# Design of Bandlimited Signals for Binary Communication Using Simple Correlation Detection*

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*This paper considers the design of binary bandlimited signals for transmission over a channel with additive white Gaussian noise, the signals to be received by a memoryless correlation detector. A signal waveform is found which allows communication at the Nyquist rate without intersymbol interference and with 1.3 db degradation compared to an optimum communication system. Other waveforms, consisting of the sum of a few prolate spheroidal functions, are also investigated.*

## I. INTRODUCTION

In the reception of serial binary data transmitted over a noisy bandlimited channel, errors result from the combined effects of intersymbol interference and noise. Minimization of the error rate involves appropriate design of both the transmitted signal and the method of detection, taking into account the effects of both causes of degradation.

Nyquist has shown how bandlimited signals may be designed so as to eliminate intersymbol interference when detection is accomplished by periodic instantaneous sampling.[1] Sunde has shown that optimum performance over a channel with white Gaussian noise is achieved when the shaping is divided equally between the transmitter and receiver.[2] Tufts has developed a technique of long memory detection which eliminates intersymbol interference and optimizes noise performance subject to that constraint, for an arbitrary transmitted signal.[3] Kurz and Trabka have studied the design of signals for transmission in the presence of nonwhite noise without the problem of intersymbol interference.[4,5]

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This paper discusses the design of bandlimited signals for communication in the presence of white Gaussian noise, when the detector is a memoryless correlator. Memoryless correlation is a widely used suboptimum means of detection. It will be shown in Section III that we can communicate without intersymbol interference at the Nyquist rate using memoryless correlation. In Section IV we investigate another form of signaling for communication with memoryless correlation. Here signals are chosen which do not eliminate intersymbol interference, but lead to low error probability for the most adverse message sequence.

## II. PRELIMINARIES

In serial binary transmission, the $n$th binary digit of the message is transmitted by sending either $s_0(t - nT)$ or $s_1(t - nT)$. We will assume that the a priori probabilities of $s_0$ and $s_1$ are 1/2 and that all digits are independent. The transmitted information rate is therefore $1/T$ bits per second.

If the signal is perturbed by additive white Gaussian noise, the optimum detector is well known to be a simple correlator if $s_0(t)$ and $s_1(t)$ are time limited to an interval of length $T$.[6] Such a detector chooses $s_0$ if and only if

$$\int v(t)s_0(t - nT)dt - \frac{1}{2}\int s_0^2(t - nT)dt > \int v(t)s_1(t - nT)dt - \frac{1}{2}\int s_1^2(t - nT)dt$$

where $v(t)$ is the received signal and the integration is taken over the interval of length $T$.

A polar signal leads to minimum error probability:[7]

$$s_0(t) = -s_1(t) = f(t).$$

The correlation detector then chooses $s_0$ if and only if

$$\int v(t)f(t - nT)dt > 0.$$

If $f(t)$ is not time limited to an interval of length $T$, as is inevitable if it is bandlimited, then the memoryless correlator is a suboptimum detector because it does not make use of the signal energy outside the interval. An infinite memory correlator or, equivalently, a matched filter and sampler, is the optimum detector, provided that intersymbol interference can be eliminated. The memoryless correlator, however, has found extensive practical application. With proper choice of $f(t)$, the degradation as compared with optimum detection need not be too large.

Aein and Hancock have shown that some improvement of the memoryless correlator can be obtained in the presence of intersymbol interference by modifying the correlating function.[8] However, this procedure is sensitive to amplitude variations of both the signal and the noise, whereas the simple correlation detector is not. We will therefore use the simple correlator and seek to minimize error probability through the choice of $f(t)$.

We will shift the time axis so that the origin is in the center of the bit to be detected, and assume that an infinite number of bits has been transmitted both before and after the bit currently being detected.

$$v(t) = \sum_{k=-\infty}^{\infty} a_k f(t + kT) + n(t)$$

where $a_k = \pm 1$ and $n(t)$ is a member function of a stationary Gaussian random process with autocorrelation $N[\delta(t)]/2$. The one-sided spectral density of the noise is therefore $N$.

