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A Note on a Special Class of One-Sided Distribution Sums

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I. INTRODUCTION

Occasionally encountered in the calculation of power spectra are limits of the form[1]

$$\lim_{N\to\infty} \sum_{n=0}^{N} [1 + F(n,N)]e^{inx},$$

where

$$F(n,N) \equiv \sum_{m=0}^{M} a_m(N)n^m \qquad M < \infty$$

$$a_m(N) = o(1) \qquad (N \to \infty) \qquad \forall m$$

$$x \in (-\infty,\infty), \qquad i = \sqrt{-1}.$$

It is shown here that these limits exist as distributions, or generalized functions,[2,3,4] and have several simple and useful representations. Specifically, we prove the following

Theorem:

$$\lim_{N\to\infty}{}^{(D)} \sum_{n=0}^{N} [1 + F(n,N)]e^{inx} = \frac{1}{2} + \pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n) + \frac{i}{2}\cot\frac{x}{2}$$

$$= \lim_{\substack{\alpha\to 0 \\ \mathrm{Re}\,\alpha>0}}{}^{(D)} [1 - e^{-\alpha}e^{ix}]^{-1},$$

where $\lim^{(D)}$ and $\delta(\cdot)$ denote respectively a distribution limit[2,3] and the Dirac delta function.

II. ANALYSIS

Concerning notation, let C^∞ represent the space of infinitely differentiable scalar functions defined on the real line $(-\infty,\infty)$; C_d, the space of "rapidly decaying" test functions, viz., the linear vector space

$$C_d = \{\varphi \mid \varphi \in C^\infty, x^j\varphi^{(k)}(x) \to 0(\,|x| \to \infty)\forall j, k \geqq 0\};$$

and G, the space of generalized functions defined relative to the test functions of C_d. Finally, let Fg signify the generalized Fourier transform[2,3] of $g \in G$ with

$$F\varphi \equiv \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i y x} dx \qquad \varphi \in C_d.$$

The theorem under discussion is now established in terms of the following three lemmas:

Lemma I:

$$\lim_{N\to\infty}{}^{(\mathrm{D})} \sum_{n=0}^{N} [1 + F(n,N)] e^{inx} = \lim_{N\to\infty}{}^{(\mathrm{D})} \sum_{n=0}^{N} e^{inx} \equiv h(x) \in G.$$

Proof: Inasmuch as $Fh \in G$, then $h \in G$ and

$$\lim_{N} \int_{-\infty}^{\infty} a_m(N) \left[\sum_{n=0}^{N} e^{inx} \right] \varphi(x) dx = [\lim_{N} a_m(N)] \lim_{N} \int_{-\infty}^{\infty} \left[\sum_{n=0}^{N} e^{inx} \right] \varphi(x) dx$$
$$= 0 \quad \forall m.$$

Hence,

$$\lim_{N}{}^{(\mathrm{D})} \left[a_m(N) \sum_{n=0}^{N} e^{inx} \right] = 0 \quad \forall m,$$

and

$$\lim_{N\to\infty}{}^{(\mathrm{D})} \sum_{n=0}^{N} [1 + F(n,N)] e^{inx} = h(x) + \sum_{m=0}^{M} \left\{ \lim_{N}{}^{(\mathrm{D})} \left[a_m(N) \sum_{n=0}^{N} n^m e^{inx} \right] \right\}$$
$$= h(x) + \sum_{m=0}^{M} \left\{ (-i)^m \frac{d^m}{dx^m} \right.$$
$$\left. \cdot \left[\lim_{N}{}^{(\mathrm{D})} a_m(N) \sum_{n=0}^{N} e^{inx} \right] \right\} = h(x).$$

Lemma II:

$$h(x) = \lim_{\substack{\alpha\to 0 \\ \mathrm{Re}\ \alpha>0}}{}^{(\mathrm{D})} [1 - e^{-\alpha} e^{ix}]^{-1}.$$

