Nonlinear *RLC* Networks

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This article considers the question of existence and uniqueness of the response of nonlinear time-varying RLC networks driven by independent voltage and current sources. It is proved that under certain conditions the response exists, is unique, and is defined by a set of ordinary differential equations satisfying some Lipschitz conditions. These conditions are of two types: (1) the network elements must have characteristics which satisfy suitable Lipschitz conditions and (2) the network must satisfy certain topological conditions. It should be noted that elements with nonmonotonic characteristics are allowed and that the element characteristics need to be continuous but not differentiable.

I. INTRODUCTION

This article considers the questions of existence and uniqueness of the response of nonlinear time-varying *RLC* networks. It is proved that under conditions imposed on the network elements and the network topology the response exists, is unique, and is defined by a set of ordinary differential equations satisfying some Lipschitz conditions. Thus, from the conditions imposed on the network it follows that the response of a network of this class is continuous whenever the sources applied to the network are continuous functions of time. In other words, for the class of networks under consideration, jump phenomena (of the type that occur in relaxation oscillators) are excluded.[1]

One motivation for studying this problem is the construction of nonlinear network models for physical devices and processes. The behavior of these models is often investigated by simulation studies performed on digital computers. It is clear that in order to get meaningful answers the existence and uniqueness of the model's response have to be assured. The simulation study requires the setting up of an appropriate set of differential equations and their integration. As networks of the class

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considered here do not have jump phenomena, their equations can be integrated by some standard subroutines.

This article may be viewed as an extension to the *RLC* case of the articles by R. J. Duffin[2,3,4] and G. Birkhoff and J. B. Diaz[5] which were devoted to nonlinear resistive networks. We make heavy use of topological considerations and had to extend the techniques developed for the linear case by many people[6,7,8] P. R. Bryant in particular.[9,10] For further references see Ref. 16.

In the next section, we classify the network elements and exhibit the basic assumptions which hold for the remainder of the article. Some simple nonlinear circuits are also considered. Section III presents some standard reductions of sources and the definition of determinateness. Section IV deals with one-element-kind networks; its theorems are generalizations of Duffin's work and include some of his theorems as corollaries. The main result of the article is Theorem IV in Section V, which states the conditions under which a nonlinear *RLC* network is determinate. The conditions are of two types: (*i*) every characteristic has to satisfy suitable Lipschitz conditions and (*ii*) the network has to satisfy certain topological conditions. It has to be noted that, first, elements with nonmonotonic characteristics are allowed and, second, that each characteristic has to be representable by a function which is continuous but not necessarily differentiable. Finally, in Section VI we introduce a symbolic notation which allows us to write the differential equations for the nonlinear case in a manner which resembles that of the linear case.

II. ELEMENTS AND SIMPLE CIRCUITS

2.1 *Elements*

We assume that the reader has some familiarity with network theory, so that the basic concepts need not be defined.[11,12] A network may be considered as a set of points, called *nodes*, and a set of connecting *branches*. Each branch represents a physical *two-pole*. We assume that the voltage drop across each two-pole and the current through each two-pole can be measured at any time. The sign conventions are shown in Fig. 1: if, with respect to some arbitrary reference, the potential of A is larger (smaller) than the potential of B, then v is positive (negative); if the current actually flows in the direction of the arrow (opposite to the arrow) then i is positive (negative). Thus the product vi gives the power delivered by the outside world to the two-pole under consideration.

In most of the following, the branches consist of either a single source or a single *element* such as a resistor, an inductor or a capacitor. For each of these elements we shall adopt very broad definitions which we will narrow down in stating specific results. A two-pole is called a *resistor* if it is defined, for each t, by a set of ordered pairs (v, i), where v and i are finite numbers representing all the possible values, at time t, of the voltage and the current associated with the resistor. If the set of ordered pairs is independent of t, the resistor is said to be *time-invariant*. The set of (v, i) is called the *characteristic* of the resistor; for example, the characteristic of an *ideal diode* is given by

