natural to expect that for nonblocking ν,

$$|x_t|$$

is itself a Markov process, indeed, a birth-and-death process. It will be shown that these conjectures are true, and that they have important practical consequences.

Theorem 4: If ν is nonblocking, then knowledge of the actual state x_0 is irrelevant to $|x_t|$ if $|x_0|$ is known, i.e.,

$$p_{xk}(t) = p_{|x|k}(t), \qquad \text{for all } x.$$

Proof: The backward Kolmogorov equation for the process is

$$\frac{d}{dt} p_{xy} = -[|x| + \lambda s(x)] p_{xy} + \sum_{u \in B_x} p_{uy} + \lambda \sum_{u \in A_x} r_{xu} p_{uy} .$$

Summing on $|y| = k$ gives

$$\frac{d}{dt} p_{xk} = -[|x| + \lambda s(x)] p_{xk} + \sum_{u \in B_x} p_{uk} + \lambda \sum_{u \in A_x} r_{xu} p_{uk} .$$

Since $u \in B_x$ for exactly $(|x| - 1)$ values of u, and since

$$\sum_{x \in A_u} r_{xu} = s(x) = \binom{T - 2|x|}{2},$$

it is enough to show that the result is true in a neighborhood of $t = 0$. Evidently, though,

$$p_{xk}(0) = \begin{cases} 1 & |x| = k \\ 0 & |x| \neq k \end{cases}$$

$$\frac{d}{dt} p_{xk}(0) = \begin{cases} -\left[|x| + \lambda\binom{T - 2|x|}{2}\right] & |x| = k \\ 0 & |x| \neq k \end{cases}$$

and

$$p_{xk}^{(n)}(0) = \begin{cases} -\left[|x| + \lambda\binom{T - 2|x|}{2}\right] p_{xk}^{(n-1)}(0) + \sum_{u \in B_x} p_{uk}^{(n-1)}(0) \\ \qquad + \lambda \sum_{x \in A_u} r_{xu} p_{uk}^{(n-1)}(0), & |x| = k \\ 0 & |x| \neq k. \end{cases}$$

Since $p_{xk}(\cdot)$ is analytic in a neighborhood of $t = 0$, the theorem follows.

Theorem 5: If ν is nonblocking, then

$$|x_t|$$

is a Markov stochastic process.

Proof: Set $y_t = |x_t|$. Since x_t is a Markov process, for $t_1 < t_2 < \cdots < t_n < t$ we have a.e.

$$\Pr\{y_t = k \mid x_{t_i}, i = 1, \cdots, n\} = \Pr\{y_t = k \mid x_{t_n}\}, \\ = \Pr\{y_t = k \mid y_{t_n}\}$$

by Theorem 4.

X. TRANSITION PROBABILITIES OF $|x_t|$

It follows from Theorem 5 and the *forward* Kolmogorov equation for for x_t that the transition probabilities $p_{ij}(\cdot)$ of $y_t = |x_t|$ satisfy the equations

$$\frac{d}{dt} p_{ij} = -\left[j + \lambda\binom{T-2j}{2}\right] p_{ij} + (j+1)p_{i(j+1)} + \lambda\binom{T-2j+2}{2} p_{i(j-1)}, \tag{6}$$

with obvious conventions at the (reflecting) boundaries $j = 0$ and $j = [\frac{1}{2}T]$. These are the equations of a birth-and-death process on a finite number of states, and so the known results of Karlin and McGregor[6] can be carried over at once, as summarized below.

The matrix $A(T,\lambda)$ governing the system (6) is given by

$$a_{ij} = \begin{cases} 0 & |i-j| > 1 \\ i & j+1 = i \\ -i - \lambda\binom{T-2i}{2} & i = j \\ \lambda\binom{T-2i}{2} & i+1 = j. \end{cases} \tag{7}$$

With

$$\pi_k = \lambda^k a_k \qquad k = 0, 1, \cdots, [\tfrac{1}{2}T],$$

and

$$Q_0(x) \equiv 1,$$

$$-xQ_0(x) = -\lambda\binom{T}{2}Q_0(x) + \lambda\binom{T}{2}Q_1(x),$$

$$-xQ_k(x) = kQ_{k-1}(x) - \left[k + \lambda\binom{T-2k}{2}\right]Q_k(x) + \lambda\binom{T-2k}{2}Q_{k+1}(x), \qquad 1 < k < [\tfrac{1}{2}T],$$

there is a unique[6,7] positive regular measure ψ on $0 \leqq x < \infty$ such that

$$\int_0^\infty Q_i(x)Q_j(x)d\psi(x) = \frac{\delta_{ij}}{\pi_j} \qquad i,j = 0, 1, \cdots, [\tfrac{1}{2}T].$$

The transition probabilities of $| x_t |$ are represented by the formula

$$p_{ij}(t) = \pi_j \int_0^\infty e^{-xt} Q_i(x)Q_j(x)d\psi(x) \tag{8}$$

XI. THE COVARIANCE OF $| x_t |$

As has been pointed out,[3,8] the covariance function of the number of calls in progress is of great practical interest in connection with estimates of sampling error in telephone traffic averages. This covariance is defined as

$$R(t) = E\{ | x_{t+s} | | x_s | \} - E^2\{ | x_s | \},$$

and does not depend on s, since it is understood that x_s has its equilibrium distribution. The variance of the continuous time-average

$$\frac{1}{T} \int_0^T | x_t | dt$$

is then

$$2T^{-2} \int_0^T (T - t)R(t)dt,$$

while that of the periodic scanned average

$$\frac{1}{n} \sum_{j=1}^{n} | x_{j\tau} |, \qquad \tau > 0,$$

with scanning interval τ is

$$\sum_{j=-n}^{n} (n - | j |)R(j\tau).$$

It is easily seen from the integral representation (8) that the covariance of $| x_t |$ is

$$R(t) = \sum_{i,j=1}^{w} ij\pi_i\pi_j \int_0^\infty e^{-xt} Q_i(x)Q_j(x)d\psi(x) - m^2,$$

$$w = [\tfrac{1}{2}T] = \max_{x \in S} | x |.$$

The orthogonality of the $Q_i(\cdot)$ with respect to $\psi(\cdot)$ allows the simplification of this formula to

$$R(t) = \int_0^\infty e^{-xt} \left[\sum_{i=1}^{w} i\pi_i Q_i(x) \right]^2 d\psi(x) - m^2.$$

