The Existence of Eigenvalues for the Integral Equations of Laser Theory

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In this paper the general integral equations governing the mode spectra of optical masers are investigated from a point-of-view based upon certain theoretical results for Hölder continuous kernels. Using an estimation originally performed by Fredholm, it is proved that the homogeneous integral equation

$$\varphi(x) = \lambda \int_a^b K(x,y)\, \varphi(y)\, dy$$

has at least one eigenvalue for Hölder continuous kernels K with exponent $\alpha > \frac{1}{2}$ and with nonvanishing trace. All the integral equations which have been treated in laser theory so far can be "factored" into one-dimensional equations with continuously differentiable kernels, to which this result applies directly.

Although in practice the vanishing of the trace is the exception rather than the rule, the later sections of this paper are devoted to demonstrations of the nonvanishing character of the trace of several of the common "laser kernels" associated with practical reflector configurations. These results provide in almost all cases the first rigorous proofs of the existence of eigenvalues and eigenfunctions for the integral equations of the optical maser.

I. INTRODUCTION

Homogeneous linear Fredholm integral equations with nonsingular kernels of normal type (which includes Hermitian kernels as a special case), i.e., kernels for which

$$\int_a^b K(x,z)\, \bar{K}(y,z)\, dz = \int_a^b \bar{K}(z,x)\, K(z,y)\, dz,$$

have been extensively studied. Within the framework of complex-valued $\mathfrak{L}^2$ functions, questions of existence, uniqueness, and representation of solutions can be largely answered for such equations. Recently, however, a number of integral equations involving kernels which are neither Hermitian nor normal have arisen in laser theory. These kernels, of which

$$K(x,y) = e^{ik(x-y)^2}, \tag{1}$$

with k a given complex constant, may be considered representative, do have the seemingly beneficial property of being complex-symmetric, viz.

$$K(x,y) = K(y,x).$$

Unfortunately, due to the lack of a sufficient analytic theory for such kernels, this "advantage" has yet to be adequately exploited. Thus, although it is well-known that every Hermitian kernel distinct from the zero transformation has at least one eigenvalue, the existence of eigenvalues for general complex-symmetric or other non-normal kernels still remains an open mathematical question.

These remarks are not meant to imply that there has been a paucity of theoretical investigation to complement the widespread experimental work with masers and lasers of various geometries. Quite the contrary! Boyd and Gordon,[1] for instance, have shown that for the confocal geometry the resultant integral equation is equivalent to that considered earlier by Slepian and Pollak[2] and has prolate spheroidal wave functions as eigenfunctions. Somewhat later, Boyd and Kogelnik[3] generalized this work to resonators with unequal reflector apertures and curvatures. Moreover, iterative computational methods have been applied by Fox and Li[4,5] and Li[6] to integral equations arising from a wide range of interferometer geometries. Their techniques have produced plausible numerical descriptions of the characteristic low-order modes and eigenvalues for the configurations considered.

Even with the contributions represented by the above papers, however, there still remains a dearth of knowledge, in a mathematical sense, about the eigenfunctions and eigenvalues (if any) of the homogeneous integral equations encountered in the general theory of the optical maser. The nature of some of the mathematical questions yet to be answered in this area was considered in an early 1963 paper by S. P. Morgan.[7] Since that time some progress has been made regarding the existence of eigenvalues for certain "laser kernels." Newman and Morgan,[8] by means of lengthy Taylor series techniques, have proved that kernels of the form

$$K(x,y) = G(x)\, F(xy)\, H(y)$$

with rather general G, F, H and with nonvanishing trace possess at least one nontrivial eigenvalue. Other recent work[9,10] has centered around the use of the natural Hilbert-Schmidt expansions of "planar" and "near-confocal" kernels in terms of their singular systems [Ref. 11, p. 142 ff.].

In this paper we want to assume a somewhat different approach based upon certain theoretical results for Hölder continuous kernels. We first prove (in Section III) the following

Theorem: Let the kernel $K(x,y)$ *be Hölder continuous in either variable, with exponent* $\alpha > \frac{1}{2}$, *for* $a \leqq x,y \leqq b$. *Then if the trace of* K *does not vanish, the homogeneous integral equation*

$$\varphi(x) = \lambda \int_a^b K(x,y)\, \varphi(y)\, dy$$

has at least one eigenvalue.

