Interaction of Adaptive Antenna Arrays in an Arbitrary Environment

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This paper deals with adaptive transmitting arrays in which the excitations of the elements are varied in response to a pilot field incident on the array from a distant source. General theorems, some quite simple, are obtained relating to optimal power transfer from an adaptive array in an arbitrary reciprocal medium to either a single receiver or a receiving array. We assume first that the amplitudes and phases of the transmitting elements are separately adjustable, and afterward that only the phases are adjustable. The results involve in particular the matrix which represents the pilot fields produced at the elements of the transmitting array by currents at the locations of the receiving elements. In some important special cases, optimal power transfer results from making the phase of each transmitting element equal to the negative of the phase of the pilot field at that element.

We also consider the dynamic behavior of two adaptive arrays which simultaneously transmit and receive, the phases on transmission being made equal to the negatives of the received phases. Analysis of an idealized model indicates that the arrays will reach a unique steady state which is in practical cases identical with or very close to the condition for optimal power transfer. Some numerical simulations of 2- and 3-element interacting arrays have been made to show how such arrays approach an essentially steady state under moderately realistic assumptions.

I. INTRODUCTION AND SUMMARY

A number of recent papers[1,2] have dealt with adaptive antenna arrays, also called self-steering or retrodirective arrays. In an adaptive transmitting array, the excitations of the individual elements are electronically varied in response to a pilot field incident on the array from a distant terminal, in order to steer the beam to the terminal which is originating the pilot signal. It is easy to see that in free space the required steering can be accomplished by making the phase of each element equal to the negative of the phase of the pilot beam at the given element. Cutler, Kompfner, and Tillotson[1] and others[2] have shown how phase reversal can be obtained using frequency conversion techniques.

This paper deals with general adaptive transmitting arrays in an arbitrary environment. The transmission medium need not be homogeneous or isotropic, but it is assumed to be linear and symmetric and to be time-invariant, at least over intervals comparable to the propagation time between transmitter and receiver. We are concerned in particular with the conditions for optimal power transfer from an adaptive transmitting array to either a single receiver or a receiving array. We shall also investigate the transient and steady-state behavior of two interacting adaptive arrays, each of which simultaneously transmits and adjusts the excitations of its elements in response to the field received from the other array.

In Section II we consider an adaptive array in which the amplitudes and phases of the excitations of the individual elements are separately adjustable, but the total radiated power is fixed. Such an array is easier to treat mathematically than one in which the excitation amplitudes are all fixed and only the phases are variable, even though the latter array might be easier to build. In the most general case, the power radiated by the transmitting array is a positive definite Hermitian form in the element excitations, and the received power is a positive definite Hermitian form in the electric fields at the elements of the receiving array. The distribution of excitations which maximizes the ratio of received to radiated power is the eigenvector corresponding to the largest eigenvalue of a certain pencil of Hermitian matrices. The matrices in question are constructed from the impedance matrix of the transmitting array, the admittance matrix of the receiving array, and a Green's function matrix of pilot fields produced at the transmitting elements by currents at the receiver locations. The results simplify considerably if the elements of each array are uncoupled and are identical among themselves. In particular, if the receiver consists of but a single element, and if the transmitter elements are identical and uncoupled, then the optimal excitation of each element is merely proportional to the complex conjugate of the pilot field at that element.

Section III contains a brief discussion of the problem of maximizing the power transferred from an arbitrary transmitting array to an arbitrary receiving array when the excitation amplitudes are fixed and only the phases are adjustable. Maximum power is always conveyed to a *single* receiver by reversing the phase of the pilot field at each element of the transmitting array. This phase reversal principle, first recognized for free-space transmission, is thus shown to be valid for an arbitrary transmission medium. For multielement receivers an explicit solution is not given; but an example shows that even when each array consists of identical, uncoupled elements, maximum power transfer does *not* generally correspond to reversing the phase of the total pilot field at the transmitter elements.

Section IV deals with the interaction of two adaptive arrays or, in principle, the interaction of an adaptive radar array with itself. A mathematical model is set up, in which each array transmits constant power and continuously adjusts the excitations of its own elements to be proportional to the complex conjugate of the incident field. A single delay time is taken to represent the transmission delay between the two arrays. The transient behavior of this model turns out to be quite simple, and it is shown that in general, excluding mathematically pathological cases, the two arrays reach an equilibrium configuration which depends only on the Green's function (pilot field) matrix corresponding to the given geometry and transmission medium. In the most general case the equilibrium configuration is not the same as the condition for optimal power transfer derived in Section II; but it *is* the condition for optimal power transfer in the important special case when the elements of each array are identical among themselves and the interelement coupling is zero. If the elements are nearly identical and the mutual impedances are small compared to the self-impedances, then the equilibrium configuration should be nearly the same as the configuration for optimal power transfer.

Numerical simulations of the transient behavior of 2- and 3-element interacting adaptive arrays are described in Section V, both for the case of simultaneous phase and amplitude variations, and for the case of phase variations only. The simulations also include the effects of small differences in the interelement delay times compared to the average delay between the arrays. Random choices are made for the elements of the Green's function matrix and for all pairs of interelement delays. Simulations of 50 pairs of 2-element arrays and 25 pairs of 3-element arrays indicate that arrays with only phase adjustment approach a steady state about as quickly as arrays with both phase and amplitude adjustment (of course, the two steady states are not the same). Interelement delay differences which are small compared to the average interelement delay produce small fluctuations about the steady state which would be achieved for equal delays.

The results obtained in this paper depend only on the linearity, symmetry, and time-invariance of the transmission medium; in particular, they do not involve calculating any antenna patterns. Pattern calculations would be necessary if one wished to get numerical values for maximal power transfer, or to estimate the radiated fields in unwanted directions. Furthermore, the analysis is essentially for a single frequency; variations in phase and amplitude are assumed to be very slow compared to the transmission times involved. It would be worthwhile to study the behavior of adaptive arrays over a finite frequency band, but such a study is outside the scope of the present paper.

II. OPTIMAL POWER TRANSFER BETWEEN ARBITRARY ANTENNA ARRAYS

Consider a transmitting array and a receiving array embedded in an arbitrary linear, time-invariant medium, as in Fig. 1. The medium may be inhomogeneous and anisotropic, but the permeability, permittivity, and conductivity tensors at any point are assumed to be symmetric. (This rules out ferrites and plasmas in the presence of a magnetic field.) All fields are assumed to be time-harmonic with angular frequency ω, the time dependence $\exp i\omega t$ being suppressed. For simplicity the individual radiators and receivers are taken to be elemental electric dipoles, although they could equally well be elemental current loops. The assumption of dipole sources is not a major restriction, since the dipoles could be used, for example, together with microwave circuitry to feed aperture-type radiators such as elemental horns.

Let the transmitting array have M elements and let the complex excitation of the ith element be $I_{1,i}$. Physically $I_{1,i}$ may be regarded as the electric moment of an elemental current, having the dimensions of ampere-meters. The M-component vector

$$\mathbf{I}_1 = (I_{1,1}, I_{1,2}, \cdots, I_{1,M}), \tag{1}$$

whose components are complex scalars, will be called the excitation of Array 1. Similarly let the receiving array have N elements, and let the

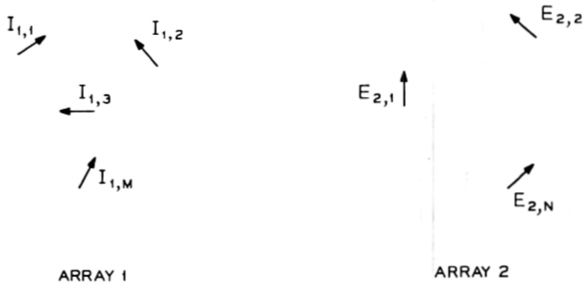

Fig. 1 — Schematic representation of arbitrary transmitting and receiving arrays of electric dipoles.

complex electric field component at the location and in the direction of the jth element be $E_{2,j}$. Then the electric field at Array 2 is the N-component vector

$$\mathbf{E}_2 = (E_{2,1}, E_{2,2}, \cdots, E_{2,N}). \tag{2}$$

If Array 2 is transmitting and Array 1 is receiving, we define the vectors $\mathbf{I}_2$ and $\mathbf{E}_1$ in an analogous way.

The total power P_T radiated by the transmitter over all space is given by the Hermitian form

$$P_T = \tfrac{1}{2}(\mathbf{Z}_1\mathbf{I}_1, \mathbf{I}_1), \tag{3}$$

where $\mathbf{Z}_1$ is an $M \times M$ positive definite Hermitian matrix and $(\mathbf{x},\mathbf{y})$ represents the scalar product of two vectors $\mathbf{x}$ and $\mathbf{y}$. A fuller discussion of notation and of the properties of Hermitian forms is given in Appendix A.