$$\{(0,i): 0 \leqq i < \infty\} \, U \, \{(v,0): -\infty < v \leqq 0\} .$$

A resistor is called *current-controlled* if, for all time and all currents in the interval $(-\infty, \infty)$, the voltage $v(t)$ is a function* of the current $i(t)$ and time t (we shall write $v(t) = \mathcal{R}(i(t),t))$, and the function $\mathcal{R}(i,t)$ is a piecewise-continuous function† of t for each fixed number i. A *voltage-controlled* resistor is defined in the dual manner. For example, a voltage source is a current-controlled resistor and a current source is a voltage-controlled resistor. If a resistor is *current-controlled* and *time-invariant* then the characteristic can be represented by a function $v = \mathcal{R}(i)$. A resistor is called a *one-to-one resistor* if, for each t, the voltage is related to the current by a one-to-one mapping from $(-\infty, \infty)$ onto $(-\infty, \infty)$ which may depend on time.

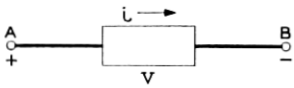

Fig. 1 — Sign conventions for two-pole.

A two-pole is called an *inductor* if it is defined, for each t, by a set of ordered pairs (φ, i) which represent the instantaneous flux and current associated with the inductor. The voltage across the inductor is given by $v = d\varphi/dt$. The *current-controlled* inductor, the *flux-controlled* inductor and the *one-to-one* inductor are defined as in the case of resistors. In the first two cases, if the elements are time-invariant, we shall write $\varphi = \mathcal{L}(i)$ and $i = \Gamma(\varphi)$, respectively.

* Unless specifically indicated, we follow modern usage: each function is single-valued; i.e., to each element of its domain it associates one and only one element of its range.

† A vector-valued function of time is said to be piecewise-continuous if it is continuous in every finite interval except at a finite number of points where it is discontinuous. At these points the function has a finite limit on the left as well as on the right.

A two-pole is called a *capacitor* if it is defined, for each t, by a set of ordered pairs (q,v) which represent the instantaneous charge and voltage associated with the capacitor. The current through the capacitor is given by $i = dq/dt$. The *charge-controlled capacitor*, the *voltage-controlled capacitor* and the *one-to-one capacitor* are defined as in the case of resistors. In the first two cases, if the elements are time invariant, we shall write $v = \mathfrak{D}(q)$ and $q = \mathfrak{C}(v)$, respectively.

Throughout the article we consider only elements whose characteristics can be represented, at all times, by a function defined on the interval $(-\infty, \infty)$. For example, Fig. 2(a), (b) and (c) represents the characteristics at time t of three time-varying resistors; we consider only resistors of the type shown in Fig. 2(a) and (b), since they are current- and

