On the Boundedness of Solutions of Nonlinear Integral Equations

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Sufficient conditions are presented for the boundedness of the solutions of a vector nonlinear Volterra integral equation of the second kind that frequently arises in the study of automatic control systems containing an arbitrary finite number of time-varying nonlinear elements. Similar conditions are given for the boundedness of the solutions of the discrete analog of the integral equation.

A direct application of the results yields a Nyquist-like frequency-domain condition for the "bounded-input implies bounded-output stability" of a large class of feedback systems containing a single time-varying nonlinear element.

I. NOTATION AND DEFINITIONS

Let M denote an arbitrary matrix. We shall denote by M', M^*, and M^{-1}, respectively, the transpose, the complex-conjugate transpose, and the inverse of M. The positive square root of the largest eigenvalue of M^*M is denoted by $\Lambda\{M\}$, and 1_N denotes the identity matrix of order N.

The set of real measurable N-vector-valued functions of the real variable t defined on $[0, \infty)$ is denoted by $\mathcal{K}_N(0, \infty)$ and the jth component of $f \,\varepsilon\, \mathcal{K}_N(0, \infty)$ is denoted by f_j.

The sets $\mathcal{L}_{\infty N}(0, \infty)$ and $\mathcal{L}_{2N}(0, \infty)$ are defined by

$$\mathcal{L}_{\infty N}(0, \infty) = \{f \mid f \,\varepsilon\, \mathcal{K}_N(0, \infty), \sup_{t \geqq 0} [f'(t)f(t)] < \infty\}$$

$$\mathcal{L}_{2N}(0, \infty) = \left\{f \mid f \,\varepsilon\, \mathcal{K}_N(0, \infty), \int_0^\infty f'(t)f(t)dt < \infty\right\}$$

The norm of $f \,\varepsilon\, \mathcal{L}_{2N}(0, \infty)$ is denoted by $\| f \|$ and is defined by

$$\| f \| = \left(\int_0^\infty f'(t) f(t) dt \right)^{\frac{1}{2}}.$$

With this norm $\mathcal{L}_{2N}\ (0, \infty)$ is a Banach space.

Let $y\ \varepsilon\ (0, \infty)$ and define f_y by

$$\begin{aligned} f_y(t) &= f(t) \quad \text{for} \quad t\ \varepsilon\ [0,y] \\ &= 0 \qquad \text{for} \quad t > y \end{aligned}$$

for any $f\ \varepsilon\ \mathcal{H}_N\ (0, \infty)$, and let

$$\mathcal{E}_N = \{ f \mid f\ \varepsilon\ \mathcal{H}_N\ (0, \infty), f_y\ \varepsilon\ \mathcal{L}_{2N}\ (0, \infty) \text{ for } 0 < y < \infty \}.$$

With A an arbitrary real measurable $N \times N$ matrix-valued function of t with elements $\{a_{nm}\}$ defined on $[0, \infty)$, let $\mathcal{K}_{pN}\ (p = 1,2)$ denote

$$\left\{ A \mid \int_0^\infty | a_{nm}(t) |^p dt < \infty \ (n,m = 1, 2, \cdots, N) \right\}.$$

Let $\psi[f(t),t]$ denote

$$(\psi_1[f_1(t),t], \psi_2[f_2(t),t], \cdots, \psi_N[f_N(t),t])', \qquad f\ \varepsilon\ \mathcal{H}_N\ (0, \infty)$$

where $\psi_1(w,t)$, $\psi_2(w,t)$, $\cdots$, $\psi_N(w,t)$ are real-valued functions of the real variables w and t for $-\infty < w < \infty$ and $0 \leqq t < \infty$ such that

(*i*) $\psi_n\ (0,t) = 0$ for $t\ \varepsilon\ [0, \infty)$ and $n = 1, 2, \cdots, N$

(*ii*) there exist real numbers α and β with the property that

$$\alpha \leqq \frac{\psi_n(w,t)}{w} \leqq \beta \qquad (n = 1, 2, \cdots, N)$$

for $t\ \varepsilon\ [0, \infty)$ and all real $w \neq 0$.

(*iii*) $\psi_n\ [w(t),t]\ (n = 1, 2, \cdots, N)$ is a measurable function of t whenever $w(t)$ is measurable.

The symbol s denotes a scalar complex variable with $\sigma = \text{Re}\ [s]$ and $\omega = \text{Im}\ [s]$.

II. INTRODUCTION AND SUMMARY

In the study of physical systems such as nonlinear automatic control systems containing an arbitrary finite number of time-varying nonlinear elements, attention is frequently focused on the properties of the equation

$$g(t) = f(t) + \int_0^t k(t - \tau) \psi[f(\tau),\tau] d\tau, \qquad t \geqq 0$$

in which $g \varepsilon \mathcal{E}_N$, $f \varepsilon \mathcal{E}_N$, $k(\cdot) \varepsilon \mathcal{K}_{1N}$, and $\psi[\cdot,\cdot]$ is as defined in the previous section.