Since we are using a simple correlation detector, $a_0$ will be chosen as

$$\{a_0\} = \operatorname{sgn} \int_{-T/2}^{T/2} v(t)f(t)dt$$

where

$$\operatorname{sgn} x = \frac{x}{|x|}.$$

The choice of $a_0$ when

$$\int_{-T/2}^{T/2} v(t)f(t)dt = 0$$

is not important, since this event occurs with zero probability.

$$Q = \int_{-T/2}^{T/2} v(t)f(t)dt$$

is a linear functional of a Gaussian process and is therefore itself normally distributed for a given sequence $[a_k]$. Its expected value is

$$E(Q) = \sum_{k=-\infty}^{\infty} a_k \int_{-T/2}^{T/2} f(t + kT)f(t)dt,$$

which may be written as

$$E(Q) = a_0 d^2 + \sum_{k \neq 0} a_k \rho_k d^2$$

where

$$d^2 = \int_{-T/2}^{T/2} f^2(t)\,dt \tag{1}$$

and

$$\rho_k = \frac{1}{d^2} \int_{-T/2}^{T/2} f(t + kT) f(t)\,dt. \tag{2}$$

The variance of $Q$ is

$$\mathrm{Var}\,(Q) = \frac{N}{2} d^2.$$

The probability density of $Q$ is therefore

$$p(Q) = \frac{1}{\sqrt{\pi N}\, d} \exp\left[ -\frac{1}{N d^2} \left(Q - a_0 d^2 - \sum_{k \neq 0} a_k \rho_k d^2\right)^2 \right].$$

We may now calculate the probability of error as

$$p(e \mid a_0 = +1) = \frac{1}{\sqrt{\pi N}\, d} \int_{-\infty}^{0} \exp\left[ -\frac{1}{N d^2} \left(Q - d^2 - \sum_{k \neq 0} a_k \rho_k d^2\right)^2 \right] dQ$$

$$p(e \mid a_0 = -1) = \frac{1}{\sqrt{\pi N}\, d} \int_{0}^{\infty} \exp\left[ -\frac{1}{N d^2} \left(Q + d^2 - \sum_{k \neq 0} a_k \rho_k d^2\right)^2 \right] dQ.$$

These expressions reduce to

$$p(e \mid a_0 = +1) = \frac{1}{2} \operatorname{erfc}\left[ \frac{d}{\sqrt{N}} \left(1 + \sum_{k \neq 0} a_k \rho_k\right) \right]$$

$$p(e \mid a_0 = -1) = \frac{1}{2} \operatorname{erfc}\left[ \frac{d}{\sqrt{N}} \left(1 - \sum_{k \neq 0} a_k \rho_k\right) \right]$$

where

$$\operatorname{erfc}\,(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\,dt.$$

The maximum probability of error occurs when

$$a_k = -a_0 \operatorname{sgn} \rho_k \quad \text{for all} \quad k \neq 0 \tag{3}$$

$$p_{\max} = \frac{1}{2} \operatorname{erfc}\left[ \frac{d}{\sqrt{N}} \left(1 - \sum_{k \neq 0} \mid \rho_k \mid\right) \right]. \tag{4}$$

It may be noted that if $\rho_k = 0$ for all $k \neq 0$, then the probability of error is independent of the message sequence and is equal to

$$\tfrac{1}{2} \operatorname{erfc} (d/\sqrt{N}).$$

This is the case of no intersymbol interference, and the error probability is a monotone decreasing function of $d$. If intersymbol interference does exist, the error probability is greatest for the sequence (3) and is given by (4). The average error probability is tedious to calculate, but may be readily approximated.[9]

Equation (4) may be compared with the error probability for optimum detection[10]

$$p_e = \tfrac{1}{2} \operatorname{erfc} (A/\sqrt{N})$$

where

$$A^2 = \int_{-\infty}^{\infty} f^2(t) dt).$$

It is extremely desirable that the system perform error-free in the absence of noise, $N = 0$. Since

$$\begin{aligned} \lim_{N \to 0} p_{\max} &= 0, \qquad \text{if } \sum_{k \neq 0} |\rho_k| < 1 \\ &= \tfrac{1}{2}, \qquad \text{if } \sum_{k \neq 0} |\rho_k| = 1 \\ &= 1, \qquad \text{if } \sum_{k \neq 0} |\rho_k| > 1, \end{aligned}$$

we will reject any system for which $\sum_{k \neq 0} |\rho_k| \geqq 1$, since in this case there will be some sequence of binary digits that cannot be received without error.

## III. SIGNALS WITHOUT INTERSYMBOL INTERFERENCE

In order to avoid intersymbol interference with memoryless correlation detection, it is necessary that

$$\rho_k = \int_{-T/2}^{T/2} f(t) f(t + kT) dt = d^2 \delta_{0k}. \tag{5}$$

We will seek bandlimited functions $f(t)$ which satisfy (5) by using an unpublished method of H. O. Pollak.