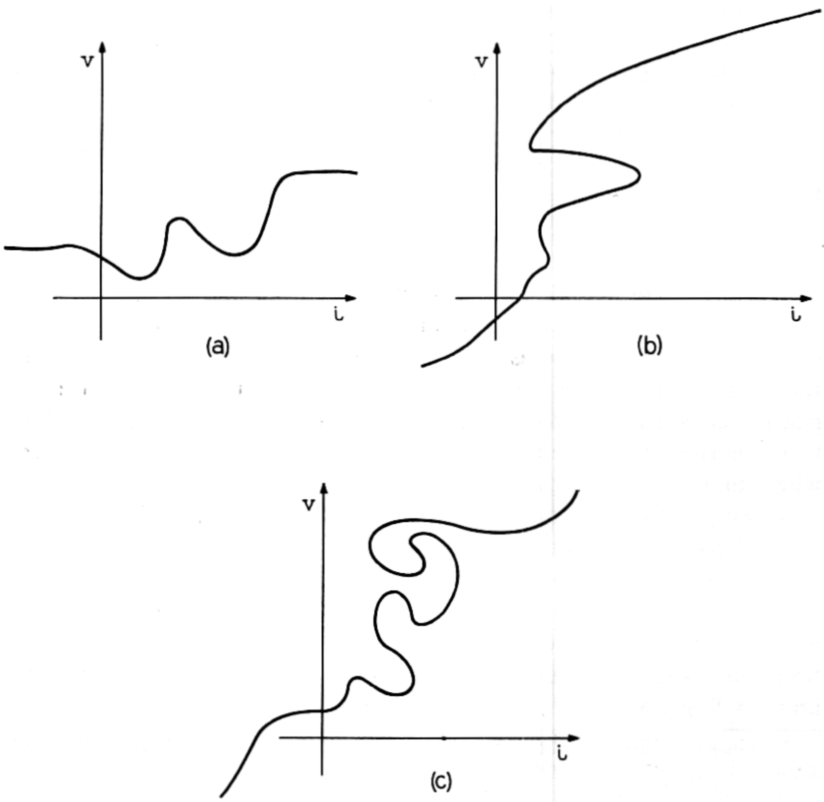

Fig. 2 — Characteristics at time t of three time-varying resistors: (a) and (b) are current- and voltage-controlled, respectively, while (c) is neither current- nor voltage-controlled.

voltage-controlled; these characteristics can be represented by

$$v(t) = \mathcal{R}(i(t),t), \quad \text{and} \quad i(t) = \mathcal{G}(v(t),t),$$

respectively. The characteristics of Fig. 2(c) cannot be represented in this way, and resistors of this type will not be considered.

Throughout the paper, whenever time-varying network elements are considered, it is assumed that the functions $\mathcal{R}(\cdot,t)$, $\mathcal{G}(\cdot,t)$, $\mathcal{L}(\cdot,t)$, $\Gamma(\cdot,t)$, $\mathcal{D}(\cdot,t)$, $\mathcal{C}(\cdot,t)$ are piecewise-continuous functions of t for all fixed values of their first argument.

In addition to resistors, capacitors and inductors, our networks include voltage and current sources. Throughout this article we shall assume that the voltages of the voltage sources and currents of current sources are regulated functions of time.* For convenience we shall say that an element is continuous and monotonically increasing, when we mean that its characteristic is represented by a continuous monotonically increasing function which is defined on $(-\infty, \infty)$.

It is convenient to refer to functions like $\mathcal{R}(\cdot,t)$ and $\mathcal{D}(\cdot,t)$, which represent the characteristics of some elements, as the characteristics of the elements. This slight misuse of the concept of a function and a relation will be used only when there is no danger of confusion between the two.

2.2 *Two-Poles and Simple Connections of Two-Poles*

A two-pole is called *voltage-controlled* [current-controlled] if, for any initial time t_0 and for any initial state, the voltage $v(\cdot)$ [the current $i(\cdot)$] from t_0 on across its terminals determines uniquely the current $i(\cdot)$ through [the voltage $v(\cdot)$ across] the two-pole for $t \geqq t_0$.

A two-pole is said to be *one-to-one* if (a) it is both current-controlled and voltage-controlled and (b) it satisfies the following condition: for any initial state s_0, any initial time t_0, and any input current $i(\cdot)$, let $f(s_0, i)$ be the voltage appearing at the terminals; for any initial state s_0, any initial time t_0, and any input voltage $v(\cdot)$, let $g(s_0, v)$ be the current — then it is required that

$$g(s_0, f(s_0, i)) = i$$

for all initial states s_0 and all input currents $i(\cdot)$.

An immediate consequence of these definitions is that *any parallel connection of a finite number of voltage-controlled two-poles is voltage-controlled.*

* A vector-valued function of time is said to be regulated when, for all t, it has a limit on the left as well as a limit on the right.[13] A step function and a rectangular wave are regulated functions.

Consider the case where there are only two two-poles in the parallel connection. Let them be characterized by the functions

$$i_k = F_k(v,s_k(t_0)), \qquad (k = 1,2),$$

where v is the voltage across the parallel connection, i_k is the current through the kth two-pole, $s_k(t_0)$ is the state of the kth two-pole at time t_0. The v_k and i_k are real-valued functions defined on $[t_0, \infty)$. Kirchoff's current law implies that the current i through the parallel connection is given by

$$F_1(v,s_1(t_0)) + F_2(v,s_2(t_0));$$

hence, for fixed $(s_1(t_0),s_2(t_0))$, i is a function of v. This argument obviously extends, by induction, to the case where there are a finite number of two-poles.