In Ref. 1, the following theorem is proved.

Theorem 1: Let $k \varepsilon \mathcal{K}_{1N}$, and let

$$v(t) = u(t) + \int_0^t k(t-\tau)\psi[u(\tau),\tau]d\tau, \qquad t \geqq 0$$

where $v \varepsilon \mathcal{L}_{2N}(0,\infty)$ and $u \varepsilon \mathcal{E}_N$. Let

$$K(s) = \int_0^\infty k(t)e^{-st}dt, \qquad \sigma \geqq 0.$$

Suppose that

(i) det $[1_N + \frac{1}{2}(\alpha+\beta)K(s)] \neq 0$ *for* $\sigma \geqq 0$

(ii) $\frac{1}{2}(\beta-\alpha) \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \frac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}K(i\omega)\} < 1.$

Then $u \varepsilon \mathcal{L}_{2N}(0,\infty)$, and there exists a positive constant ρ which depends only on k, α, and β such that

$$\| u \| \leqq \rho \| v \|.$$

The primary purpose of this paper is to prove the following related result.

Theorem 2: Let $t^p k \varepsilon \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$ for $p = 0, 1, 2$. Let

$$g(t) = f(t) + \int_0^t k(t-\tau)\psi[f(\tau),\tau]d\tau, \qquad t \geqq 0$$

where $g \varepsilon \mathcal{L}_{\infty N}(0,\infty)$ and $f \varepsilon \mathcal{E}_N$. Let

$$K(s) = \int_0^\infty k(t)e^{-st}dt, \qquad \sigma \geqq 0.$$

Suppose that

(i) det $[1_N + \frac{1}{2}(\alpha+\beta)K(s)] \neq 0$ *for* $\sigma \geqq 0$

(ii) $\frac{1}{2}(\beta-\alpha) \sup_{-\infty<\omega<\infty} \Lambda\{[1_N + \frac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}K(i\omega)\} < 1.$

Then $f \varepsilon \mathcal{L}_{\infty N}(0,\infty)$, there exists a positive constant c which depends only on k, α, and β such that

$$\max_j \sup_{t\geqq 0} |f_j(t)| \leqq c \max_j \sup_{t\geqq 0} |g_j(t)|,$$

and $f_j(t) \to 0$ as $t \to \infty$ for $j = 1, 2, \cdots, N$ whenever $g_j(t) \to 0$ as $t \to \infty$ for $j = 1, 2, \cdots, N$.

A direct application of Theorem 2 yields a *frequency-domain* condition for the $\mathfrak{L}_\infty$-stability[2] of a well-known type of feedback system. This is discussed in Section IV. In Section V, sufficient conditions are stated for the boundedness of the solutions of the discrete analog of the nonlinear integral equation considered in Theorem 2. In Section VI, we describe some additional results that can be proved by combining the methods of this paper with the $\mathfrak{L}_{2N}(0,\infty)$ arguments of Ref. 1 and another earlier paper.

III. PROOF OF THEOREM 2

Assume throughout this section that the hypotheses of Theorem 2 are satisfied.

Let $q_j(t)$ be defined on $[0,\infty)$ by

$$q_j(t) = \frac{\psi_j[f_j(t),t]}{f_j(t)}, \qquad t \,\varepsilon\, \{t \mid t \geqq 0, f_j(t) \neq 0\}$$
$$= \tfrac{1}{2}(\alpha + \beta), \qquad t \,\varepsilon\, \{t \mid t \geqq 0, f_j(t) = 0\}$$

for $j = 1, 2, \cdots, N$; and let $q(t)$ denote the diagonal matrix $\operatorname{diag}[q_1(t), q_2(t), \cdots, q_N(t)]$. Then

$$g(t) = f(t) + \int_0^t k(t-\tau)q(\tau)f(\tau)d\tau, \qquad t \geqq 0.$$

Let a be an arbitrary positive number, and for each nonnegative integer n let $g^{(n)}(t)$ be defined on $[0,\infty)$ by

$$g^{(n)}(t) = g(t), \quad na \leqq t < (n+1)a$$
$$= 0, \qquad 0 \leqq t < na \text{ and } t \geqq (n+1)a.$$

Lemma 1: For each integer $n \geqq 0$, $\mathfrak{L}_{2N}(0,\infty)$ contains a unique element $f^{(n)}$ such that

(*i*) $f^{(n)}(t) = 0, \qquad 0 \leqq t < na$

(*ii*) $g^{(n)}(t) = f^{(n)}(t) + \int_0^t k(t-\tau)q(\tau)f^{(n)}(\tau)d\tau, \qquad t \geqq 0.$