In principle the radiation impedance matrix $\mathbf{Z}_1$ may be determined from experimental measurements, or it may be calculated from the fields of the radiating elements. For example, if the field due to unit excitation of the ith element at a great distance R from all currents and material media is $\mathcal{E}_{1,i}/R$, then by integrating the Poynting vector due to the whole array over a large sphere we find that the total radiated power is given by an expression of the form (3), with

$$Z_{1,ij} = \frac{1}{\eta}\int \mathcal{E}_{1,i}\cdot\mathcal{E}_{1,j}{}^{*}d\Omega, \qquad i = 1,2, \cdots, M; \qquad j = 1,2, \cdots, M, \tag{4}$$

where η is the characteristic impedance of free space and $d\Omega$ is an element of solid angle.

We now assume that the power P_R received by Array 2 is given by a Hermitian form in $\mathbf{E}_2$, the electric field which would exist at Array 2 if its elements were open-circuited. Thus we write

$$P_R = \tfrac{1}{2}(\mathbf{Y}_2\mathbf{E}_2, \mathbf{E}_2), \tag{5}$$

where $\mathbf{Y}_2$ is an $N \times N$ positive definite Hermitian matrix. Equation (5) is equivalent to the assumption that the transmitter field is independent of whether or not currents are flowing in the elements of Array 2, i.e., that the back reaction of Array 2 on Array 1 is negligible. This will be a very good approximation in the practical case where the arrays are far apart, so that P_R is a very small fraction of P_T.

The field at Array 2 is related to the excitation of Array 1 by the Green's function matrix $\mathbf{\Gamma}$; thus

$$\mathbf{E}_2 = \mathbf{\Gamma}\mathbf{I}_1, \tag{6}$$

where $\mathbf{\Gamma}$ is an $N \times M$ matrix and Γ_{ij} represents the field at the ith element of Array 2 due to unit excitation of the jth element of Array 1. A basic reciprocity theorem for linear, time-invariant, symmetric media, proved in Appendix B, guarantees that Γ_{ij} also represents the field at the jth element of Array 1 due to unit excitation of the ith element of Array 2. Thus if Array 2 has the excitation $\mathbf{I}_2$, the field at Array 1 is given by

$$\mathbf{E}_1 = \mathbf{\Gamma}'\mathbf{I}_2 , \tag{7}$$

where $\mathbf{\Gamma}'$ is the transpose of $\mathbf{\Gamma}$.

From (5) and (6), the power received by Array 2 is

$$P_R = \tfrac{1}{2}(\mathbf{Y}_2\mathbf{\Gamma}\mathbf{I}_1 , \mathbf{\Gamma}\mathbf{I}_1) = \tfrac{1}{2}(\mathbf{\Gamma}^\dagger\mathbf{Y}_2\mathbf{\Gamma}\mathbf{I}_1 , \mathbf{I}_1), \tag{8}$$

where $\mathbf{\Gamma}^\dagger$ is the adjoint (= conjugate transpose) of $\mathbf{\Gamma}$. We wish to maximize the ratio of received power to transmitted power, which is

$$\frac{P_R}{P_T} = \frac{(\mathbf{\Gamma}^\dagger\mathbf{Y}_2\mathbf{\Gamma}\mathbf{I}_1,\mathbf{I}_1)}{(\mathbf{Z}_1\mathbf{I}_1,\mathbf{I}_1)} . \tag{9}$$

But the right side is the quotient of two Hermitian forms in which the denominator is positive definite; and it is well known (see Appendix A) that the maximum value of the quotient is the largest eigenvalue λ_M of the pencil of matrices $\mathbf{\Gamma}^\dagger\mathbf{Y}_2\mathbf{\Gamma} - \lambda\mathbf{Z}_1$. The desired eigenvalue is the largest root of the equation

$$\det (\mathbf{\Gamma}^\dagger\mathbf{Y}_2\mathbf{\Gamma} - \lambda\mathbf{Z}_1) = 0; \tag{10}$$

and the corresponding eigenvector, which maximizes the right side of (9), is any nonzero solution of the system of equations

$$\mathbf{\Gamma}^\dagger\mathbf{Y}_2\mathbf{\Gamma}\mathbf{I}_1 - \lambda_M\mathbf{Z}_1\mathbf{I}_1 = 0. \tag{11}$$

The foregoing equations simplify in a special case which will be important in what follows, namely when all the self-impedances and self-admittances are equal and all the mutual impedances and admittances are zero. In this case we may write

$$\mathbf{Z}_1 = R_1\mathbf{1}_M , \qquad \mathbf{Y}_2 = G_2\mathbf{1}_N , \tag{12}$$

where R_1 and G_2 are real scalars and $\mathbf{1}_M$ and $\mathbf{1}_N$ are unit matrices of orders M and N respectively. Then (9) becomes

$$\frac{P_R}{P_T} = \frac{G_2}{R_1}\frac{(\mathbf{\Gamma}^\dagger\mathbf{\Gamma}\mathbf{I}_1,\mathbf{I}_1)}{(\mathbf{I}_1,\mathbf{I}_1)} , \tag{13}$$

and the maximum value of the ratio is proportional to the largest eigenvalue of the matrix $\mathbf{\Gamma}^\dagger \mathbf{\Gamma}$, that is, the largest root λ_M of

$$\det\left(\mathbf{\Gamma}^\dagger \mathbf{\Gamma} - \lambda \mathbf{1}_M\right) = 0. \tag{14}$$

The excitation corresponding to maximum power transfer is any nonzero solution of

$$\mathbf{\Gamma}^\dagger \mathbf{\Gamma} \mathbf{I}_1 - \lambda_M \mathbf{I}_1 = 0. \tag{15}$$

The optimal transmitter excitation given by (14) and (15) is one which can exist when both arrays are transmitting and the excitation of each element of each array is proportional to the complex conjugate of the field incident on the element from the other array. Suppose, for example, that

$$\mathbf{I}_1 = M_1 \mathbf{E}_1^*, \qquad \mathbf{I}_2 = M_2 \mathbf{E}_2^*, \tag{16}$$

where M_1 and M_2 are complex scalars and $\mathbf{E}_1$ and $\mathbf{E}_2$ are related to $\mathbf{I}_2$ and $\mathbf{I}_1$ by the Green's function matrix, as in (6) and (7). Then it is easy to show that $\mathbf{I}_1$ must satisfy

$$\mathbf{I}_1 = M_1 M_2^* \mathbf{\Gamma}^\dagger \mathbf{\Gamma} \mathbf{I}_1 . \tag{17}$$

A nonvanishing solution of (17) exists if and only if

$$M_1 M_2^* = 1/\lambda, \tag{18}$$

where λ is an eigenvalue of $\mathbf{\Gamma}^\dagger \mathbf{\Gamma}$, that is, a root of the determinantal equation (14). Although steady-state excitations satisfying (16) are mathematically possible when λ is any eigenvalue of $\mathbf{\Gamma}^\dagger \mathbf{\Gamma}$, it is shown in Section IV that the system is unstable unless λ is the largest eigenvalue, and that the excitations corresponding to the largest eigenvalue are in fact the excitations toward which two interacting adaptive arrays tend.

We shall now consider optimal power transfer in the special case where Array 2 consists of but a single receiver. Then $\mathbf{Z}_1$ is an $M \times M$ matrix, $\mathbf{Y}_2$ is a 1×1 matrix, i.e., a scalar quantity G_2, and $\mathbf{\Gamma}$ is a $1 \times M$ matrix. Equation (10) is therefore equivalent to

$$\begin{aligned}\det\left(\lambda G_2^{-1} \mathbf{Z}_1 - \mathbf{\Gamma}^\dagger \mathbf{\Gamma}\right) &= \det\left(\lambda G_2^{-1} \mathbf{Z}_1\right) \det\left(\mathbf{1}_M - \{\lambda G_2^{-1} \mathbf{Z}_1\}^{-1} \mathbf{\Gamma}^\dagger \mathbf{\Gamma}\right) \\ &= \lambda^M G_2^{-M} \det \mathbf{Z}_1 \det\left(\mathbf{1}_1 - \mathbf{\Gamma}\{\lambda G_2^{-1} \mathbf{Z}_1\}^{-1} \mathbf{\Gamma}^\dagger\right) \\ &= (\lambda/G_2)^{M-1}(\lambda/G_2 - \mathbf{\Gamma} \mathbf{Z}_1^{-1} \mathbf{\Gamma}^\dagger) \det \mathbf{Z}_1 \\ &= 0,\end{aligned} \tag{19}$$

where in the second step we have made use of a lemma due to Sandberg,[3] which is stated and proved in Appendix C. It follows that the only nonzero eigenvalue of (19) is

$$\lambda_M = G_2 \mathbf{\Gamma} \mathbf{Z}_1^{-1} \mathbf{\Gamma}^\dagger. \tag{20}$$