Let $F(\omega)$ be the Fourier transform of $f(t)$. If $f(t)$ is bandlimited to $|\omega| < \omega_c$, then

$$f(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} F(\omega) e^{j\omega t} d\omega$$

and

$$\rho_k = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \int_{-\omega_c}^{\omega_c} F(\omega)F^*(x) \frac{\sin(\omega - x)\frac{T}{2}}{\pi(\omega - x)} e^{j\omega kT} d\omega\, dx.$$

Let

$$G(\omega) = \int_{-\omega_c}^{\omega_c} F^*(x) \frac{\sin(\omega - x)\frac{T}{2}}{\pi(\omega - x)} dx. \tag{6}$$

Then

$$\rho_k = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} F(\omega)G(\omega)e^{j\omega kT} d\omega.$$

We now divide the interval $(-\omega_c, \omega_c)$ into subintervals of length $2\pi/T$

$$\rho_k = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{(2n-1)\pi/T}^{(2n+1)\pi/T} F(\omega)G(\omega)e^{j\omega kT} d\omega$$

where

$$N \geqq \frac{1}{2}\left(\frac{\omega_c T}{\pi} - 1\right)$$

$$\rho_k = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \sum_{n=-N}^{N} F\left(\omega + \frac{2n\pi}{T}\right) G\left(\omega + \frac{2n\pi}{T}\right) e^{j\omega kT} d\omega. \tag{7}$$

Equation (7) indicates that the $\rho_k$'s are the Fourier coefficients of the function

$$H(\omega) = \sum_{n=-N}^{N} F\left(\omega + \frac{2n\pi}{T}\right) G\left(\omega + \frac{2n\pi}{T}\right). \tag{8}$$

Since $\rho_k = 0$ for all $k \neq 0$, $H(\omega)$ must be a constant independent of $\omega$. Using (5) and (7), we find that

$$\sum_{n=-N}^{N} F\left(\omega + \frac{2n\pi}{T}\right) G\left(\omega + \frac{2n\pi}{T}\right) = Td^2. \tag{9}$$

If $\omega_c T \leqq \pi$, then we may choose $N = 0$, and (9) reduces to

$$F(\omega)G(\omega) = Td^2, \qquad -\frac{\pi}{T} \leqq \omega \leqq \frac{\pi}{T}. \tag{10}$$

Equation (10) cannot be satisfied if $F(\omega) = 0$ for any $\omega$ in the interval $(-\pi/T,\pi/T)$. Therefore intersymbol interference cannot be avoided if $\omega_c < \pi/T$.

Let us now investigate the case $\omega_c = \pi/T$. Substituting (6) into (10)

$$F(\omega) \int_{-\pi/T}^{\pi/T} F^*(x) \frac{\sin (\omega - x) \frac{T}{2}}{\pi(\omega - x)} dx = Td^2. \tag{11}$$

In an unpublished work, Pedro Nowosad has proved that the quadratic integral equation (11) has a continuous, real, positive solution $F(\omega)$.

Equation (11) has been solved numerically by assuming an arbitrary $F_0(\omega)$ and iteratively finding

$$\frac{1}{F_n(\omega)} = \int_{-\pi/T}^{\pi/T} F_{n-1}(x) \frac{\sin (\omega - x) \frac{T}{2}}{\pi(\omega - x)} dx.$$

The resultant amplitude spectrum $F(\omega)$ is plotted in Fig. 1. The corresponding time function $f(t)$ is plotted in Fig. 2. Since both $F(\omega)$ and $f(t)$ are even functions, only the positive abscissas are shown. The

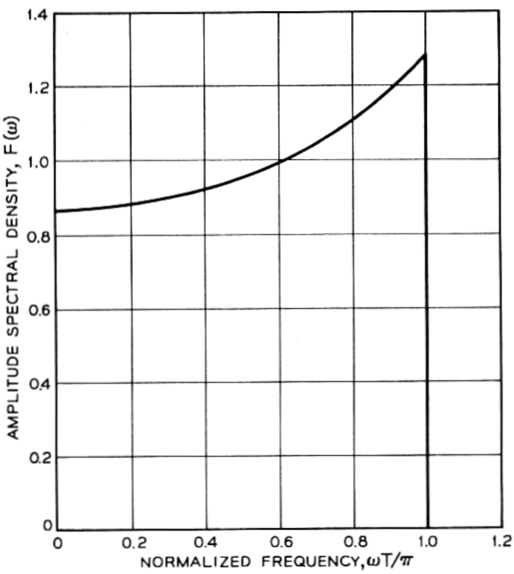


Fig. 1 — Spectrum of the signal which permits transmission without intersymbol interference.