Proof of Lemma 1:

Clearly $g^{(n)} \,\varepsilon\, \mathfrak{L}_{2N}(0,\infty)$ for $n \geqq 0$. Let **I** denote the identity operator on $\mathfrak{L}_{2N}(0,\infty)$, and let **K** and **Q** denote the mappings of $\mathfrak{L}_{2N}(0,\infty)$ into itself defined by[3]

$$(\mathbf{K}h)(t) = \int_0^t k(t-\tau)h(\tau)d\tau, \qquad t \geqq 0$$

$$(\mathbf{Q}h)(t) = (q_1(t)h_1(t), q_2(t)h_2(t), \cdots, q_N(t)h_N(t))', \qquad t \geqq 0$$

where h is an arbitrary element of $\mathfrak{L}_{2N}(0, \infty)$.

According to Lemma 5 of Ref. 1, the operator $[\mathbf{I} + \frac{1}{2}(\alpha + \beta)\mathbf{K}]$ possesses an inverse on $\mathfrak{L}_{2N}(0, \infty)$. Thus the functional equation

$$g^{(n)} = h^{(n)} + \mathbf{KQ}h^{(n)}, \qquad h^{(n)} \varepsilon\, \mathfrak{L}_{2N}(0, \infty)$$

can be written as $h^{(n)} = \mathbf{T}h^{(n)}$, in which $\mathbf{T}$ is defined by

$$\mathbf{T}h^{(n)} = [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}g^{(n)} - [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}\mathbf{K}[\mathbf{Q} - \tfrac{1}{2}(\alpha + \beta)\mathbf{I}]h^{(n)}.$$

Using the bounds of Lemma 5 of Ref. 1, and the fact that $\alpha \leqq q_j(t) \leqq \beta$ for $j = 1, 2, \cdots, N$ and $t \geqq 0$, it can easily be shown that $\mathbf{T}$ is a *contraction* mapping of $\mathfrak{L}_{2N}(0, \infty)$ into itself. Thus, it follows from the contraction-mapping fixed-point theorem that $\mathfrak{L}_{2N}(0, \infty)$ contains a unique element $f^{(n)}$ which satisfies condition (*ii*) of the lemma.

Since $[\mathbf{I} + \frac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}$ is necessarily causal, and

$$f^{(n)} = \lim_{m \to \infty} \mathbf{T}^m \theta,$$

in which θ is the zero-element of $\mathfrak{L}_{2N}(0, \infty)$, we see that $f^{(n)} = 0$ for $0 \leqq t < na$ and $n > 0$.

Lemma 2: Let $f^{(n)}$ be the associate of $g^{(n)}$ in accordance with Lemma 1. Then

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(t), \qquad t \geqq 0.$$

Proof of Lemma 2:

Let

$$\hat{f}(t) = \sum_{n=0}^{\infty} f^{(n)}(t), \qquad t \geqq 0.$$

Then

$$g(t) = \hat{f}(t) + \int_0^t k(t-\tau)q(\tau)\hat{f}(\tau)\, d\tau, \qquad t \geqq 0$$

and hence

$$0 = [f(t) - \hat{f}(t)] + \int_0^t k(t - \tau)q(\tau)[f(\tau) - \hat{f}(\tau)]\, d\tau, \qquad t \geqq 0. \quad (1)$$

Theorem 1 implies that $(f - \hat{f}) \varepsilon \mathfrak{L}_{2N}(0,\infty)$ and that $\| f - \hat{f} \| = 0$. Since the integral in (1) must therefore vanish for $t \geqq 0$, we have

$$f(t) = \hat{f}(t) \quad \text{for} \quad t \geqq 0.$$

Lemma 3: Let $f^{(n)}$ be the associate of $g^{(n)}$ in accordance with Lemma 1. Then there exists a positive constant Ω which depends only on k, α, and β such that

$$| f_j^{(n)}(t)| \leqq | g_j^{(n)}(t)| + (1 + t - na)^{-2}\Omega(1 + a)^2(Na)^{\frac{1}{2}} \max_j \sup_{t \geqq 0} | g_j^{(n)}(t)|, \qquad t \geqq na$$

for $j = 1, 2, \cdots, N$ and every $n \geqq 0$.

Before proceeding to the proof of Lemma 3, it is convenient to state the following result, which is easily provable with the aid of Parseval's identity, the well-known extremal property of the largest eigenvalue of a Hermitian matrix, and the Schwarz inequality.