The corresponding eigenvector is

$$\mathbf{I}_1^{(M)} = \mathbf{Z}_1^{-1} \mathbf{\Gamma}^\dagger \tag{21}$$

up to a constant factor, since it is easy to see that

$$\begin{aligned} \mathbf{\Gamma}^\dagger G_2 \mathbf{\Gamma} \mathbf{I}_1^{(M)} - \lambda_M \mathbf{Z}_1 \mathbf{I}_1^{(M)} &= \mathbf{\Gamma}^\dagger G_2 \mathbf{\Gamma} \mathbf{Z}_1^{-1} \mathbf{\Gamma}^\dagger - \lambda_M \mathbf{Z}_1 \mathbf{Z}_1^{-1} \mathbf{\Gamma}^\dagger \\ &= \mathbf{\Gamma}^\dagger \lambda_M - \lambda_M \mathbf{\Gamma}^\dagger = 0. \end{aligned} \tag{22}$$

If the elements of the transmitting array are uncoupled, then the mutual radiation impedances vanish, and $\mathbf{Z}_1$ and $\mathbf{Z}_1^{-1}$ are diagonal matrices. The optimal excitation of the jth element is then

$$I_{1,j} = \Gamma_{1j}^*/R_{jj}, \qquad j = 1, 2, \cdots, M, \tag{23}$$

up to a constant factor, where Γ_{1j}^* is the complex conjugate of the pilot field produced at the jth element by a dipole at the location of the receiver, and R_{jj} is the radiation resistance of the jth element. If all the radiation resistances are equal, then since the eigenvector is determined only up to a multiplicative constant, we may take

$$I_{1,j} = \Gamma_{1j}^*, \qquad j = 1, 2, \cdots, M; \tag{24}$$

in other words, the excitation is merely proportional to the complex conjugate of the pilot field.

We have shown that if the mutual radiation impedances of the transmitter elements are zero, then the field at the receiver is maximized, for constant radiated power, when the phase of the excitation of each transmitter element is the negative of the phase of the pilot field. If, however, the transmitter elements are coupled by their radiation fields, so that the impedance matrix $\mathbf{Z}_1$ is not diagonal, the optimal excitations are given by (21) and do not generally satisfy the phase reversal condition.

III. OPTIMAL POWER TRANSFER WITH PHASE ADJUSTMENTS ONLY

If the amplitudes of the transmitter excitations are fixed but the phases are adjustable, we wish to maximize the received power,

$$P_R = \tfrac{1}{2}(\mathbf{\Gamma}^\dagger \mathbf{Y}_2 \mathbf{\Gamma} \mathbf{I}_1, \mathbf{I}_1) = \tfrac{1}{2}(\mathbf{Y}_2 \mathbf{\Gamma} \mathbf{I}_1, \mathbf{\Gamma} \mathbf{I}_1), \tag{25}$$

when

$$I_{1,j} = r_j e^{i\theta_j}, \qquad j = 1, 2, \cdots, M, \tag{26}$$

and the r_j are fixed but the θ_j are at our disposal.

If there is only one receiver ($N = 1$), the solution is immediate. We have to maximize

$$P_R = \tfrac{1}{2} G_2 \left| \sum_{j=1}^{M} \Gamma_{1j} I_{1,j} \right|^2 \tag{27}$$

by adjusting the phases of the $I_{1,j}$; and it is clear that the modulus of the sum will be greatest when the phases of all the summands are equal, that is, when

$$\arg I_{1,j} \equiv \theta_j = -\arg \Gamma_{1j} + \text{constant}, \qquad j = 1, 2, \cdots, M. \tag{28}$$

In other words, the phase of the jth transmitting element should be the negative of the phase of the pilot field produced at that element by a radiating element at the position of the receiver. This result is independent of the nature of the transmission medium, subject only to the requirements of linearity, time-invariance, and symmetry, and it is independent of the position of the receiver relative to the transmitting array.

For a two-element transmitter ($M = 2$) and an arbitrary receiving array, we have

$$2P_R = (\mathbf{\Gamma}^\dagger \mathbf{Y}_2 \mathbf{\Gamma})_{11} r_1^2 + (\mathbf{\Gamma}^\dagger \mathbf{Y}_2 \mathbf{\Gamma})_{12} r_1 r_2 e^{i(\theta_2 - \theta_1)} + (\mathbf{\Gamma}^\dagger \mathbf{Y}_2 \mathbf{\Gamma})_{21} r_1 r_2 e^{i(\theta_1 - \theta_2)} + (\mathbf{\Gamma}^\dagger \mathbf{Y}_2 \mathbf{\Gamma})_{22} r_2^2. \tag{29}$$

Since $\mathbf{\Gamma}^\dagger \mathbf{Y}_2 \mathbf{\Gamma}$ is Hermitian, the right side of (29) is maximized by taking

$$\arg I_{1,j} - \arg I_{2,j} \equiv \theta_1 - \theta_2 = \arg (\mathbf{\Gamma}^\dagger \mathbf{Y}_2 \mathbf{\Gamma})_{12}, \tag{30}$$

and this is the condition for optimal power transfer if the transmitting array has only two elements.

A complete analytic solution of the problem of maximizing P_R for an arbitrary transmitter with $M \geqq 3$ and an arbitrary receiver with $N \geqq 2$ has not been found, although since P_R is a continuous, periodic function of each of the θ_j, it is obvious that a maximum exists and could be located as accurately as desired by an iterative numerical procedure.

In contrast to the situation for arrays with both amplitudes and phases adjustable, the condition for optimal power transfer between multielement arrays with fixed excitation amplitudes is not generally satisfied by making the phase of each element equal to the negative of the phase of the field incident from the other array, even if it be assumed that the array elements are uncoupled and are identical among themselves. As a counterexample, consider the case in which each array has two elements, and $\mathbf{Y}_2$ is a multiple of the unit matrix. Let the element currents be

$$\begin{aligned} I_{1,1} &= re^{i\theta_1}, \qquad I_{2,1} = \rho e^{i\varphi_1}, \\ I_{1,2} &= re^{i\theta_2}, \qquad I_{2,2} = \rho e^{i\varphi_2}, \end{aligned} \tag{31}$$

where r and ρ are real and positive, and the phases are at our disposal. Equations (6) and (7) give

$$\begin{aligned} E_{2,1} &= r(\Gamma_{11}e^{i\theta_1} + \Gamma_{12}e^{i\theta_2}), \\ E_{2,2} &= r(\Gamma_{21}e^{i\theta_1} + \Gamma_{22}e^{i\theta_2}), \\ E_{1,1} &= \rho(\Gamma_{11}e^{i\varphi_1} + \Gamma_{21}e^{i\varphi_2}), \\ E_{1,2} &= \rho(\Gamma_{12}e^{i\varphi_1} + \Gamma_{22}e^{i\varphi_2}). \end{aligned} \tag{32}$$

The phase reversal condition leads to the following pair of simultaneous equations:

$$\begin{aligned} \theta_1 - \theta_2 &= \arg E_{1,2} - \arg E_{1,1} = \arg \frac{\Gamma_{12}e^{i(\varphi_1-\varphi_2)} + \Gamma_{22}}{\Gamma_{11}e^{i(\varphi_1-\varphi_2)} + \Gamma_{21}} \\ \varphi_1 - \varphi_2 &= \arg E_{2,2} - \arg E_{2,1} = \arg \frac{\Gamma_{21}e^{i(\theta_1-\theta_2)} + \Gamma_{22}}{\Gamma_{11}e^{i(\theta_1-\theta_2)} + \Gamma_{12}}. \end{aligned} \tag{33}$$

On the other hand, the condition (30) for maximum power transfer reduces to

$$\theta_1 - \theta_2 = \arg\,(\mathbf{\Gamma}^\dagger\mathbf{\Gamma})_{12} = \arg\,(\Gamma_{11}^*\Gamma_{12} + \Gamma_{21}^*\Gamma_{22}). \tag{34}$$

Since $\mathbf{\Gamma}$ is an essentially arbitrary complex matrix, equations (33) are not equivalent to (34), although it is possible that in practical cases the two conditions will yield values of $\theta_1 - \theta_2$ which do not differ by very much.