The essential step in the proof is an estimation of the coefficients in the classical series representation for the Fredholm determinant of the kernel $K(x,y)$, an estimation originally carried out by Fredholm himself.[12]

We next observe that all the integral equations which have actually been treated in laser theory so far can be "factored" into one-dimensional equations with continuously differentiable kernels, to which the above theorem applies directly. Although we expect that in practice the vanishing of the trace is the exception rather than the rule, we devote the latter sections of this paper to demonstrations of the nonvanishing character of the trace of several of the common "laser kernels" associated with practical reflector configurations. These examples are indicative of the ease with which the existence of eigenvalues and eigenfunctions can be rigorously established for many of the one-dimensional kernels arising in the theory of the optical maser.

II. MATHEMATICAL PRELIMINARIES

In general we shall consider complex-valued kernels $K(x,y)$ defined on the bounded real domain $a \leqq x,y \leqq b$. Thus, where limits of integration on integrals are not specified, the integrations are to be performed over the interval $[a,b]$. We shall also assume that K belongs to the class $\mathfrak{L}^2$, i.e.,

$$\text{norm } K = \| K \| = \left[\int_a^b \int_a^b |K(x,y)|^2\, dx\, dy \right]^{\frac{1}{2}} < \infty ,$$

and that $K(x,y)$ is a square-summable function of y for each value of x and conversely.

Our notation for composite kernels shall be

$$KL = \int_a^b K(x,z)\, L(z,y)\, dz.$$

Iterated kernels will be denoted by

$$\begin{aligned} K^\nu &= KK^{\nu-1} \\ &= \int_a^b K(x,z)\, K^{\nu-1}(z,y)\, dz \qquad \nu \geqq 2 \end{aligned}$$

with $K^1 = K(x,y)$. In the same manner

$$\text{trace } K = \text{tr}(K) = \int_a^b K(x,x)\, dx$$

and

$$\begin{aligned} k_\nu = \text{tr}(K^\nu) &= \int_a^b K^\nu(x,x)\, dx \\ &= \int_a^b \int_a^b K(x,z)\, K^{\nu-1}(z,x)\, dz\, dx. \end{aligned}$$

Reference should be made to Smithies[11] for further definitions and standard theorems on integral equations as needed.

Certain notions regarding the characterization of entire or integral functions will also be of value in our work. In particular, recall that the *order* μ of entire $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is given by

$$\mu = \limsup_{n \to \infty} \frac{n \log n}{\log (1/|a_n|)}. \qquad (2)$$

Other definitions and results may be found in texts such as Boas.[13]

The property of Hölder continuity is indicative of the smoothness of a given function. Kernels for which there exist positive constants A and α such that either

$$|K(x,y) - K(z,y)| < A\,|x - z|^\alpha \qquad \text{for all } x,y,z \text{ in } [a,b]$$

or

$$|K(x,y) - K(x,z)| < A\,|y - z|^\alpha \qquad \text{for all } x,y,z \text{ in } [a,b]$$

are termed Hölder continuous in x or y respectively with exponent α. If $\alpha = 1$ the functions are said to satisfy a Lipschitz condition, and thus Hölder continuity is occasionally designated Lip_α. It should be noted that continuously differentiable functions automatically satisfy Lipschitz conditions with $\alpha = 1$.

III. THE MAIN THEOREM

Theorem: Let the kernel $K(x,y)$ *be Hölder continuous in either variable, with exponent* $\alpha > \frac{1}{2}$, *for* $a \leqq x,y \leqq b$. *Then if the trace of* K *does not vanish, the homogeneous integral equation*

$$\varphi(x) = \lambda \int_a^b K(x,y)\, \varphi(y)\, dy \tag{3}$$

has at least one eigenvalue.