Lemma 4: Let $w \varepsilon \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$, $z \varepsilon \mathfrak{L}_{2N}(0,\infty)$, and

$$y(t) = \int_0^t w(t - \tau)z(\tau)\, d\tau \quad \text{for} \quad t \geqq 0.$$

Then:

(*i*) $y \varepsilon \mathfrak{L}_{2N}(0,\infty)$

(*ii*) with $W(j\omega) = \int_0^\infty w(t)\, e^{-i\omega t}\, dt \qquad (-\infty < \omega < \infty)$,

$$\| y \| \leqq \sup_{-\infty<\omega<\infty} \Lambda\{ W(j\omega)\} \| z \|$$

(*iii*) $| y_n(t) | \leqq \left(\sum_{m=1}^N \int_0^\infty | w_{nm}(t) |^2\, dt \right)^{\frac{1}{2}} \| z \|$

for $t \geqq 0$ and $n = 1, 2, \cdots, N$.

Proof of Lemma 3

Let n denote an arbitrary nonnegative integer.

Since

$$g^{(n)}(t) = f^{(n)}(t) + \int_0^t k(t - \tau)q(\tau)f^{(n)}(\tau)\, d\tau, \qquad t \geqq 0$$

it is certainly true that for each positive integer p

$$(1 + t - na)^p g^{(n)}(t) = (1 + t - na)^p f^{(n)}(t) + \int_0^t k(t - \tau)[(1 + \tau - na) + (t - \tau)]^p q(\tau) f^{(n)}(\tau)\, d\tau, \qquad t \geqq 0$$

or, what is the same thing,

$$h(p,t) = (1 + t - na)^p f^{(n)}(t) + \int_0^t k(t - \tau) q(\tau)(1 + \tau - na)^p f^{(n)}(\tau)\, d\tau, \qquad t \geqq 0 \tag{2}$$

in which

$$h(p,t) = (1 + t - na)^p g^{(n)}(t) - \sum_{m=0}^{p-1} \frac{p!}{(p - m)!m!} \int_0^t (t - \tau)^{p-m} k(t - \tau) \cdot q(\tau)(1 + \tau - na)^m f^{(n)}(\tau)\, d\tau, \qquad t \geqq 0.$$

From Lemma 4, our assumption that $tk \in \mathcal{K}_{1N}$, and the fact that $f^{(n)} \in \mathfrak{L}_{2N}(0, \infty)$, it is clear that $h(1,\cdot) \in \mathfrak{L}_{2N}(0, \infty)$. A direct application of Theorem 1 to (2) with $p = 1$ shows [recall that $\alpha \leqq q_j(t) \leqq \beta$ for $t \geqq 0$ and $j = 1, 2, \cdots, N$] that $(1 + t - na)f^{(n)} \in \mathfrak{L}_{2N}(0, \infty)$ and that there exists a positive constant c_1 that depends only on k, α, and β such that

$$\| (1 + t - na)f^{(n)} \| \leqq c_1 \| h(1,\cdot) \|.$$

Since by assumption $t^2 k \in \mathcal{K}_{1N}$, this argument can be repeated for $p = 2$. Thus, $(1 + t - na)^2 f^{(n)} \in \mathfrak{L}_{2N}(0, \infty)$ and

$$\| (1 + t - na)^2 f^{(n)} \| \leqq c_1 \| h(2,\cdot) \|.$$

Using Lemma 4, our assumption that $t^r k \in \mathcal{K}_{2N}$ $(r = 0,1,2)$, (2) with $p = 2$, our bounds on $\| (1 + t - na)f^{(n)} \|$ and $\| (1 + t - na)^2 f^{(n)} \|$, and the fact that $\| f^{(n)} \| \leqq c_1 \| g^{(n)} \|$, it is a simple matter to show that there exist positive constants c_2, c_3, and c_4, each depending only on k, α, and β, such that

$$| f_j^{(n)}(t) | \leqq | g_j^{(n)}(t) | + (1 + t - na)^{-2}[c_2 \| (1 + t - na)^2 g^{(n)} \| + c_3 \| (1 + t - na) g^{(n)} \| + c_4 \| g^{(n)} \|], \qquad t \geqq na$$

for $j = 1, 2, \cdots, N$.

Since

$$\| g^{(n)} \| \leqq \| (1 + t - na) g^{(n)} \| \leqq \| (1 + t - na)^2 g^{(n)} \|,$$

and

$$\begin{aligned} \| (1 + t - na)^2 g^{(n)} \| &\leqq (1 + a)^2 \| g^{(n)} \| \\ &= (1 + a)^2 \left(\int_{na}^{(n+1)a} g'(t) g(t) \, dt \right)^{\frac{1}{2}} \\ &\leqq (1 + a)^2 (Na)^{\frac{1}{2}} \max_j \sup_{t \geqq 0} | g_j^{(n)}(t) | , \end{aligned}$$

we have, with $\Omega = c_2 + c_3 + c_4$,

$$| f_j^{(n)}(t) | \leqq | g_j^{(n)}(t) | + (1 + t - na)^{-2} \Omega (1 + a)^2 (Na)^{\frac{1}{2}} \max_j \sup_{t \geqq 0} | g_j^{(n)}(t) |, \qquad t \geqq na$$

for $j = 1, 2, \cdots, N$. This proves Lemma 3.