IV. DYNAMIC BEHAVIOR OF INTERACTING ADAPTIVE ARRAYS

In this section we set up a simple model of the dynamic behavior of two adaptive arrays, each of which continuously adjusts the excitations of its own elements in response to the fields from the other array. In principle the same equations would apply to a single array interacting with itself, as a combined radar transmitter and receiver. The fundamental assumption is that the amplitudes and phases of the element currents vary so slowly, compared with the transmission time between the arrays, that a single-frequency analysis is valid.

Since in this model the excitation of an adaptive array is indeterminate in the absence of an external field, we have to use an auxiliary antenna or beacon to turn the system on. The steps are as follows: First the beacon is turned on, illuminating at least Array 1. Then Array 1 is turned on, Array 2 is turned on, and the beacon is turned off, leaving Arrays 1 and 2 to interact only with each other. It is convenient to assume that when a transmitter is switched on or off, its radiated power changes continuously, during a finite time interval, from one steady-state value to another.

First we consider arrays in which the excitation of each element is proportional to the complex conjugate of the field incident on that element, and the total radiated power is a prescribed function of time. Thus let $\mathbf{I}_1(t)$ and $\mathbf{I}_2(t)$ be the (slowly varying) complex excitations of the two arrays. We assume that the dynamic behavior of the arrays is described by the following equations:

$$I_{1,j}(t) = \mu_1(t)e^{i\vartheta_1}\left[\sum_{k=1}^{N}\Gamma_{kj}^{*}I_{2,k}^{*}(t-\tau_{kj}-\tau_1) + B_{1,j}^{*}(t-\tau_1)\right], \quad j = 1, 2, \cdots, M; \tag{35}$$

$$I_{2,k}(t) = \mu_2(t)e^{i\vartheta_2}\left[\sum_{j=1}^{M}\Gamma_{kj}^{*}I_{1,j}^{*}(t-\tau_{kj}-\tau_2) + B_{2,k}^{*}(t-\tau_2)\right], \quad k = 1, 2, \cdots, N. \tag{36}$$

In these equations, $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$ are the beacon fields, if any, at Arrays 1 and 2, τ_{kj} is the transmission delay between the jth element of Array 1 and the kth element of Array 2, τ_1 and τ_2 are constant time delays in the amplifiers of Arrays 1 and 2, ϑ_1 and ϑ_2 are constant phase shifts, and $\mu_1(t)$ and $\mu_2(t)$ are real normalization factors determined by

$$\tfrac{1}{2}(\mathbf{Z}_1\mathbf{I}_1(t),\mathbf{I}_1(t)) = P_1(t), \tag{37}$$

$$\tfrac{1}{2}(\mathbf{Z}_2\mathbf{I}_2(t),\mathbf{I}_2(t)) = P_2(t), \tag{38}$$

where the radiated powers $P_1(t)$ and $P_2(t)$ are given functions of time.

Similarly, the equations describing two arrays in which the excitation amplitudes $|I_{1,j}(t)|$ and $|I_{2,k}(t)|$ are prescribed functions of time, while the phases are continuously adjusted to satisfy the phase reversal condition, are as follows:

$$\arg I_{1,j}(t) = \vartheta_1 - \arg\left[\sum_{k=1}^{N}\Gamma_{kj}I_{2,k}(t-\tau_{kj}-\tau_1) + B_{1,j}(t-\tau_1)\right], \quad j = 1, 2, \cdots, M; \tag{39}$$

$$\arg I_{2,k}(t) = \vartheta_2 - \arg\left[\sum_{j=1}^{M} \Gamma_{kj} I_{1,j}(t - \tau_{kj} - \tau_2) + B_{2,k}(t - \tau_2)\right], \quad k = 1, 2, \cdots, N. \tag{40}$$

Before undertaking numerical simulations of the dynamic behavior of adaptive arrays, we consider an example which can be handled analytically, namely the special case of two power-limited arrays in which all the interelement delay times are equal. We obtain this case from (35) and (36) by setting

$$\tau_{kj} = \tau_3 = \text{constant}. \tag{41}$$

If we assume for simplicity that the beacon has been turned off, (35) and (36) take the form

$$\mathbf{I}_1(t) = \mu_1(t) e^{i\vartheta_1} \mathbf{\Gamma}^\dagger \mathbf{I}_2^*(t - \tau_3 - \tau_1), \tag{42}$$

$$\mathbf{I}_2(t) = \mu_2(t) e^{i\vartheta_2} \mathbf{\Gamma}^* \mathbf{I}_1^*(t - \tau_3 - \tau_2). \tag{43}$$

Eliminating $\mathbf{I}_2$ yields

$$\mathbf{I}_1(t) = \mu_1(t) e^{i\vartheta} \mathbf{\Gamma}^\dagger \mathbf{\Gamma} \mathbf{I}_1(t - \tau), \tag{44}$$

where

$$\vartheta = \vartheta_1 - \vartheta_2, \qquad \tau = \tau_1 + \tau_2 + 2\tau_3, \tag{45}$$

and the normalizing factor $\mu_1(t)$ may be expressed, if needed, in terms of the radiated power $P_1(t)$ by (37).

The Hermitian matrix $\mathbf{\Gamma}^\dagger\mathbf{\Gamma}$ is at least positive semidefinite and will have M real eigenvalues. We suppose that the eigenvalues are numbered in order of increasing size and that the largest eigenvalue is *unique;* that is,

$$0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_{M-1} < \lambda_M. \tag{46}$$

The corresponding eigenvectors $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \cdots, \mathbf{z}^{(M)}$ satisfy

$$\mathbf{\Gamma}^\dagger \mathbf{\Gamma} \mathbf{z}^{(i)} = \lambda_i \mathbf{z}^{(i)}, \qquad i = 1, 2, \cdots, M, \tag{47}$$

and may be taken as orthonormal, i.e.,

$$(\mathbf{z}^{(i)}, \mathbf{z}^{(j)}) = \delta_{ij}. \tag{48}$$

Let $\mathbf{I}_1(t)$ be expanded in terms of the $\mathbf{z}^{(i)}$, with coefficients depending, of course, on time; thus

$$\mathbf{I}_1(t) = \sum_{i=1}^{M} c_i(t) \mathbf{z}^{(i)}. \tag{49}$$

Repeated application of (44) and (47) gives

$$\mathbf{I}_1(t + n\tau) = N_n(t)e^{in\vartheta} \sum_{j=1}^{M} c_j(t)\lambda_j^n \mathbf{z}^{(j)}, \tag{50}$$

where $N_n(t)$ is again a normalization factor chosen to satisfy (37).

Now suppose that there is an interval of length τ, say $t_0 \leqq t < t_0 + \tau$, in which all the $c_i(t)$ are bounded and $c_M(t)$ is bounded away from zero. It follows from (46) that the term in $\lambda_M{}^n$ in (50) will eventually dominate all the others, and we shall have

$$\mathbf{I}_1(t + n\tau) \underset{n\to\infty}{\longrightarrow} \frac{[2P_1(t)]^{\frac{1}{2}} \exp\,[in\vartheta + \arg c_M(t)]}{(\mathbf{Z}_1\mathbf{z}^{(M)},\mathbf{z}^{(M)})^{\frac{1}{2}}}\,\mathbf{z}^{(M)}, \qquad \text{for} \quad t_0 \leqq t < t_0 + \tau. \tag{51}$$

It is easy to verify that the phase of $\mathbf{I}_1(t)$, as given by (51), is continuous at $t = t_0 + (n + 1)\tau$ if the phase of $c_M(t)$ is continuous at $t = t_0 + \tau$.

We have just proved that two power-limited adaptive arrays with equal interelement delays will reach an equilibrium state in which the excitation of Array 1 is proportional to the eigenvector belonging to the largest eigenvalue of $\mathbf{\Gamma}^\dagger\mathbf{\Gamma}$. Similarly, the equilibrium excitation of Array 2 is proportional to the eigenvector belonging to the largest eigenvalue of $\mathbf{\Gamma}\mathbf{\Gamma}^\dagger$ (again λ_M). But it was shown in Section II that the eigenvector belonging to the largest eigenvalue of $\mathbf{\Gamma}^\dagger\mathbf{\Gamma}$ corresponds to maximum power transfer when $\mathbf{Z}_1$ and $\mathbf{Y}_2$ are multiples of the unit matrix; that is, when all the mutual impedances and admittances are zero and all the self-impedances and self-admittances are equal. If this condition is approximately satisfied, as will often be the case in practice, then the equilibrium excitation should be nearly the same as the excitation for optimal power transfer.