A dual argument would show that *any series connection of a finite number of current-controlled two-poles is current-controlled.*

A parallel connection of current-controlled two-poles is not necessarily current-controlled. Refer to Fig. 3, which shows the characteristics of two current-controlled resistors. The dashed line shows the characteristic of the parallel connection: depending on the operating point there may be three distinct values of the voltage for the same input current. Dually,

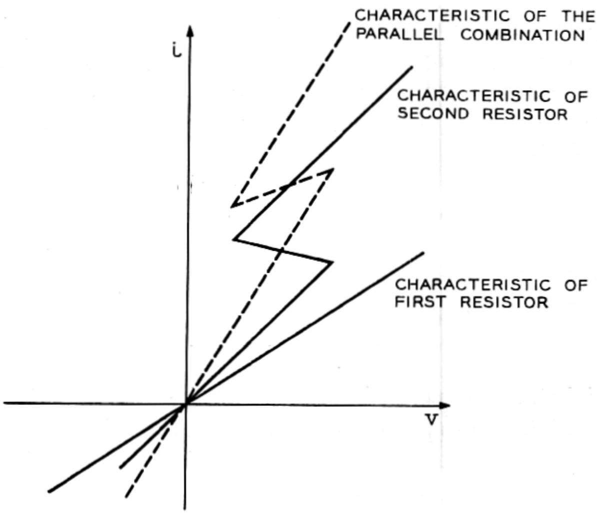

Fig. 3 — Parallel connection of two current-controlled resistors.

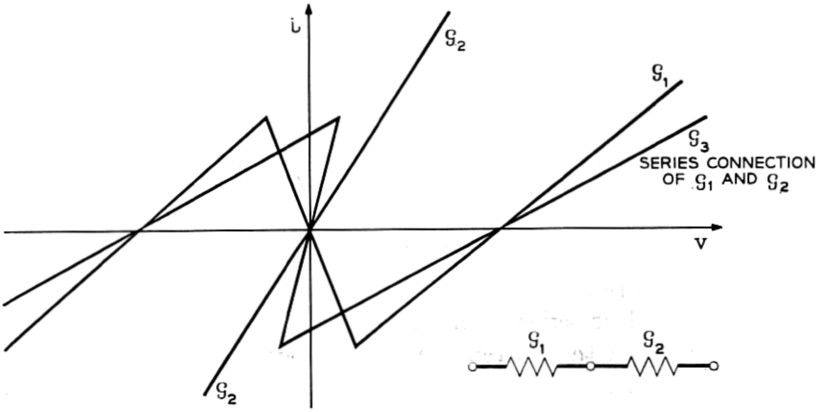

Fig. 4 — Series connection of two voltage-controlled resistors.

a series connection of voltage-controlled two-poles is not necessarily voltage-controlled.

To assume that each two-pole is one-to-one is not enough to cause both arbitrary parallel connections and arbitrary series connections to be one-to-one. Indeed, the well known characterization of continuous functions of bounded variation[14] implies that *any voltage-controlled resistor characteristic,* $i(t) = \mathcal{R}(v(t),t)$, *that is continuous and of bounded variation in* v *can be obtained by connecting in parallel two one-to-one resistors whose characteristics are continuous and strictly monotonic.* (One resistor is monotonically increasing and the other is monotonically decreasing.) A dual statement holds for current-controlled resistors.

In fact, there are combined series and parallel connections of one-to-one two-poles that are neither voltage-controlled nor current-controlled. Refer to Fig. 4, which shows the series connection of $\mathcal{G}_1$ and $\mathcal{G}_2$. Fig. 5 shows how a voltage-controlled characteristic such as $\mathcal{G}_1$ may be obtained by connecting in parallel two one-to-one resistors. Putting the two resistors of characteristic $\mathcal{G}_1$ and $\mathcal{G}_2$ in series, we obtain (see Fig. 4) the characteristic $\mathcal{G}_3$, which is neither voltage-controlled nor current-controlled.