Let t satisfy $ma \leqq t < (m + 1)a$ where m is an arbitrary nonnegative integer. Then, by Lemmas 1, 2, and 3

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(t) = \sum_{n=0}^{m} f^{(n)}(t),$$

and

$$\begin{aligned} | f_j(t) | &\leqq \sum_{n=0}^{m} | f_j^{(n)}(t) | \\ &\leqq | g_j^{(m)}(t) | \\ &\quad + c_5(a) \max_j \sup_{t \geqq 0} | g_j(t) | \sum_{n=0}^{m} (1 + ma - na)^{-2} \end{aligned}$$

for $j = 1, 2, \cdots, N$, in which

$$c_5(a) = \Omega (1 + a)^2 (Na)^{\frac{1}{2}}.$$

Let

$$c_6(a) = \sum_{n=0}^{\infty} (1 + na)^{-2}.$$

Since

$$\sum_{n=0}^{m} (1 + ma - na)^{-2} < \sum_{n=0}^{\infty} (1 + na)^{-2},$$

we have

$$|f_j(t)| \leqq \sup_{t \geqq 0} |g_j(t)| + c_5(a)c_6(a) \max_j \sup_{t \geqq 0} |g_j(t)|$$

for *every* integer $m \geqq 0$ (and hence *every* $t \geqq 0$) and $j = 1, 2, \cdots, N$. Therefore

$$\max_j \sup_{t \geqq 0} |f_j(t)| \leqq [1 + c_5(a)c_6(a)] \max_j \sup_{t \geqq 0} |g_j(t)|.$$

Now suppose that $g_j(t) \to 0$ as $t \to \infty$ for $j = 1, 2, \cdots, N$. We will show that for each $\epsilon > 0$ there exists a $t_\epsilon > 0$ such that $|f_j(t)| < \epsilon$ for $t > t_\epsilon$ and $j = 1, 2, \cdots, N$.

Let $\epsilon > 0$ be given, and again consider the relation

$$f(t) = \sum_{n=0}^{\infty} f^{(n)}(t).$$

Since

$$\begin{aligned}\sum_{n=n_1}^{\infty} |f_j^{(n)}(t)| &\leqq \max_j \sup_{t \geqq n_1 a} |g_j(t)| \left[1 + c_5(a) \sum_{n=n_1}^{n_2} (1 + n_2 a - na)^{-2}\right] \\ &\leqq \max_j \sup_{t \geqq n_1 a} |g_j(t)| [1 + c_5(a)c_6(a)]\end{aligned}$$

for $n_1 a \leqq n_2 a \leqq t < (n_2 + 1)a$, with n_1 and n_2 positive integers, it is clear that there exists a positive integer n_3 such that

$$\sum_{n=n_3}^{\infty} |f_j^{(n)}(t)| < \tfrac{1}{2}\epsilon \quad \text{for} \quad t \geqq n_3 a \quad \text{and} \quad j = 1, 2, \cdots, N.$$

From the inequality

$$\sum_{n=0}^{(n_3-1)} |f_j^{(n)}(t)| \leqq c_5(a) \max_j \sup_{t \geqq 0} |g_j(t)| \sum_{n=0}^{(n_3-1)} (1 + t - na)^{-2}, \qquad t \geqq n_3 a$$

it is evident that there exists a positive integer $n_4 > n_3$ such that

$$\sum_{n=0}^{(n_3-1)} |f_j^{(n)}(t)| < \tfrac{1}{2}\epsilon \quad \text{for} \quad t > n_4 a \quad \text{and} \quad j = 1, 2, \cdots, N.$$

Thus

$$|f_j(t)| \leqq \sum_{n=0}^{\infty} |f_j^{(n)}(t)| < \epsilon \qquad \text{for} \quad t > n_4 a \quad \text{and} \quad j = 1, 2, \cdots, N$$

This completes the proof of Theorem 2.

Remarks:

With regard to the hypotheses of Theorem 2, it can easily be verified that if the elements of k are uniformly bounded on $[0, \infty)$, then the assumption that $f \in \mathcal{E}_N$ can be replaced by $f \in \mathcal{H}_N(0, \infty)$ with locally integrable elements.

In most cases of interest the elements of $t^p k$ are uniformly bounded on $[0, \infty)$ for $p = 0,1,2$. In such cases $t^p k \in \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$ for $p = 0,1,2$ provided that $t^2 k \in \mathcal{K}_{1N}$.