We observe that the equilibrium excitation is unique except in the pathological case where $\mathbf{\Gamma}^\dagger\mathbf{\Gamma}$ has two or more equal eigenvalues which are larger than all the rest. Steady states in which the current distribution corresponds to one of the smaller eigenvalues of $\mathbf{\Gamma}^\dagger\mathbf{\Gamma}$ are mathematically possible, but are unstable. Also, if the arrays are moved with respect to each other or if the transmission medium changes (either case would correspond to changing the Green's function matrix), the final equilibrium state depends only on the final positions of the arrays and the properties of the transmission medium, and not at all on how the situation was reached.

It should be pointed out that the foregoing argument does not apply, at least in its present form, to fixed-amplitude arrays with only phases adjustable. Clearly there are extreme cases in which the behavior of fixed-amplitude arrays will be qualitatively different from that of power-limited arrays. For example, if $\mathbf{\Gamma}$ is diagonal, so that each element of Array 1 is coupled to only one element of Array 2, then power-limited arrays will ultimately cut out all of the elements except for the pair which is most closely coupled; but the elements of fixed-amplitude arrays will go on indefinitely talking to each other in pairs, with no particular phase relationship between the elements of different pairs. Nevertheless, the numerical simulations of the next section indicate that in typical cases fixed-amplitude arrays do settle down to a steady state about as quickly as power-limited ones. As yet, however, no mathematical theorem has been proved about the steady-state behavior of fixed-amplitude arrays.

V. NUMERICAL SIMULATIONS

Because the equations describing the dynamic behavior of interacting adaptive arrays generally do not lend themselves to analytic treatment, we have made a few numerical simulations of 2- and 3-element arrays on an IBM 7094, in order to get some feeling for the possible behavior of interacting adaptive arrays in practice. Since these simulations were only computational experiments, no physical significance is to be attached to the specific numerical results.

We shall first describe the method of simulation, then show the outcome of a typical calculation, and finally summarize the results of the whole study.

For each simulation we selected the elements of the 2×2 or 3×3 matrix $\mathbf{\Gamma}$ according to the following scheme: We set

$$\Gamma_{jk} = G_{jk} \exp\left(-i\pi n_{jk}/5\right), \tag{52}$$

where G_{jk} was a random number selected with equal probability from the set $\{1,2,4,8\}$, and n_{jk} was selected with equal probability from the set $\{-5, -4, \cdots, 5\}$. For the interelement delay times we took

$$\tau_{jk} = 20 + n_{jk} . \tag{53}$$

As a matter of interest, we also computed the Hermitian matrix $\mathbf{\Gamma}^\dagger \mathbf{\Gamma}$ and its eigenvalues and eigenvectors.

If the time delays are all commensurable, (35) and (36) or (39) and (40) can easily be solved recursively on a digital computer. To start the system off, Array 1 was supposed to be illuminated initially by a constant beacon field. Array 2 was turned on linearly during a period of 20 time units, and the beacon was turned off linearly during a similar period. Four cases were run with each choice of $\mathbf{\Gamma}$.

Case I. Power limited, equal delays. The condition

$$(\mathbf{I}_1, \mathbf{I}_1) = (\mathbf{I}_2, \mathbf{I}_2) = 1 \tag{54}$$

was imposed, and all interelement delays were set equal to 20 units. This case must approach a steady state, according to Section IV, provided only that the largest eigenvalue of $\mathbf{\Gamma}^\dagger\mathbf{\Gamma}$ is unique.

Case II. Power limited, unequal delays. Same as Case I, except that the time delays given by (53) were used.

Case III. Fixed amplitudes, equal delays. The condition

$$| I_{1,j} | = | I_{2,k} | = 1 \tag{55}$$

was imposed, and all interelement delays were set equal to 20 units.

Case IV. Fixed amplitudes, unequal delays. Same as Case III, except that the time delays given by (53) were used.

In a typical run, the random number generator produced:

$$(n_{jk}) = \begin{pmatrix} -1 & -1 & -2 \\ 0 & -5 & -5 \\ -1 & 1 & 3 \end{pmatrix}, \tag{56}$$

$$(\Gamma_{jk}) = \begin{pmatrix} 8\underline{/36^\circ} & 2\underline{/36^\circ} & 8\underline{/72^\circ} \\ 2\underline{/0^\circ} & 8\underline{/180^\circ} & 1\underline{/180^\circ} \\ 8\underline{/36^\circ} & 8\underline{/-36^\circ} & 2\underline{/-108^\circ} \end{pmatrix}. \tag{57}$$

It follows that

$$\mathbf{\Gamma}^\dagger\mathbf{\Gamma} = \begin{pmatrix} 132.0\underline{/0^\circ} & 64.0\underline{/-72.0^\circ} & 46.4\underline{/37.5^\circ} \\ 64.0\underline{/72.0^\circ} & 132.0\underline{/0^\circ} & 26.5\underline{/-12.7^\circ} \\ 46.4\underline{/-37.5^\circ} & 26.5\underline{/12.7^\circ} & 69.0\underline{/0^\circ} \end{pmatrix}; \tag{58}$$

$$\lambda_1 = 20.6, \qquad \lambda_2 = 109.8, \qquad \lambda_3 = 202.6. \tag{59}$$

The eigenvector corresponding to λ_3 is

$$\mathbf{z}^{(3)} = (0.719\underline{/0^\circ}, 0.656\underline{/64.5^\circ}, 0.229\underline{/-6.2^\circ}). \tag{60}$$

Figs. 2 through 5 show the results of running Cases I through IV over the time interval $0 \leqq t \leqq 400$. In the figures the phases are referred to the phase of the first element of each array, and the following notation is used:

$$\mathbf{I}_1 = (A_1, A_2\underline{/\alpha_2}, A_3\underline{/\alpha_3}), \tag{61}$$

$$\mathbf{I}_2 = (B_2, B_2\underline{/\beta_2}, B_3\underline{/\beta_3}). \tag{62}$$

The initial behavior of the two arrays depends on the particular way in which they were turned on and is of no great importance; what we are really interested in is the behavior at large times. In Cases I and III (Figs. 2 and 4), under the assumption of equal interelement delay times, the system appears to settle down to a perfectly steady state. It is easy to verify that in Case I the steady-state excitation of Array 1 corresponds to the eigenvector $\mathbf{z}^{(3)}$ given by (60). On the other hand, in Cases II and IV (Figs. 3 and 5), where the delays are not all equal, the array excitations continue to show small residual fluctuations about the steady-state solutions of Cases I and III. These fluctuations are quite apparent in the original plots from which the present figures were redrawn.

In the numerical study, 50 pairs of 2-element arrays were simulated and four cases run for each pair. The ratio of eigenvalues λ_2/λ_1 of $\mathbf{\Gamma}^\dagger\mathbf{\Gamma}$ ranged from 293.5 to 1.385. As expected, the larger values of λ_2/λ_1 generally produced quicker convergence; but only one case, out of all those tried, failed to reach essentially steady values by $t = 400$. In this particular example λ_2/λ_1 was 7.37, and the interelement delays happened to range all the way from 15 to 25. Cases I, II, and III settled down relatively quickly, but Case IV (fixed amplitudes, unequal delays) went into a large-amplitude oscillation which was obviously not dying out at $t = 1000$. A similar, subsequent run in which the extreme interelement delay times were changed to 16 and 24 settled down normally.

Twenty-five pairs of 3-element arrays were simulated, with eigenvalue ratios λ_3/λ_2 ranging from 26.02 to 1.458. Every one of these cases appeared to have reached an essentially steady state at $t = 400$. The example shown in Figs. 2 through 5 is entirely typical.

From the numerical simulations it is clear that sufficiently large delay differences (perhaps ± 25 per cent of the average delay time) can make a pair of interacting adaptive arrays fail to settle down. We conjecture, however, that the arrays will always reach an essentially steady state if the delay differences are a sufficiently small fraction of the average delay. Conceivably one could put bounds on the fluctuations as a function of the deviations of the delays from the mean delay, but a more practical approach might be to do some experiments with real adaptive arrays.