It is easily verified that for $k > 0$

$$Q_k(0) = 1$$

and that

$$\psi(0+) - \psi(0-) = \left(\sum_{j=0}^{w} \pi_j\right)^{-1}.$$

Hence the contribution of $\psi(\cdot)$ at the origin (to the first term on the right of $R(t)$) gives precisely m^2, and we have proved the important result that

$$R(t) \geqq 0.$$

We note next that the matrix $A(T,\lambda)$ of the differential equations for $p_{ij}(\cdot)$ is symmetrizable, and so has real nonpositive characteristic values. In a standard way[3,8] it is deduced that one of these is zero, and that the dominant characteristic value r_1 satisfies

$$\begin{aligned} -(m/\sigma^2) &\leqq r_1 < 0, \\ R(t) &\leqq \sigma^2 e^{r_1 t}. \end{aligned} \tag{9}$$

As in the theory[8] of the finite trunk group, it is expected that this upper bound for $R(\cdot)$ will be a good approximation for low to moderate traffic levels. Together, the two inequalities suggest the alternative estimate

$$R(t) \sim \sigma^2 \exp -\left(\frac{m}{\sigma^2} t\right),$$

also used in Ref. 8.

Since the equilibrium distribution $\{p_k\}$ of the number of calls in progress approaches Poisson's as $\lambda \to 0$ and $T \to \infty$ with λT^2 constant, it is to be expected that the characteristic values of the matrix $A(T,\lambda)$ of the system (6) will concentrate at the nonpositive integers in this same limit. In this connection it is instructive to see how the lower bound $-m/\sigma^2$ to r_1 behaves in the above limit. With $\lambda T^2 \equiv 2a > 0$, we find

$$\begin{aligned} \frac{m}{\sigma^2} &= 2\lambda + \frac{\lambda}{2} \frac{(T - 2m)^2}{\sigma^2} + \frac{\lambda}{2} \frac{T - 2m}{\sigma^2} \\ &= \frac{1}{1 + 2\lambda T + \lambda}\left[2\lambda + \frac{a}{\sigma^2}(1 + T^{-1}) + 2\lambda\left(\frac{m}{\sigma}\right)^2\right]. \end{aligned} \tag{10}$$

Since the variance of a Poisson distribution equals its mean, $\sigma^2 \to a$, and it is easily verified that σ^2/a depends only on T and not on λ so that

$$\sigma^2/a = 1 + o(1)$$

with $o(1)$ depending only on T. It follows that for any $a > 0$,

$$\liminf_{\substack{\lambda \to 0 \\ T \to \infty \\ \lambda T^2 = 2a}} r_1 \geqq -1,$$

i.e., the lower limit of the dominant characteristic value is at least -1. If we retain only terms of order λT in (10) we obtain

$$-1 - \frac{4a - 1}{T} \tag{11}$$

as an approximate lower bound for r_1, indicating that r_1 actually approaches -1 from above or below according as $a < \frac{1}{4}$ or $a > \frac{1}{4}$, the latter case being overwhelmingly prevalent in practice.

Actually it is not necessary that $T \to \infty$ in order that the lower bound in (9) approach -1. It suffices that λ be small, for with T fixed, as $\lambda \to 0$,

$$-\frac{m}{\sigma^2} = -\frac{m}{\lambda \dot{m}} = -\frac{\lambda\binom{T}{2} + o(\lambda)}{\lambda\binom{T}{2} - \lambda^2\binom{T-2}{2} + o(\lambda)}$$

$$= -1 + \lambda\binom{T-2}{2} + o(\lambda).$$

We note that the correction term is quite different from that in (11).

XII. STOCHASTIC CHARACTERIZATION OF WIDE AND STRICT SENSES OF "NONBLOCKING"

In the following, we regard the process x_t defined in Section II as a function of ν, λ and the routing matrix R, $T = T(\nu)$, etc.

Theorem 6: ν is nonblocking in the wide sense if and only if for some routing matrix R, $|x_t|$ is a birth-and-death process whose semigroup of transition probabilities is generated by $A(T,\lambda)$.

Proof: The necessity follows from Theorem 5. For the sufficiency we argue that if ν is not nonblocking in the wide sense then any choice of R gives rise to a nonzero probability of blocking. Thus by the basic formula (4)

$$\frac{1}{\lambda}\frac{2m}{(T-2m)^2-T+2m+4\sigma^2}<1$$

for any R, which contradicts the condition that for some R, $p = \{p_k\}$ satisfies

$$Ap = 0,$$

with the convention $(Ap)_j = \sum_i a_{ij}p_i$. In a similar way we can prove

Theorem 7: ν is nonblocking in the strict sense if and only if for every R, $|x_t|$ is a birth-and-death process whose semigroup of transition probabilities is generated by $A(T,\lambda)$.

The proof is a minor modification of that of Theorem 6, and is omitted.

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