IV. AN APPLICATION: A FREQUENCY-DOMAIN CONDITION FOR THE $\mathcal{L}_\infty$-STABILITY OF FEEDBACK SYSTEMS CONTAINING A SINGLE TIME-VARYING NONLINEAR ELEMENT

In a recent brief,[2] a two-part sufficient condition is given for the $\mathcal{L}_\infty$-stability of a well-known type of feedback system containing a single time-varying nonlinear element. In another publication,[4] conditions are presented for the $\mathcal{L}_2$-stability of the same type of feedback system. Unlike the conditions for $\mathcal{L}_2$-stability of Ref. 4, which are expressed entirely in the frequency domain, the key condition of Ref. 2 for $\mathcal{L}_\infty$-stability is that the integral of the modulus of a certain function be less than unity.

A direct application of Theorem 2 shows that under somewhat stronger assumptions than those of Ref. 2 or Ref. 4 concerning $k(\cdot)$, there the impulse-response function of the linear time-invariant portion of the forward path, the conditions given for $\mathcal{L}_2$-stability are also sufficient conditions for $\mathcal{L}_\infty$-stability. Specifically, the following result is a direct consequence of Theorem 2.

Theorem 3: The feedback system described in Ref. 2 is $\mathcal{L}_\infty$-stable if

$$(i) \int_0^\infty | t^p k(t) | \, dt < \infty \text{ and } \int_0^\infty | t^p k(t) |^2 \, dt < \infty \text{ for } p = 0, 1, 2$$

$$(ii) \text{ with } K(s) = \int_0^\infty k(t) e^{-st} \, dt \quad \text{for } \sigma \geqq 0,$$

$$(a)\ 1 + \tfrac{1}{2}(\alpha + \beta) K(s) \neq 0 \quad \text{for } \sigma \geqq 0$$

$$(b)\ \tfrac{1}{2}(\beta - \alpha) \max_{-\infty < \omega < \infty} | K(i\omega)[1 + \tfrac{1}{2}(\alpha + \beta) K(i\omega)]^{-1} | < 1.$$

Part (*b*) of (*ii*) above is a weaker condition than the condition of the theorem of Ref. 2 that it replaces [i.e., (*ii*) of Ref. 2]. From an engineering viewpoint condition (*ii*) above possesses an interesting frequency-domain interpretation.[4]†

V SUFFICIENT CONDITIONS FOR THE BOUNDEDNESS OF SOLUTIONS OF THE DISCRETE ANALOG OF THE INTEGRAL EQUATION CONSIDERED IN THEOREM 2

Sufficient conditions for the boundedness of the solutions of the discrete analog of the nonlinear integral equation considered in Theorem 2 can be obtained by modifying in a straightforward manner both the arguments presented in Section III and the arguments of Ref. 1 that lead to Theorem 1. In order to state the result (Theorem 2′, below) we need some notation.

Let Ξ denote the set of *nonnegative* integers. Let $\tilde{\mathfrak{IC}}_N$ be the set of real N-vector-valued functions defined on Ξ, and let the jth component of $f \varepsilon \tilde{\mathfrak{IC}}_N$ be denoted by f_j. Let

$$\tilde{\mathfrak{L}}_{\infty N} = \{f \mid f \varepsilon \tilde{\mathfrak{IC}}_N, \sup_{n \geqq 0} [f'(n)f(n)] < \infty\},$$

$$\tilde{\mathfrak{L}}_{2N} = \{f \mid f \varepsilon \tilde{\mathfrak{IC}}_N, \sum_{n=0}^{\infty} f'(n)f(n) < \infty\},$$

and

$$\| f \|_{\sim} = \left(\sum_{n=0}^{\infty} f'(n)f(n)\right)^{\frac{1}{2}} \quad \text{for} \quad f \varepsilon \tilde{\mathfrak{L}}_{2N}.$$

With B an arbitrary real $N \times N$ matrix-valued function of n with elements $\{b_{lm}(n)\}$ defined on Ξ, let $\tilde{\mathfrak{K}}_{pN}(p = 1,2)$ denote

$$\{B \mid \sum_{n=0}^{\infty} | b_{lm}(n)|^p < \infty \quad (l,m = 1, 2, \cdots, N)\}.$$

Let $\varphi[f(n),n]$ denote

$$(\varphi_1[f_1(n),n],\varphi_2[f_2(n),n], \cdots, \varphi_N[(f_N(n),n])', \qquad f \varepsilon \tilde{\mathfrak{IC}}_N$$

where $\varphi_1(w,n),\varphi_2(w,n), \cdots, \varphi_N(w,n)$ are real-valued functions of w and n for $-\infty < w < \infty$ and $n \varepsilon \Xi$ such that

(*i*) $\varphi_m(0,n) = 0$ for $n \varepsilon \Xi$ and $m = 1, 2, \cdots, N$

(*ii*) there exist real numbers α and β with the property that

† We take this opportunity to correct the result of a typographical error: In the first inequality on page 1606 of Ref. 4 the "<" sign should be replaced by "≦".

$$\alpha \leqq \frac{\varphi_m(w,n)}{w} \leqq \beta \qquad (m = 1, 2, \cdots, N)$$

for all real $w \neq 0$ and $n \varepsilon \Xi$.