VI. ACKNOWLEDGMENTS

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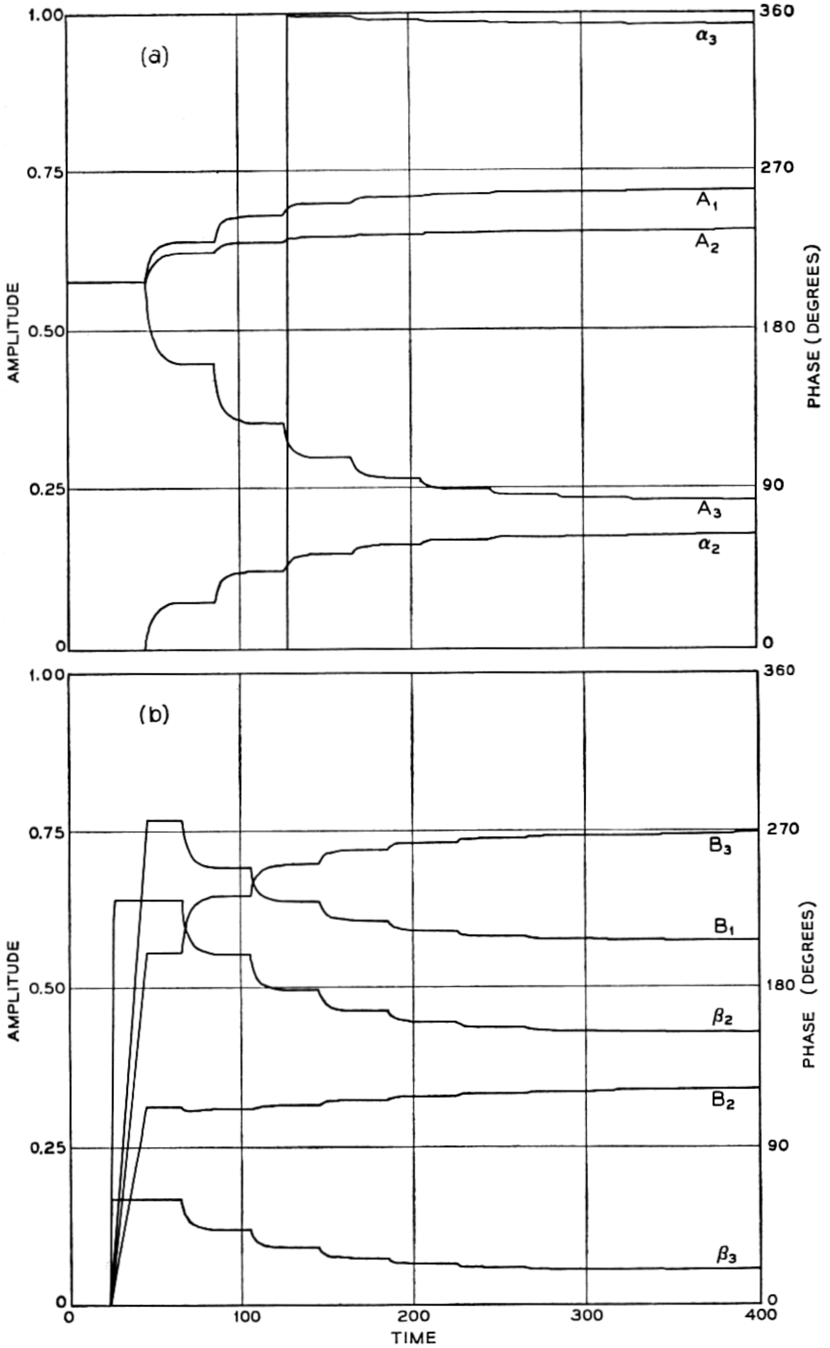

Fig. 2 — Transient behavior of amplitudes and phases in power-limited adaptive arrays with equal interelement delays: (a) Array 1, (b) Array 2.

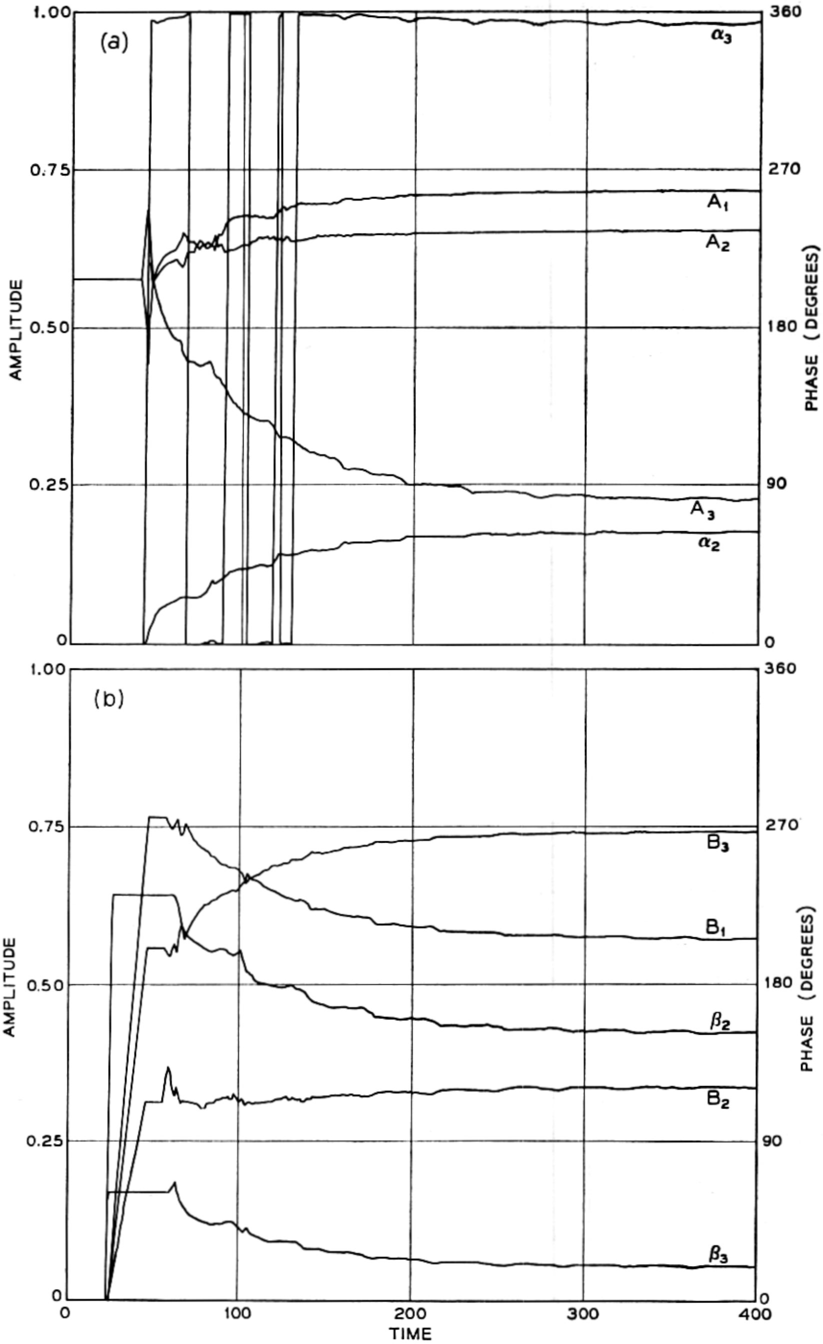

Fig. 3 — Transient behavior of amplitudes and phases in power-limited adaptive arrays with unequal interelement delays: (a) Array 1, (b) Array 2.

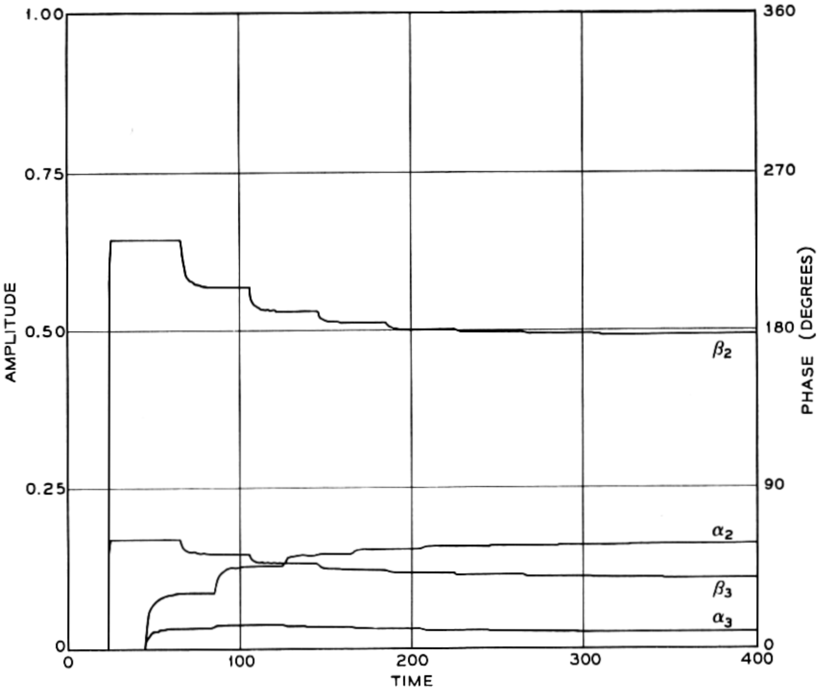

Fig. 4 — Transient behavior of phases in fixed-amplitude adaptive arrays with equal interelement delays.