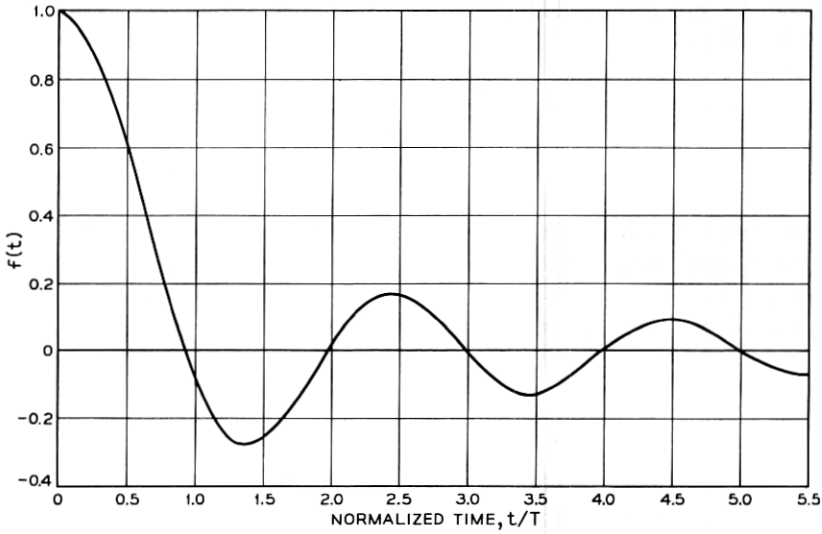


Fig. 2 — Bandlimited signal which permits transmission without intersymbol interference.

signal has been normalized for unit total energy. This time function does indeed satisfy (5) with $d^2 = 0.744$.

Digital communication using bandlimited signals and memoryless correlation detection can therefore be achieved without intersymbol interference at the Nyquist rate, $1/T = \omega_c/\pi$. The resultant degradation in the presence of white Gaussian noise, when compared with optimum detection of the signal $\sin \omega_c t/t$, is $-10 \log_{10} 0.744 = 1.3$ db.

A disadvantage which $f(t)$ as shown in Fig. 2 shares with $\sin \omega_c t/t$ is that $\Sigma f(t + nT)$ does not converge absolutely. Very large amplitudes may therefore be caused by certain sequences. If $\omega_c T > \pi$, then signals which converge more rapidly can easily be designed for detection by sampling. It is expected that solutions of (9) exist which also make such use of the additional available bandwidth. However, no such signals have as yet been found.

## IV. OTHER SIGNALS

It is not at all necessary that intersymbol interference be eliminated in order to achieve reliable digital communication. For use with memoryless correlation, a signal with some intersymbol interference may very likely lead to a lower error probability than a signal with no intersymbol interference but with less of its energy in the principal time interval.

In this section we will drop the constraint of no intersymbol interference. The probability of error is therefore dependent on the message sequence. We will use a minimax type of criterion for designing the bandlimited signal. That is, we will attempt to minimize the probability of error for the worst sequence. It is believed that the minimax criterion may be more realistic than an average error rate criterion, since the latter approach does not prevent the possibility of having some extremely sensitive message sequences. It is possible that such sensitive messages cannot be transmitted without error even over a noiseless channel. A further advantage of the minimax criterion is that it leads to a solution which does not require knowledge of the noise level.

An additional constraint that will be imposed is that the signal amplitude remain bounded for any message sequence.

We will attempt to minimize $p_{\max}$ as given by (4). From (4), $p_{\max}$ is a monotone decreasing function of

$$D = d(1 - \sum_{k \neq 0} | \rho_k |). \tag{12}$$

We can therefore satisfy the minimax criterion by maximizing the separation function, $D$. It is convenient to scale the amplitude of $f(t)$ so that $d = 1$. Such scaling, of course, affects the total energy of the signal. However, the quantity $D/\sqrt{E}$ remains invariant under such scaling, and we may accordingly maximize the quantity

$$D' = \frac{1}{\sqrt{E}} \left[ 1 - \sum_{k \neq 0} \left| \int_{-T/2}^{T/2} f(t) f(t + kT) dt \right| \right] \tag{13}$$

and the resultant $f(t)$ may later be multiplied by the factor $\sqrt{A^2/E}$ in order to satisfy the fixed energy requirement. Here, $A^2$ is the required energy, while $E$ is the energy of the scaled signal.