Theorem 2′: Let $n^2 k \varepsilon \tilde{\mathfrak{K}}_{1N}$. *Let*

$$g(n) = f(n) + \sum_{m=0}^{n} k(n - m)\varphi[f(m),m], \qquad n \varepsilon \Xi$$

where $g \varepsilon \tilde{\mathfrak{L}}_{\infty N}$ *and* $f \varepsilon \tilde{\mathfrak{K}}_N$. *Let*

$$K(s) = \sum_{n=0}^{\infty} k(n)e^{-sn}, \qquad \sigma \geqq 0.$$

Suppose that

(*i*) $\det [1_N + \frac{1}{2}(\alpha + \beta)k(0)] \neq 0$, *and*

$\det [1_N + \frac{1}{2}(\alpha + \beta)K(s)] \neq 0$ *for* $\sigma \geqq 0$

(*ii*) $\frac{1}{2}(\beta - \alpha) \sup_{-\pi \leqq \omega \leqq \pi} \Lambda\{[1_N + \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\} < 1.$

Then $f \varepsilon \tilde{\mathfrak{L}}_{\infty N}$, *there exists a positive constant c which depends only on* k, α, *and* β *such that*

$$\max_j \sup_{n \geqq 0} |f_j(n)| \leqq c \max_j \sup_{n \geqq 0} |g_j(n)|,$$

and $f_j(n) \to 0$ *as* $n \to \infty$ *for* $j = 1, 2, \cdots, N$ *whenever* $g_j(n) \to 0$ *as* $n \to \infty$ *for* $j = 1, 2, \cdots, N$.

In the statement of Theorem 2′ we have used the fact that $n^p k \varepsilon \tilde{\mathfrak{K}}_{1N} \cap \tilde{\mathfrak{K}}_{2N}$ for $p = 0, 1, 2$ provided that $n^2 k \varepsilon \tilde{\mathfrak{K}}_{1N}$.

The result analogous to Theorem 1 is the following theorem.

Theorem 1′: Let $k \varepsilon \tilde{\mathfrak{K}}_{1N}$, *and let*

$$g(n) = f(n) + \sum_{m=0}^{n} k(n - m)\varphi[f(m),m], \qquad n \varepsilon \Xi$$

where $g \varepsilon \tilde{\mathfrak{L}}_{2N}$ *and* $f \varepsilon \tilde{\mathfrak{K}}_N$. *Let*

$$K(s) = \sum_{n=0}^{\infty} k(n)e^{-sn}, \qquad \sigma \geqq 0.$$

Suppose that

(*i*) $\det [1_N + \frac{1}{2}(\alpha + \beta)k(0)] \neq 0$, *and*

$\det [1_N + \frac{1}{2}(\alpha + \beta)K(s)] \neq 0$ *for* $\sigma \geqq 0$.

(*ii*) $\frac{1}{2}(\beta - \alpha) \sup_{-\pi \leqq \omega \leqq \pi} \Lambda\{[1_N + \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\} < 1.$

Then $f \varepsilon \tilde{\mathfrak{L}}_{2N}$, *and there exists a positive constant* ρ *which depends only on* k, α, *and* β *such that*

$$\| f \|_{\sim} \leqq \rho \| g \|_{\sim} .$$

VI. SOME ADDITIONAL RESULTS

Arguments very similar to those of Section III and the proof of the lemma of Ref. 5 can be used to establish the following result, which is of direct interest in the study of the properties of solutions of systems of differential equations.

Theorem 3: Let $t^p k \,\varepsilon\, \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$ *for* $p = 0, 1, 2$. *Let* $Q(\cdot)$ *denote a real measurable* $N \times N$ *matrix-valued function of* t *defined on* $[0, \infty)$, *and let the elements of* $Q(t)$ *be uniformly bounded on* $[0, \infty)$. *Let*

$$g(t) = f(t) + \int_0^t k(t - \tau)Q(\tau)f(\tau)\, d\tau, \qquad t \geqq 0$$

where $g \,\varepsilon\, \mathfrak{L}_{\infty N}(0, \infty)$ *and* $f \,\varepsilon\, \mathcal{E}_N$. *With*

$$K(i\omega) = \int_0^\infty k(t)e^{-i\omega t}\, dt \quad \textit{for} \quad -\infty < \omega < \infty,$$

let

$$\sup_{t \geqq 0} \Lambda\{Q(t)\} \sup_{-\infty<\omega<\infty} \Lambda\{K(i\omega)\} < 1.$$