APPENDIX A

Vectors, Matrices, and Hermitian Forms

We summarize here the notation used in this paper, as well as some properties of Hermitian forms which are proved in textbooks like that of Gantmacher.[4]

A *vector* in n-dimensional complex space is an ordered array of n complex numbers:

$$\mathbf{x} = (x_1, x_2, \cdots, x_n). \tag{63}$$

The *scalar product* of the vectors $\mathbf{x}$ and $\mathbf{y}$ is written $(\mathbf{x},\mathbf{y})$ and is defined by

$$(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} x_i y_i^*, \tag{64}$$

where the asterisk denotes complex conjugate.

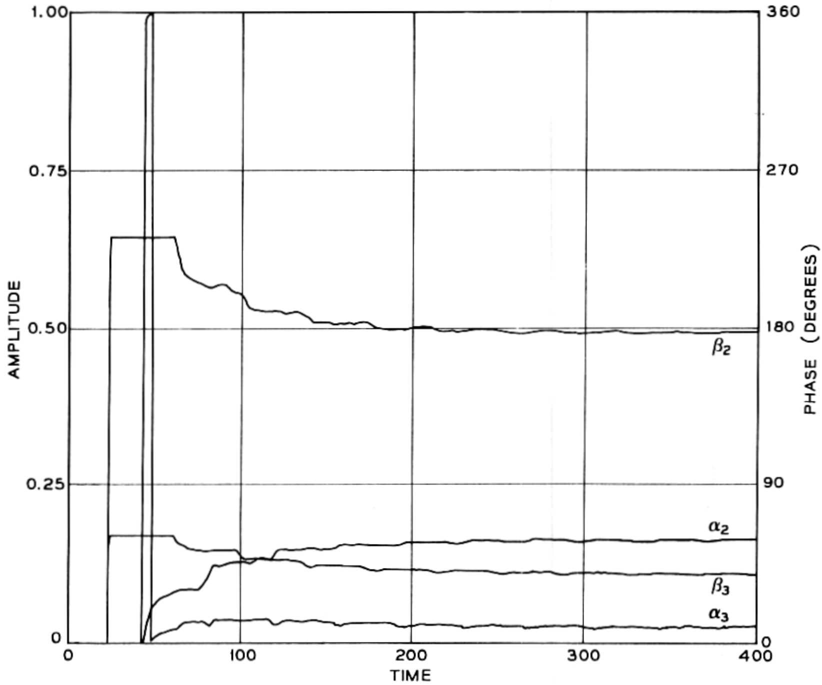

Fig. 5 — Transient behavior of phases in fixed-amplitude adaptive arrays with unequal interelement delays.

A *matrix* is an $m \times n$ array of complex numbers:

$$\mathbf{A} = (A_{ij}), \qquad i = 1, 2, \cdots, m; \qquad j = 1, 2, \cdots, n. \tag{65}$$

Associated with a given matrix are the following matrices:

$$\begin{aligned} &\textit{Conjugate} \ (\mathbf{A}^*)_{ij} = A_{ij}^*, \\ &\textit{Transpose} \ (\mathbf{A}')_{ij} = A_{ji}, \\ &\textit{Adjoint} \quad (\mathbf{A}^\dagger)_{ij} = A_{ji}^*. \end{aligned} \tag{66}$$

Note that this definition of the adjoint, while in accord with modern usage, differs from the definitions given in some older textbooks.

A *Hermitian matrix* is one which is equal to its own adjoint:

$$\mathbf{H} = \mathbf{H}^\dagger \quad \text{or} \quad H_{ij} = H_{ji}^*. \tag{67}$$

The *product* of an $m \times n$ matrix and an n-component vector is an m-component vector, written

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad \text{or} \quad y_i = \sum_{j=1}^{n} A_{ij}x_j\,, \qquad i = 1, 2, \cdots, m. \tag{68}$$

If $\mathbf{x}$ is an m-component vector, $\mathbf{y}$ an n-component vector, and $\mathbf{A}$ an $m \times n$ matrix, then from (64) and (66),

$$(\mathbf{x},\mathbf{A}\mathbf{y}) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i A_{ij}^* y_j^* = (\mathbf{A}^\dagger\mathbf{x},\mathbf{y}). \tag{69}$$

A *Hermitian form* is the scalar product of $\mathbf{H}\mathbf{x}$ with $\mathbf{x}$, where $\mathbf{H}$ is a Hermitian matrix:

$$(\mathbf{H}\mathbf{x},\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i^* H_{ij} x_j\,. \tag{70}$$

Hermitian forms are *real-valued*, since in view of (67),

$$(\mathbf{H}\mathbf{x},\mathbf{x})^* = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i H_{ij}^* x_j^* = \sum_{i=1}^{n}\sum_{j=1}^{n} x_j^* H_{ji} x_i = (\mathbf{H}\mathbf{x},\mathbf{x}). \tag{71}$$

A Hermitian form is *positive definite* if

$$(\mathbf{H}\mathbf{x},\mathbf{x}) > 0 \quad \text{whenever} \quad (\mathbf{x},\mathbf{x}) \neq 0. \tag{72}$$

If the $>$ sign is replaced by $\geqq$, the form is called *positive semidefinite*.

The *product* of an $m \times n$ matrix $\mathbf{A}$ and an $n \times p$ matrix $\mathbf{B}$ is an $m \times p$ matrix $\mathbf{C}$ whose elements are given by

$$C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}\,, \qquad i = 1, 2, \cdots, m; \qquad j = 1, 2, \cdots, p. \tag{73}$$

A square matrix whose determinant vanishes is called *singular*. If the determinant does not vanish, the matrix is called *nonsingular*. The matrix of a positive definite Hermitian form is nonsingular.

The *inverse* of a nonsingular $n \times n$ square matrix $\mathbf{A}$ is the $n \times n$ square matrix $\mathbf{A}^{-1}$ which satisfies

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{1}_n\,, \tag{74}$$

where $\mathbf{1}_n$ is the $n \times n$ *unit matrix* with 1's on the main diagonal and 0's elsewhere. The elements of $\mathbf{A}^{-1}$ are given by

$$(\mathbf{A}^{-1})_{ij} = \frac{\mathcal{A}_{ji}}{\det \mathbf{A}}, \tag{75}$$

where $\mathcal{A}_{ji}$ is the cofactor of the element A_{ji} in the determinant of $\mathbf{A}$.

If $\mathbf{\Gamma}$ is an arbitrary $m \times n$ matrix, $\mathbf{\Gamma}^\dagger\mathbf{\Gamma}$ is an $n \times n$ Hermitian matrix, since

$$(\mathbf{\Gamma}^\dagger \mathbf{\Gamma})_{ij} = \sum_{k=1}^{m} \Gamma_{ki}^* \Gamma_{kj} = \sum_{k=1}^{m} \Gamma_{kj} \Gamma_{ki}^* = (\mathbf{\Gamma}^\dagger \mathbf{\Gamma})_{ji}^*, \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, n. \tag{76}$$

If **A** and **B** are $n \times n$ Hermitian matrices and λ is a complex parameter, then $\mathbf{A} - \lambda\mathbf{B}$ is called a *pencil* of matrices. If **B** is positive definite, the pencil is called *regular*. The characteristic equation of a regular pencil, namely

$$\det(\mathbf{A} - \lambda\mathbf{B}) = 0, \tag{77}$$

always has n real roots $\lambda_1, \lambda_2, \cdots, \lambda_n$, which are called the *eigenvalues* of the pencil. The eigenvalues correspond to *eigenvectors* $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \cdots, \mathbf{z}^{(n)}$, which satisfy the homogeneous equations

$$\mathbf{A}\mathbf{z}^{(k)} = \lambda_k \mathbf{B}\mathbf{z}^{(k)}, \qquad k = 1, 2, \cdots, n. \tag{78}$$

The eigenvectors may be chosen to satisfy

$$(\mathbf{B}\mathbf{z}^{(i)}, \mathbf{z}^{(j)}) = \delta_{ij}, \tag{79}$$

where δ_{ij} is the Kronecker delta.

The largest eigenvalue λ_n of the regular pencil $\mathbf{A} - \lambda\mathbf{B}$ satisfies

$$\lambda_n = \max_{\mathbf{x} \neq 0} \frac{(\mathbf{A}\mathbf{x}, \mathbf{x})}{(\mathbf{B}\mathbf{x}, \mathbf{x})}, \tag{80}$$

and this maximum is assumed only for eigenvectors of the pencil corresponding to the eigenvalue λ_n.