Proof: Setting

$$u(y) \equiv \begin{cases} 1, & y \geqq 0 \\ 0, & y < 0 \end{cases}$$

$$g_\alpha(y) \equiv \sum_{n=0}^{\infty} e^{-\alpha n} u\left(y - \frac{n}{2\pi}\right)$$

$$g_0(y) \equiv \lim_{\alpha} g_\alpha = \left[\sum_{n=0}^{\infty} u\left(y - \frac{n}{2\pi}\right)\right] \in G$$

and noting that

$$\begin{aligned} &\int_{-\infty}^{\infty} | g_0(y)\varphi(y) | \, dy < \infty \qquad \forall \varphi \in C_d \\ &| g_\alpha | \leqq g_0 \qquad \forall y \in (-\infty, \infty), \qquad \forall \operatorname{Re} \alpha > 0, \end{aligned} \tag{1}$$

one obtains by means of Lebesgue's dominated convergence theorem[5] the condition

$$\lim_{\alpha} \int_{-\infty}^{\infty} g_\alpha(y)\varphi(y)dy = \int_{-\infty}^{\infty} g_0(y)\varphi(y)dy \qquad \forall \varphi \in C_d. \tag{2}$$

Consequently,

$$\lim_{\alpha}{}^{(\mathrm{D})} g_\alpha = g_0 , \tag{3}$$

and

$$\begin{aligned} h(x) &= F^{-1} \cdot \sum_{n=0}^{\infty} \delta\left(y - \frac{n}{2\pi}\right) = F^{-1} \cdot \frac{d}{dy} \cdot g_0 = F^{-1} \cdot \frac{d}{dy} \cdot \lim_{\alpha}{}^{(\mathrm{D})} g_\alpha \\ &= \lim_{\alpha}{}^{(\mathrm{D})} \cdot \sum_{n=0}^{\infty} e^{-\alpha n} e^{inx} = \lim_{\alpha}{}^{(\mathrm{D})} [1 - e^{-\alpha} e^{ix}]^{-1}. \end{aligned}$$

Lemma III:

$$\lim_{\substack{\alpha \to 0 \\ \operatorname{Re} \alpha > 0}}{}^{(\mathrm{D})} [1 - e^{-\alpha} e^{ix}]^{-1} = \frac{1}{2} + \pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n) + \frac{i}{2} \cot \frac{x}{2}$$

Proof: From the definitions

$$C_\alpha(x) \equiv \frac{1}{2} \log \left[e^{-\alpha} \sin^2 \frac{x}{2} + \left(\frac{1 - e^{-\alpha}}{2}\right)^2 \right]$$

$$d_\alpha(x) \equiv \tan^{-1} \left[\frac{e^{-\alpha} \sin x}{1 - e^{-\alpha} \cos x} \right]$$

$$f_0(x) \equiv \pi \left[\sum_{n=0}^{\infty} u(x - 2\pi n) - \sum_{n=1}^{\infty} u(-x - 2\pi n) \right]$$

it is found that

$$| C_\alpha | \leqq \left| \log \left| \sin \frac{x}{2} \right| - \frac{1}{2} \right|$$

$$| d_\alpha | \leqq \frac{\pi}{2}$$

$$\lim_\alpha C_\alpha = \left[\log \left| \sin \frac{x}{2} \right| \right] \in G$$

$$\lim_\alpha d_\alpha = \left[f_0(x) - \frac{x}{2} \right] \in G$$

for all $\alpha \in (0,1)$ and almost all $x \in (-\infty, \infty)$. Therefore, as in (1), (2), and (3),

$$\lim_\alpha{}^{(\mathrm{D})} C_\alpha = \log \left| \sin \frac{x}{2} \right|$$

$$\lim_\alpha{}^{(\mathrm{D})} d_\alpha = f_0(x) - \frac{x}{2},$$

and

$$\begin{aligned} \lim_\alpha{}^{(\mathrm{D})} [1 - e^{-\alpha} e^{ix}]^{-1} &= \frac{d}{dx} \cdot \lim_\alpha{}^{(\mathrm{D})} [x + i \log (1 - e^{-\alpha} e^{ix})] \\ &= \frac{d}{dx} \cdot \lim_\alpha{}^{(\mathrm{D})} [x + d_\alpha(x) + i C_\alpha(x)] \\ &= \frac{1}{2} + \pi \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n) + \frac{i}{2} \cot \frac{x}{2}. \end{aligned}$$

REFERENCES

1. Salz, J., The Spectral Density Function of Multilevel, Continuous-Phase FM, to be published.
2. Barnard, R. D., On the Spectral Properties of Single-Sideband Angle-Modulated Signals, B.S.T.J., **43**, Nov., 1964, p. 2811.
3. Temple, G., Generalized Functions, Proc. Roy. Soc. (London), **A228**, 1955, pp. 175–190.
4. Dunford, N., and Schwartz, J., *Linear Operators—Part II*, Interscience, New York, 1963.
5. Burkill, J. C., *The Lebesgue Integral*, Cambridge University Press, London, 1961, p. 41.