Then $f \,\varepsilon\, \mathfrak{L}_{\infty N}(0, \infty)$, *there exists a positive constant* c *which depends only on* $k(\cdot)$ *and* $Q(\cdot)$ *such that*

$$\max_j \sup_{t \geqq 0} | f_j(t)| \leqq c \max_j \sup_{t \geqq 0} | g_j(t)|,$$

and $f_j(t) \to 0$ *as* $t \to \infty$ *for* $j = 1, 2, \cdots, N$ *whenever* $g_j(t) \to 0$ *as* $t \to \infty$ *for* $j = 1, 2, \cdots, N$.

Theorem 3 remains valid if the sets $\mathcal{K}_{1N}$, $\mathcal{K}_{2N}$, $\mathfrak{L}_{\infty N}(0, \infty)$, and $\mathcal{E}_N$ are replaced with their natural complex extensions, and $Q(\cdot)$ is permitted to be complex valued.

A result that can easily be proved with the aid of Theorem 3 (see the proofs of the theorem and corollary of Ref. 5) is as follows.

Theorem 4: Let $\psi(\cdot, \cdot)$ *be as defined in Section I with* $N = 1$ *and* $\alpha > 0$, *and let* f *be any real-valued function of* t *defined and twice differentiable on* $[0, \infty)$ *such that*

$$\frac{d^2 f}{dt^2} + a \frac{df}{dt} + \psi[f, t] = g, \qquad t \geqq 0$$

where $g(t)$ is uniformly bounded on $[0, \infty)$. Suppose that a is a real constant such that $a > \sqrt{\beta} - \sqrt{\alpha}$. Then $f(t)$ is uniformly bounded on $[0, \infty)$, and $f(t) \to 0$ as $t \to \infty$ if $g(t) \to 0$ as $t \to \infty$.

The following theorem, which can be proved with arguments very similar to those of Section III and the proof of Theorem 5 of Ref. 1, is of immediate interest in the theory of stability of electrical networks containing time-varying capacitors.[6]

Theorem 5: Let $t^p k \in \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$ for $p = 0, 1, 2$. Let B denote a constant real $N \times N$ matrix, and let $a_1(t), a_2(t), \cdots, a_N(t)$ denote real-valued measurable functions of the real variable t for $t \geqq 0$ with the property that there exist real constants α and β such that

$$\alpha \leqq a_n(t) \leqq \beta \qquad (n = 1, 2, \cdots, N)$$

for $t \geqq 0$. Let $A(t) = \operatorname{diag}[a_1(t), a_2(t), \cdots, a_N(t)]$ for $t \geqq 0$, and let

$$g(t) = A(t)f(t) + Bf(t) + \int_0^t k(t - \tau)f(\tau)\, d\tau, \qquad t \geqq 0$$

where $g \in \mathcal{L}_{\infty N}(0, \infty)$ and $f \in \mathcal{E}_N$. Suppose that

(*i*) $\det[\frac{1}{2}(\alpha + \beta)1_N + B] \neq 0$, $\det[A(t) + B] \neq 0$ *for* $t \geqq 0$, *and* $\sup_{t \geqq 0} \Lambda\{[A(t) + B]^{-1}\} < \infty$;

and that, with

$$K(s) = \int_0^\infty k(t)e^{-st}\, dt \quad \text{for} \quad \sigma \geqq 0,$$

(*ii*) $\det[\frac{1}{2}(\alpha + \beta)1_N + B + K(s)] \neq 0$ *for* $\sigma \geqq 0$

(*iii*) $\frac{1}{2}(\beta - \alpha) \sup_{-\infty<\omega<\infty} \Lambda\{[\frac{1}{2}(\alpha + \beta)1_N + B + K(i\omega)]^{-1}\} < 1.$

Then $f \in \mathcal{L}_{\infty N}(0, \infty)$, there exists a positive constant c which depends only on $A(\cdot)$, B, and k such that

$$\max_j \sup_{t \geqq 0} |f_j(t)| \leqq c \max_j \sup_{t \geqq 0} |g_j(t)|,$$

and $f_j(t) \to 0$ as $t \to \infty$ for $j = 1, 2, \cdots, N$ whenever $g_j(t) \to 0$ as $t \to \infty$ for $j = 1, 2, \cdots, N$.

Theorem 5 remains valid if the sets $\mathcal{K}_{1N}$, $\mathcal{K}_{2N}$, $\mathcal{L}_{\infty N}(0, \infty)$ and $\mathcal{E}_N$ are replaced with their natural complex extensions and B is permitted to be complex valued.

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