Proof: The eigenvalues of (3) are the zeros, if any, of the Fredholm determinant $D(\lambda)$ associated with the kernel $K(x,y)$. The classical series representation[12] for this entire function $D(\lambda)$ is

$$D(\lambda) = \sum_{\nu=0}^{\infty} d_\nu \lambda^\nu \tag{4}$$

where $d_0 = 1$ and

$$d_\nu = \frac{(-1)^\nu}{\nu!} \iint \cdots \int K\begin{pmatrix} s_1, s_2, \cdots, s_\nu \\ s_1, s_2, \cdots, s_\nu \end{pmatrix} ds_1 ds_2 \cdots ds_\nu \qquad (\nu \geqq 1) \tag{5}$$

with

$$K\begin{pmatrix} s_1, s_2, \cdots, s_\nu \\ s_1, s_2, \cdots, s_\nu \end{pmatrix} = \det\,(K(s_i, s_j)) = \begin{vmatrix} K(s_1,s_1) & K(s_1,s_2) & \cdots & K(s_1,s_\nu) \\ K(s_2,s_1) & K(s_2,s_2) & \cdots & K(s_2,s_\nu) \\ \cdots & \cdots & \cdots & \cdots \\ K(s_\nu,s_1) & K(s_\nu,s_2) & \cdots & K(s_\nu,s_\nu) \end{vmatrix}. \tag{6}$$

We want to determine the order μ of $D(\lambda)$ under the above hypotheses on the kernel $K(x,y)$. Let us assume, therefore, that K is uniformly Hölder continuous with respect to the second variable, that is

$$|\, K(x,y) - K(x,z)\, | < A\, |\, y - z\, |^{\alpha} \tag{7}$$

with $\alpha > \frac{1}{2}$.

To estimate the coefficients d_ν,* we first transform the determinant in (6) by subtracting the second column from the first, the third column from the second, etc., thus obtaining

* This estimation was originally performed by Fredholm in 1903.[12]

$$\det\,(K(s_i\,,s_j)) = [(s_1 - s_2)(s_2 - s_3) \cdots (s_{\nu-1} - s_\nu)]^\alpha \cdot \begin{vmatrix} \epsilon_{11} & \epsilon_{12} & \cdots & \epsilon_{1,\nu-1} & K(s_1\,,s_\nu) \\ \epsilon_{21} & \epsilon_{22} & \cdots & \epsilon_{2,\nu-1} & K(s_2\,,s_\nu) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \epsilon_{\nu 1} & \epsilon_{\nu 2} & \cdots & \epsilon_{\nu,\nu-1} & K(s_\nu\,,s_\nu) \end{vmatrix}$$

where

$$\epsilon_{mn} = \frac{K(s_m\,,s_n) - K(s_m\,,s_{n+1})}{(s_n - s_{n+1})^\alpha}$$

$$m = 1,2,\cdots,\nu; \qquad n = 1,2,\cdots,\nu - 1.$$

Since $|\,\epsilon_{mn}\,| < A$ by (7), and $K(x,y)$ itself is bounded by continuity, a simple application of Hadamard's inequality yields

$$|\det\,(K(s_i\,,s_j))\,| < C^\nu \nu^{\nu/2}\,|\,(s_1 - s_2)(s_2 - s_3)\cdots(s_{\nu-1} - s_\nu)\,|^\alpha, \qquad (8)$$

where C is a constant. Inasmuch as the determinant of (6) is symmetric in the s_i, we may assume in deriving a further upper bound that

$$b \geqq s_1 \geqq s_2 \geqq \cdots \geqq s_\nu \geqq a.$$

The right side of (8) is then maximized by spacing the s_i uniformly between a and b. As a consequence we obtain

$$|\det\,(K(s_i\,,s_j))\,| < C^\nu \nu^{\nu/2}\left[\frac{b-a}{\nu-1}\right]^{\alpha(\nu-1)} \qquad (\nu > 1)$$

from which it follows that the estimate*

$$|\,d_\nu\,| < (\text{const.})^\nu\,\nu^{-\nu(\alpha+\frac{1}{2})} \qquad (9)$$

is valid for the coefficients of the power series of (4).