It is also convenient to scale the time axis so that $T = 2$ and $\omega_c = c$, where the normalized bandwidth, $c = \frac{1}{2} \omega_c T$. Note that $c = \pi/2$ corresponds to transmission at the Nyquist rate.

We will make use of the properties of the prolate spheroidal functions, $\psi_i(t)$, which are extensively discussed and plotted by Slepian, Landau and Pollak.[11,12] Some of these properties are

$$\int_{-\infty}^{\infty} \psi_i(t) \psi_j(t) dt = \delta_{ij}$$

$$\int_{-1}^{1} \psi_i(t) \psi_j(t) dt = \lambda_i \delta_{ij},$$

where $\lambda_i$ is the $(i + 1)$th largest eigenvalue of

$$\lambda\psi(t) = \int_{-1}^{1} \psi(v) \frac{\sin c(t - v)}{\pi(t - v)} dv.$$

$\psi_i(t)$ is the eigenfunction corresponding to $\lambda_i$. Both $\psi_i$ and $\lambda_i$ depend on the parameter $c$.

Since $f(t)$ is a bandlimited function, it may be expressed as a series of prolate spheroidal functions[11]

$$f(t) = \sum_{n=0}^{\infty} \gamma_n \psi_n(t). \tag{14}$$

If we set

$$\gamma_n = \frac{\beta_n}{\sqrt{\lambda_n}},$$

then

$$f(t) = \sum_{n=0}^{\infty} \beta_n \frac{\psi_n(t)}{\sqrt{\lambda_n}}. \tag{15}$$

The functions $\psi_n(t)/\sqrt{\lambda_n}$ are orthonormal over the interval $(-1,1)$. $f(t)$ can be expressed as a vector $F = [\beta_0, \beta_1, \cdots]$, with orthonormal basis

$$\left[\frac{\psi_0(t)}{\sqrt{\lambda_0}}, \frac{\psi_1(t)}{\sqrt{\lambda_1}}, \cdots\right].$$

The energy in the interval $(-1,1)$ is equal to

$$\int_{-1}^{1} f^2(t)dt = FF^t = \beta_0^2 + \beta_1^2 + \cdots = 1,$$

where $F^t$ is the transpose of $F$. The total energy is equal to

$$E = \int_{-\infty}^{\infty} f^2(t)dt = \frac{\beta_0^2}{\lambda_0} + \frac{\beta_1^2}{\lambda_1} + \cdots = F\Lambda F^t$$

where $\Lambda$ is a diagonal matrix with elements $\Lambda_{ij} = \delta_{ij}/\lambda_i$.

Since $f(t)$ is bandlimited, $f(t)$ in the interval $(-1,1)$ determines $f(t)$ for all time.

$$f(t + 2k) = \sum_{n=0}^{\infty} \beta_n \frac{\psi_n(t + 2k)}{\sqrt{\lambda_n}}.$$

$\psi_n(t + 2k)/\sqrt{\lambda_n}$ can itself be expanded as

$$\frac{\psi_n(t+2k)}{\sqrt{\lambda_n}} = \sum_{m=0}^{\infty} t_{mnk} \frac{\psi_m(t)}{\sqrt{\lambda_m}}$$

where

$$t_{mnk} = \frac{1}{\sqrt{\lambda_m \lambda_n}} \int_{-1}^{1} \psi_m(t)\psi_n(t+2k)dt \tag{16}$$

so that

$$f(t+2k) = \sum_n \beta_n \sum_m t_{mnk} \frac{\psi_m(t)}{\sqrt{\lambda_m}}$$

or, in matrix form,

$$F_k = FT_k^t \tag{17}$$

where the elements of $T_k$ are $t_{ijk}$ as given by (16).

We can now express the intersymbol interference terms as

$$\int_{-1}^{1} f(t)f(t+2k)dt = FF_k^t = FT_kF^t. \tag{18}$$

Then

$$D' = \frac{1 - \sum_{k \neq 0} b_k F T_k F^t}{\sqrt{F \Lambda F^t}} \tag{19}$$

where

$$b_k = \text{sgn } (FT_kF^t).$$

We seek suboptimum solutions by confining $F$ to $M$ dimensions. That is, we will seek an optimum $f_M(t)$ of the form

$$f_M(t) = \sum_{n=0}^{M-1} \frac{\beta_n \psi_n(t)}{\sqrt{\lambda_n}}. \tag{20}$$

Such an approach is justified if

$$\lim_{M \to \infty} f_M(t) = f(t),$$

the true optimum solution, and this convergence is sufficiently rapid. All vectors in the previous development are now $M$-dimensional and all square matrices are $M \times M$. Note that (17) is no longer strictly correct, but instead gives the projection of $F_k$ in the $M$-dimensional space. Equations (18) and (19), however, remain valid.