APPENDIX B

Reciprocity Theorem for Time-Harmonic Fields

We shall prove the reciprocity theorem in a form convenient for use in the present paper, following an approach similar to that of Harrington.[5]

Consider a linear, time-invariant medium characterized by the permittivity tensor $\boldsymbol{\varepsilon}$, the permeability tensor $\boldsymbol{\mu}$, and the conductivity tensor $\boldsymbol{\sigma}$. All three tensors are assumed to be symmetric, although they may be functions of the space coordinates. Let

$$\mathcal{Y} = \boldsymbol{\sigma} + i\omega\boldsymbol{\varepsilon}, \qquad \mathcal{Z} = i\omega\boldsymbol{\mu}, \tag{81}$$

where ω is the angular frequency of the time-harmonic fields.

Consider two sets of electric current densities, $\mathbf{J}^a$ and $\mathbf{J}^b$, which are vector functions of position and which give rise to the fields $\mathbf{E}^a$, $\mathbf{H}^a$ and $\mathbf{E}^b$, $\mathbf{H}^b$ respectively. Maxwell's equations are

$$\begin{aligned} \nabla \times \mathbf{H}^a &= \mathcal{Y}\mathbf{E}^a + \mathbf{J}^a, & \nabla \times \mathbf{H}^b &= \mathcal{Y}\mathbf{E}^b + \mathbf{J}^b, \\ -\nabla \times \mathbf{E}^a &= \mathcal{Z}\mathbf{H}^a, & -\nabla \times \mathbf{E}^b &= \mathcal{Z}\mathbf{H}^b. \end{aligned} \tag{82}$$

From the first and fourth equations,

$$\begin{aligned} -\nabla\cdot(\mathbf{E}^b \times \mathbf{H}^a) &= \mathbf{E}^b\cdot\nabla \times \mathbf{H}^a - \mathbf{H}^a\cdot\nabla \times \mathbf{E}^b \\ &= \mathbf{E}^b\cdot\mathcal{Y}\mathbf{E}^a + \mathbf{E}^b\cdot\mathbf{J}^a + \mathbf{H}^a\cdot\mathcal{Z}\mathbf{H}^b, \end{aligned} \tag{83}$$

and from the second and third,

$$\begin{aligned} -\nabla\cdot(\mathbf{E}^a \times \mathbf{H}^b) &= \mathbf{E}^a\cdot\nabla \times \mathbf{H}^b - \mathbf{H}^b\cdot\nabla \times \mathbf{E}^a \\ &= \mathbf{E}^a\cdot\mathcal{Y}\mathbf{E}^b + \mathbf{E}^a\cdot\mathbf{J}^b + \mathbf{H}^b\cdot\mathcal{Z}\mathbf{H}^a. \end{aligned} \tag{84}$$

Subtracting (83) from (84) and using the symmetry of $\mathcal{Y}$ and $\mathcal{Z}$, we obtain

$$\nabla\cdot(\mathbf{E}^b \times \mathbf{H}^a - \mathbf{E}^a \times \mathbf{H}^b) = \mathbf{E}^a\cdot\mathbf{J}^b - \mathbf{E}^b\cdot\mathbf{J}^a. \tag{85}$$

Now integrate over a large spherical volume V bounded by the surface S, which contains all sources and matter in its interior. The divergence theorem yields

$$\int_S (\mathbf{E}^b \times \mathbf{H}^a - \mathbf{E}^a \times \mathbf{H}^b)\cdot\mathbf{n}\, dS = \int_V (\mathbf{E}^a\cdot\mathbf{J}^b - \mathbf{E}^b\cdot\mathbf{J}^a)\, dV, \tag{86}$$

where $\mathbf{n}$ is the outward normal to S. The individual fields fall off as $1/r$, where r is the radius of V, but for large r the leading terms satisfy

$$\mathbf{E}^a = \eta\mathbf{H}^a \times \mathbf{n}, \qquad \mathbf{E}^b = \eta\mathbf{H}^b \times \mathbf{n}, \tag{87}$$

where η is the characteristic impedance of free space.

Hence for the leading terms,

$$\begin{aligned} &\eta^{-1}[\mathbf{E}^b \times \mathbf{H}^a - \mathbf{E}^a \times \mathbf{H}^b]\cdot\mathbf{n} \\ &= [(\mathbf{H}^b \times \mathbf{n}) \times \mathbf{H}^a - (\mathbf{H}^a \times \mathbf{n}) \times \mathbf{H}^b]\cdot\mathbf{n} \\ &= [\mathbf{n}(\mathbf{H}^a\cdot\mathbf{H}^b) - \mathbf{H}^b(\mathbf{n}\cdot\mathbf{H}^a) - \mathbf{n}(\mathbf{H}^a\cdot\mathbf{H}^b) + \mathbf{H}^a(\mathbf{n}\cdot\mathbf{H}^b)]\cdot\mathbf{n} = 0. \end{aligned} \tag{88}$$

It follows that if all sources and matter are of finite extent, then

$$\int \mathbf{E}^a\cdot\mathbf{J}^b dV = \int \mathbf{E}^b\cdot\mathbf{J}^a dV, \tag{89}$$

where each integral is taken over the region in which the source currents are different from zero. If the medium is not symmetric, the theorem is still true provided that $\mathbf{E}^b$ represents the field produced by $\mathbf{J}^b$ in the "transposed" medium; but this generalization is not very useful in the present context.

Now let $\mathbf{J}^a$ correspond to an electric dipole of unit moment in the direction $\mathbf{u}^a$ at the point P_a, and let $\mathbf{J}^b$ correspond to an electric dipole of unit moment in the direction $\mathbf{u}^b$ at P_b. Equation (89) takes the form

$$\mathbf{u}^b \cdot \mathbf{E}^a(P_b) = \mathbf{u}^a \cdot \mathbf{E}^b(P_a), \tag{90}$$

where the left side represents the components of electric field due to source A at the location and in the direction of source B, and the right side represents the component due to source B at the location and in the direction of source A. This is the desired reciprocity theorem.

APPENDIX C

Sandberg's Lemma

We reproduce Sandberg's proof[3] of the following result.

Lemma. If $\mathbf{A}$ and $\mathbf{B}$ respectively are $n \times m$ and $m \times n$ matrices, then

$$\det\,(\mathbf{1}_n + \mathbf{AB}) = \det\,(\mathbf{1}_m + \mathbf{BA}).$$

Proof. First consider the case in which $\mathbf{A}$ and $\mathbf{B}$ are square $p \times p$ matrices. Then, if $\mathbf{A}$ is nonsingular,

$$\begin{aligned} \det\,[\mathbf{1}_p + \mathbf{AB}] &= \det\,[\mathbf{A}^{-1}(\mathbf{1}_p + \mathbf{AB})\mathbf{A}] \\ &= \det\,[\mathbf{1}_p + \mathbf{BA}]. \end{aligned} \tag{91}$$

If $\mathbf{A}$ is singular, it has a zero characteristic root, and hence there exists a positive number λ_0 such that $\mathbf{A} + \lambda\mathbf{1}_p$ is nonsingular for all real λ satisfying $0 < |\lambda| < \lambda_0$. Thus when $0 < |\lambda| < \lambda_0$,

$$\det\,[\mathbf{1}_p + (\mathbf{A} + \lambda\mathbf{1}_p)\mathbf{B}] = \det\,[\mathbf{1}_p + \mathbf{B}(\mathbf{A} + \lambda\mathbf{1}_p)]. \tag{92}$$

Both sides of (92) are polynomials in λ of degree at most p. Furthermore these polynomials must be identical since they agree throughout the real interval $(0,\lambda_0)$. Therefore (92) is valid when $\lambda = 0$.

Consider now the case in which $\mathbf{A}$ and $\mathbf{B}$ are not square. Let $p = m + n$,

$$\tilde{\mathbf{A}} = \overset{\quad m \quad n}{\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}\begin{matrix} n \\ m \end{matrix}, \qquad \tilde{\mathbf{B}} = \overset{\quad n \quad m}{\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}\begin{matrix} m \\ n \end{matrix}; \tag{93}$$

and let the symbol $\dotplus$ denote a direct sum of matrices. Observe that

$$\begin{aligned} \det [\mathbf{1}_p + \tilde{\mathbf{A}}\tilde{\mathbf{B}}] &= \det [(\mathbf{1}_n + \mathbf{AB}) \dotplus \mathbf{1}_m] = \det [\mathbf{1}_n + \mathbf{AB}], \\ \det [\mathbf{1}_p + \tilde{\mathbf{B}}\tilde{\mathbf{A}}] &= \det [(\mathbf{1}_m + \mathbf{BA}) \dotplus \mathbf{1}_n] = \det [\mathbf{1}_m + \mathbf{BA}], \end{aligned} \tag{94}$$

which proves the lemma.

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