The relation (9) implies that the order of the entire function $D(\lambda)$ satisfies

$$\mu < \frac{1}{\alpha + \frac{1}{2}}$$

which becomes less than 1 for $\alpha > \frac{1}{2}$. Since

$$d_1 = -\int_a^b K(s_1\,,s_1)\,ds_1 = -\mathrm{tr}(K) \qquad (10)$$

* We have used Stirling's expansion for the factorial function.

does not vanish by hypothesis, $D(\lambda)$ must be a *nonconstant* entire function of order less than 1 and hence must have at least one zero (see Ref. 13, p. 22 ff). It follows then that the integral equation has at least one eigenvalue.* Q.E.D.

For entire functions of finite order a general product expansion follows from the Hadamard factorization theorem. In view of the above results, therefore, $D(\lambda)$ may be written as the canonical product

$$D(\lambda) = \prod_{\nu=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_\nu}\right) \tag{11}$$

where $\lambda_1, \lambda_2, \cdots$ are the zeros of $D(\lambda)$ arranged in order of increasing modulus. This expansion converges absolutely and uniformly on compacta.

If we differentiate the two representations of $D(\lambda)$ given by (4) and (11), set $\lambda = 0$, and make use of (10) we obtain

$$\operatorname{tr}(K) = \sum_{\nu=1}^{\infty} \frac{1}{\lambda_\nu}. \qquad (12)\dagger$$

Thus kernels Hölder continuous in either variable with exponent $\alpha > \frac{1}{2}$ are one of the overlapping categories of nondegenerate kernels for which the expansion (11) and hence the relation (12) is valid. Other classes include

(*i*) Hermitian kernels with only a finite number of eigenvalues of one sign or the other (Mercer's Theorem), and

(*ii*) composite kernels of the form $K = K_1K_2$ (Lalesco's result; see Ref. 15).

IV. ONE-DIMENSIONAL EQUATIONS FOR THE OPTICAL MASER

Careful analysis of an idealized diffraction model for the optical maser results in the following coupled integral equations for typical field quantities, such as the current densities:[5]

$$\begin{aligned} \Phi_1(x_1,y_1) &= \lambda_1 \int_{S_2} K_{12}(x_1,y_1;x_2,y_2)\, \Phi_2(x_2,y_2)\, dx_2dy_2, \\ \Phi_2(x_2,y_2) &= \lambda_2 \int_{S_1} K_{21}(x_2,y_2;x_1,y_1)\Phi_1(x_1,y_1)\, dx_1dy_1, \end{aligned} \tag{13}$$

* If the order μ of the Fredholm determinant $D(\lambda)$ satisfies $0 < \mu < 1$ then we can conclude that there exists a countably infinite set of eigenvalues of the equation (3) since entire functions of nonintegral order have an infinite set of zeros.

† This relation is not generally valid for arbitrary $\mathfrak{L}^2$ kernels. In fact it may be inferred from results of Salem[14] that there are continuous symmetric kernels with denumerably many eigenvalues for which $\sum_{\nu=1}^{\infty} (1/\lambda_\nu)$ does not even exist.

where

$$K_{12}(x_1, y_1; x_2, y_2) = K_{21}(x_2, y_2; x_1, y_1)$$
$$= \exp ik\,[(x_1 - x_2)^2 + (y_1 - y_2)^2 + h_1(x_1, y_1) + h_2(x_2, y_2)]. \tag{14}$$

In these relations, S_1 and S_2 are the mirror surfaces, $h_1(x_1, y_1)$ and $h_2(x_2, y_2)$ represent their respective departures from parallel planes, and k is a dimensionless parameter containing the wavelength as well as various geometrical dimensions such as the average mirror separation.

The solutions $\Phi_{1,2}$ of (13) are eigenfunctions which describe the field distributions at the reflectors of the possible normal modes of laser oscillation; λ_1 and λ_2 are the corresponding eigenvalues that specify the loss and phase shift which a propagating wave undergoes between the reflecting surfaces. Note in particular that these coupled equations (13) are single-transit relations; that is, they give the field at each mirror in terms of the reflected field at the other. They can, of course, be combined into a single integral equation with a composite kernel.