At this point we will introduce the constraint which requires that the total signal amplitude remain bounded for any message sequence. This is highly desirable physically, because of the effects of inexact timing and the technical impossibility of handling unbounded signals.

For the worst sequence,

$$s_{\max}(t) = \sum_{k=-\infty}^{\infty} |f(t + 2k)|$$

and we wish to constrain $f(t)$ so that $s_{\max}(t)$ remain bounded. We first express $\psi_n(t)$ as a multiple of the radial prolate spheroidal function:[13]

$$\psi_n(t) = \frac{R_n(t)}{K_n}$$

where

$$K_n^2 = \int_{-\infty}^{\infty} R_n^2(t)dt.$$

Since[11]

$$\lambda_n = \frac{2c}{\pi} R_n^2(1)$$

we can also express $K_n^2$ as

$$K_n^2 = \frac{\pi \int_{-1}^{1} R_n^2(t)dt}{2cR_n^2(1)}.$$

Then

$$f(t) = \sum_{n=0}^{M-1} \frac{\beta_n}{K_n\sqrt{\lambda_n}} R_n(t)$$

$$s_{\max}(t) = \sum_{k=-\infty}^{\infty} \left| \sum_{n=0}^{M-1} \frac{\beta_n}{K_n\sqrt{\lambda_n}} R_n(t + 2k) \right|.$$

For large $|t|$, $R_n(t)$ can be expressed asymptotically by[13]

$$R_n(t) = (-1)^{n/2} \frac{\sin ct}{ct} + O(t^{-2}), \qquad n \text{ even}$$

$$R_n(t) = (-1)^{(n+1)/2} \frac{\cos ct}{ct} + O(t^{-2}), \qquad n \text{ odd}.$$

Let us examine

$$s_N(t) = \sum_{|k|=N}^{\infty} \left| \sum_{n=0}^{M-1} \frac{\beta_n}{K_n\sqrt{\lambda_n}} R_n(t + 2k) \right|$$

$$s_N(t) \leqq \sum_{|k|=N}^{\infty} \left| \sum_{n=0}^{M-1} \frac{\beta_n}{K_n\sqrt{\lambda_n}} O(k^{-2}) \right|$$
$$+ \sum_{|k|=N}^{\infty} \left| \sum_{n \text{ even}} (-1)^{n/2} \frac{\beta_n}{K_n\sqrt{\lambda_n}} \frac{\sin c(t+2k)}{c(t+2k)} \right|$$
$$+ \sum_{|k|=N}^{\infty} \left| \sum_{n \text{ odd}} (-1)^{(n+1)/2} \frac{\beta_n}{K_n\sqrt{\lambda_n}} \frac{\cos c(t+2k)}{c(t+2k)} \right|.$$

The last two series diverge, except for isolated values of $c$ and $t$. Sufficient conditions for $s_{\max}(t)$ to be bounded are therefore

$$\sum_{n \text{ even}} (-1)^{n/2} \frac{\beta_n}{K_n\sqrt{\lambda_n}} = 0 \tag{21}$$

and

$$\sum_{n \text{ odd}} (-1)^{(n+1)/2} \frac{\beta_n}{K_n\sqrt{\lambda_n}} = 0. \tag{22}$$

These equations confine $F$ to an $(M - 2)$-dimensional subspace orthogonal to the two vectors

$$V_0 = \left[\frac{1}{K_0\sqrt{\lambda_0}}, 0, -\frac{1}{K_2\sqrt{\lambda_2}}, 0, \frac{1}{K_4\sqrt{\lambda_4}}, \cdots\right]$$
$$V_1 = \left[0, \frac{1}{K_1\sqrt{\lambda_1}}, 0, -\frac{1}{K_3\sqrt{\lambda_3}}, 0, \cdots\right]$$
$$FV_0^t = FV_1^t = 0.$$

We can form an orthogonal matrix $V$ in which the first two rows are $CV_0$ and $KV_1$, and the remaining $M - 2$ rows are any vectors such that the $M$ rows form an orthonormal set. The last $M - 2$ rows may, for example, be chosen by the Gram-Schmidt orthogonalization process. We may then form

$$G = FV^t = FV^{-1}$$

since $V^t = V^{-1}$ for an orthogonal matrix. Due to the above constraints, the first two components of $G$, $g_1$ and $g_2 = 0$. Since $V$ is an orthogonal transformation, $GG^t = FF^t = 1$.

We may also form matrices $U_k$ from $T_k$. Since $T_k$ is used only in the quadratic form (18), we need only consider its symmetric component, $T_k'$, in which $t_{ijk}' = t_{jik}' = \frac{1}{2}(t_{ijk} + t_{jik})$. Then

$$FT_kF^t = FT_k'F^t = GVT_k'V^tG^t.$$

Let $U_k = VT_k'V^t$. $U_k$ is a symmetric matrix since it is congruent to $T_k'$, a symmetric matrix.

$$FT_kF^t = GU_kG^t.$$

If we also let $\Theta = V\Lambda V^t$

$$D' = \frac{1 - \sum_{k \neq 0} b_k G U_k G^t}{\sqrt{G \Theta G^t}}. \tag{23}$$

We find the optimum $M$-dimensional signal $f(t)$ by varying the unit-length, $(M - 2)$-dimensional vector $G$ so as to maximize $D'$ given by (23), and then perform the inverse transformation and scaling. Note that if $f(t)$ is constrained to be either an even or an odd function, only terms of even or odd $n$ appear in (14), and only one of the constraints (21) or (22) is needed.

The resultant $f(t)$ is of the form (20). Landgrebe and Cooper have shown that the Fourier transform of $\psi_n(t)$ is[14]

$$\begin{aligned} \mathfrak{F}[\psi_n(t)] &= j^{-n} \sqrt{\frac{2\pi}{\lambda_n c}}\, \psi_n\left(\frac{\omega}{c}\right), & |\omega| < c \\ &= 0, & |\omega| > c. \end{aligned}$$

Therefore the optimum $f(t)$ may be generated by passing an impulse through a filter whose frequency response is

$$\begin{aligned} H(\omega) &= K \sum_{n=0}^{M-1} j^{-n} \beta_n \psi_n \left(\frac{\omega}{c}\right), & |\omega| < c \\ &= 0, & |\omega| > c. \end{aligned}$$

If $M$ is reasonably small, $H(\omega)$ is well behaved, except at $\omega = c$, and may be readily approximated by a physically realizable filter.

The optimum signals and their separation functions have been computed for several low-dimensional cases, each for several values of $c$. The total energy of the signals was set to unity in all cases.

The simplest signal is a two-dimensional even or odd function. It is completely determined by its energy and constraint (21) or (22). Three such signals have been examined. The components of these three signals are $\psi_0$ and $\psi_2$, $\psi_1$ and $\psi_3$, and $\psi_2$ and $\psi_4$, respectively. For all values of $c$, it was found that the first signal led to the highest value of the separation function $D$, while the third signal gave the lowest value of $D$. This result would be anticipated by energy considerations alone.

The $\gamma_0$ and $\gamma_2$ components of the optimum two-dimensional signals are plotted in Fig. 3 as a function of the normalized bandwidth, $c$. The values of the separation function for these signals are shown in Fig. 5.

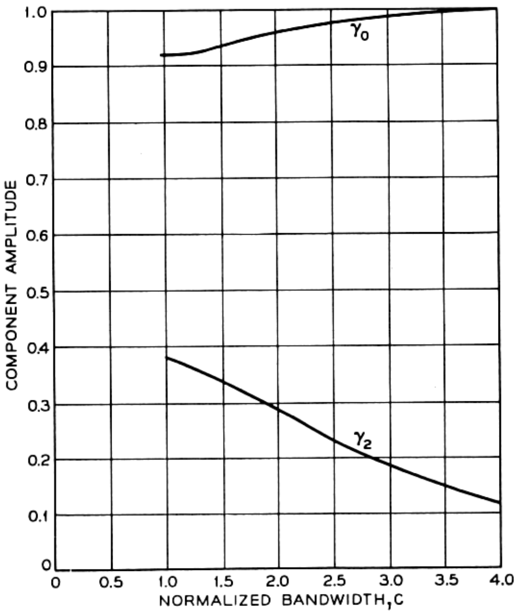


Fig. 3 — Components of optimum two-dimensional bandlimited signals.

A three-dimensional signal may be formed with $\psi_0$, $\psi_2$ and $\psi_4$ components. One degree of freedom is available for adjusting the coefficients of these components so as to maximize $D$. The optimum coefficients for signals of this form are shown in Fig. 4. The resultant values of $D$ are plotted in Fig. 5. It is seen that substantial improvement over the two-dimensional signal is obtained over a large range of $c$.