In the derivation of the above relations, the active maser material is assumed to be infinite in extent, homogeneous and isotropic. Diffraction effects at the edge of the reflecting surfaces are neglected. Moreover, the separation between the reflectors, as well as the radii of curvature of these surfaces, is taken to be large compared to typical transverse dimensions.

Although the integral equations (13) are two-dimensional, the preponderance of analytic work on this problem has been under the additional assumption that the laser kernels (14) can be adequately approximated by a product of functions of single variables. If the field quantities are correspondingly decomposable, namely if

$$\Phi(x,y) = X(x)\,Y(y)$$

or

$$\Phi(r,\theta) = R(r)\,\Theta(\theta),$$

the general problem may be reduced to consideration of integral equations in one independent variable. These equations take the form

$$\varphi_1(x_1) = \lambda_1 \int_{a_2}^{b_2} K_{12}(x_1,x_2)\,\varphi_2(x_2)\,dx_2,$$
$$\varphi_2(x_2) = \lambda_1 \int_{a_1}^{b_1} K_{21}(x_2,x_1)\,\varphi_1(x_1)\,dx_1, \tag{15}$$

with the single-transit kernels K given, for instance, by

$$K_{12}(x_1, x_2) = K_{21}(x_2, x_1) = \exp ik\,[(x_1 - x_2)^2 + p_1(x_1) + p_2(x_2)]. \quad (16)$$

Since these kernels are in general continuously differentiable functions of their arguments, we can use the main theorem of the preceding section directly to show the existence of eigenvalues whenever the trace is nonvanishing.

V. ANALYSIS OF SPECIAL REFLECTOR CONFIGURATIONS

In this section we examine the kernels associated with several well-known practical interferometer geometries and establish the general nonvanishing character of the traces.

5.1 *Plane Reflectors*

For rectangular plane reflecting surfaces which are mirror images of each other, the integral equation of interest is

$$\varphi(x) = \lambda \int_{-1}^{1} K(x,x')\, \varphi(x')\, dx'$$

with kernel

$$K(x,x') = \exp\,[ik(x - x')^2].$$

It is a trivial matter to verify that the trace of this differentiable function is 2, from which we infer the existence of eigenvalues of the above integral equation. In fact, for this kernel one can use parity arguments to show there are at least two eigenvalues for $k \neq 0$.[8]

5.2 *Circular Plane Reflectors*

When the plane reflecting surfaces have circular cross section the integral equation kernel becomes[4]

$$K(r,r') = J_n(krr')\, r' \exp\,[ik(r^2 + r'^2)/2] \qquad r,r' \text{ in } [0,1]$$

where J_n is a Bessel function of the 1st kind and nth order. The integer index n is indicative of an angular variation $e^{in\theta}$ which has been suppressed. The trace of the kernel K is proportional to

$$T_n = \int_0^1 J_n(k\tau)\, e^{ik\tau}\, d\tau,$$

and thus the nonvanishing of T_n will imply the existence of normal modes of oscillation for the system. Using an integral representation of J_0, for example, we obtain

$$\begin{aligned} T_0 &= \int_0^1 \frac{1}{\pi} \int_{-1}^1 e^{ik\tau t}(1 - t^2)^{-\frac{1}{2}} e^{ik\tau}\, dt d\tau \\ &= \frac{1}{\pi} \int_{-1}^1 (1 - t^2)^{-\frac{1}{2}} \int_0^1 e^{ik\tau(1+t)}\, d\tau dt \\ &= \frac{1}{ik\pi} \int_{-1}^1 (1 - t^2)^{-\frac{1}{2}} \left[\frac{e^{ik(1+t)} - 1}{1 + t} \right] dt. \end{aligned}$$

It is easy to see that for Im $(k) \geqq 0$, for instance, the real part of the integrand is negative almost everywhere. Hence the trace of the kernel does not vanish in this particular case, and we may draw our conclusion as to the existence of eigenvalues.