A four-dimensional signal consisting of $\psi_0$, $\psi_1$, $\psi_2$ and $\psi_3$ components was also investigated. Constraints (21) and (22) and the energy requirement permit one degree of freedom in the signal design. It was found that no significant improvement over the two-dimensional signal could be obtained by using this form of signal.

For an ideal signal which has all of its unit energy in the interval $(-1,1)$, $D = 1$. Fig. 5 may be considered to be a comparison of the worst error probabilities of bandlimited signals to the error probability of an ideal signal. If $D > 0$, then the power of the bandlimited signal must be increased by $-20 \log_{10} D$ db in order for its error probability for the worst sequence to be equal to the error probability of an ideal signal.

It should be noted that for these signals, $D < 0$ when $c < \pi/2$.

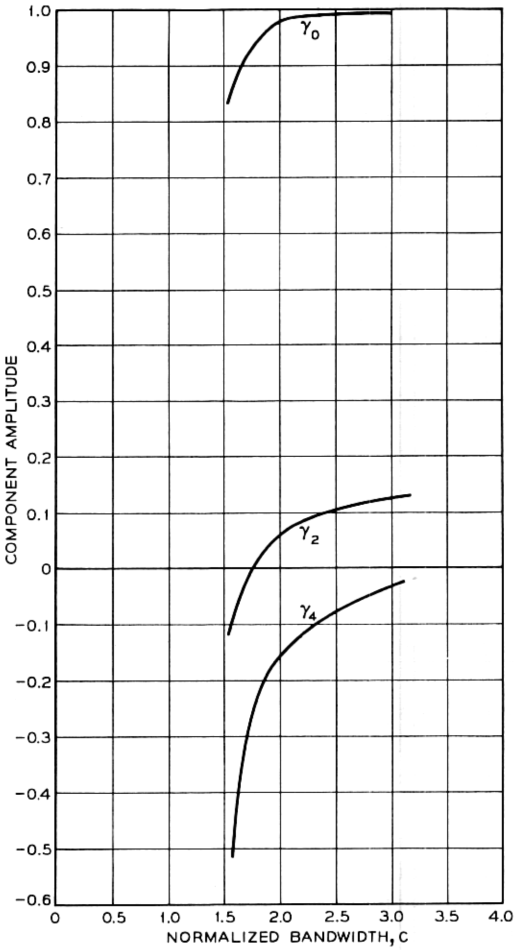


Fig. 4 — Components of optimum three-dimensional bandlimited signals.

We must therefore transmit slower than the Nyquist rate in order to achieve error-free performance in the absence of noise.

## V. CONCLUSIONS

Memoryless correlation is a suboptimum but useful method of detecting binary signals. With proper choice of the transmitted signal, the performance of a communication system using memoryless correlation can be made to be almost as good as that of an optimum system.

Communication at the Nyquist rate without intersymbol interference

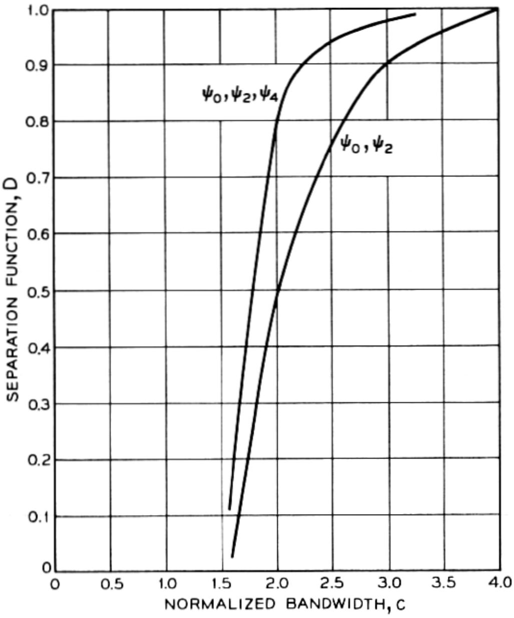


Fig. 5 — Separation functions for optimum two- and three-dimensional bandlimited signals.

using memoryless correlation detection is possible when the function shown in Fig. 2 is used as the transmitted signal. The resultant performance in the presence of noise is 1.3 db worse than that of an optimum system.

Bandlimited signals may also be designed so as to lead to low error probabilities in spite of intersymbol interference. Signals consisting of linear combinations of a finite number of prolate spheroidal functions accomplish this purpose. These signals may be designed so as to remain bounded for all message sequences.

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