5.3 *Other Reflector Configurations*

For certain kernels, of course, there is little to be learned from application of our results on Hölder continuous functions. Such is the situation regarding the kernel

$$K(x,x') = e^{ikxx'}$$

associated with mirror image reflectors of square cross section, each having the curvature of a sphere centered at the center of the other reflector. As noted previously, this particular kernel gives rise to eigenvalues and eigenfunctions which may be expressed in terms of prolate spheroidal wave functions.[2]

At the same time, however, it is advantageous that the eigenvalue existence question can be easily settled for more general reflector geometries which do not exhibit as beneficial analytic properties as the confocal configuration. In particular, the kernel

$$K(x,x') = \exp ik[(x - x')^2 + p(x) + p(x')],$$

which pertains to mirror image square reflectors with shape function proportional to $p(x)$, has eigenvalues if

$$T = \int K(x,x)\, dx = \int e^{2ik\, p(x)}\, dx \neq 0.$$

The vanishing of T for practical geometries would certainly seem to be the exception rather than the rule.

5.4 *Composite Kernels*

The above examples show how our main theorem can be used to establish simply yet rigorously the existence of eigenvalues and eigenfunctions for the one-dimensional laser kernels arising when the reflectors are mirror images of each other. For more generalized configurations in which the reflecting surfaces may be of unequal size and curvature, the applicable kernels are of a composite nature (see Refs. 3 and 5). In view of this one might choose to reason from Lalesco's results on composite kernels mentioned earlier rather than from our main theorem. This would be an acceptable method of attack. However, since (12), relating the trace to the sum of reciprocal eigenvalues, is valid in both situations, a verification of the nonvanishing character of the kernel traces is needed in either case. As a last illustrative example we shall provide this verification for the integral equations associated with asymmetric spherical reflectors of arbitrary curvature.

Let $a_1 = -b_1$, $a_2 = -b_2$, $p_1(x_1) = \alpha x_1^2$ and $p_2(x_2) = \beta x_2^2$. The one-dimensional integral equations (15) then become appropriate for analysis of an idealized interferometer having two rectangular mirrors of unequal size and unequal curvatures. As usual, these two coupled equations (15) can be combined into a single integral equation for either φ_1 or φ_2. Moreover, this new integral equation may then be split apart into two subsidiary equations according to whether the eigenfunction modes are even or odd. The kernels resulting from this division are given by

$$K_{e,o}(x,y) = 2\int_{-b_2}^{b_2} \begin{Bmatrix} \cos 2kx'y \\ i \sin 2kx'y \end{Bmatrix} \exp\{ik[(x^2 + y^2)(1 + \alpha) + 2x'^2(1 + \beta) - 2x'x]\}\, dx' \tag{17}$$

and have traces

$$\mathrm{tr}(K_{e,o}) = 2\int_0^{b_1}\int_{-b_2}^{b_2} \begin{Bmatrix} \cos 2kx'x \\ i \sin 2kx'x \end{Bmatrix} \exp\{2ik[x^2(1 + \alpha) + x'^2(1 + \beta) - x'x]\}\, dx'dx. \tag{18}$$

It is easy to show that at least one of these traces is different from zero for real k and arbitrary curvatures α, β.

Note first that

$$\mathrm{tr}(K_e) - \mathrm{tr}(K_o) = 2\int_0^{b_1}\int_{b_2}^{b_2} \exp\{2ik[x^2(1 + \alpha) + x'^2(1 + \beta)]\}\, dx'dx$$

$$= 4\left[\int_0^{b_1} \exp\{2ik(1+\alpha)x^2\}dx\right] \cdot \left[\int_0^{b_2} \exp\{2ik(1+\beta)x'^2\}\,dx'\right].$$

Now neither of the two bracketed terms on the right-hand side vanishes, since the real parts of these Fresnel integrals are positive for real k, α, β. Thus, the difference between the two traces, and hence at least one of the traces itself, is different from zero [one suspects, of course, that both of the kernels (17) have nonvanishing traces]. Although this argument gives no measure of the loss to be expected with any individual eigenmode, it does show that normal modes of oscillation exist for this arbitrary asymmetric spherical configuration.

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