A (possibly time-varying) flux-controlled inductor is a voltage-controlled two-pole and, dually, a (possibly time-varying) charge-controlled capacitor is a current-controlled two-pole. If the inductor is flux-controlled, the current i is a function of the flux φ: $i(t) = \Gamma(\varphi(t),t)$. If $v(\cdot)$ is the voltage applied to the inductor and φ_0 is the flux through it at the initial time t_0, then by Lenz's law

$$\varphi(t) = \int_{t_0}^{t} v(t')dt' + \varphi_0$$

hence,

$$i(t) = \Gamma\left(\int_{t_0}^{t} v(t')dt' + \varphi_0 , t\right) \quad \text{for all} \quad t.$$

2.3 *Examples of One-To-One Two-Poles*

We present here a set of sufficient conditions under which some elementary parallel or series connections of circuit elements constitute a two-pole which is either current-controlled, voltage-controlled or one-to-one. As the reader expects, quite specific assumptions will have to be made on the characteristics of the elements in order for the circuit to be a one-to-one two-pole.

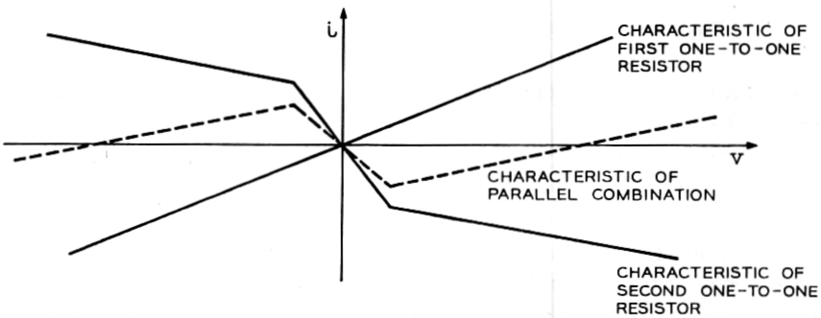

Fig. 5— Parallel connection of two one-to-one resistors.

The elements that we are going to consider are capacitors, resistors and inductors. Let us rank order these elements together with voltage sources and current sources in the following way: E,C,R,L,J. We shall say that a resistor is higher in rank than an inductor or a current source but lower in rank than a capacitor or a voltage source.

Until the end of this section, to simplify the discussion and without loss of generality, elements are assumed to be time-invariant.

Theorem: Consider the following circuits: the parallel RC, the parallel RL, the parallel LC and the parallel RLC circuit.

(*A*) *If* (*a*) *the highest-ranked element is current-controlled* (*charge-controlled in the case of the capacitor*),

(*b*) *all other elements are voltage-controlled* (*flux-controlled in the case of the inductor*), *and* (*c*) *the characteristics of all elements satisfy a Lipschitz condition according to Table I,*

then the parallel circuit is current-controlled.

(*B*) *If, in addition* (*d*) *the highest ranked element is one-to-one, then each parallel circuit is one-to-one.*

Proof: We shall consider only the *RLC* circuit, since the proofs of the simpler cases follow in a similar way.

First let us prove that (*a*), (*b*), and (*c*) imply that the circuit is current-controlled. Let i_s be the source current. Then with the usual notation

$$i_s = \dot{q} + \mathcal{G}(v) + \Gamma(\varphi)$$

or, equivalently, using the fact that the capacitor is charge-controlled

$$\begin{cases} \dot{q} = -\mathcal{G}(\mathfrak{D}(q)) - \Gamma(\varphi) + i_s . & (1) \\ \dot{\varphi} = \mathfrak{D}(q) & (2) \end{cases}$$

By assumption (*c*) $\mathfrak{D}$, Γ and $\mathcal{G}$ satisfy Lipschitz conditions. Since the composite of two Lipschitzian functions is Lipschitzian, $\mathcal{G}(\mathfrak{D}(\cdot))$ is also Lipschitzian; therefore the system (1), (2) has a unique solution for each initial state and each current source. In this case the state is (q,φ). Thus the *RLC* is current-controlled.

Second we prove that (*a*), (*b*), (*c*), (*d*) imply that the *RLC* circuit is one-to-one. It is immediate that these assumptions imply that the *RLC* circuit is voltage-controlled. It remains to show that it is one-to-one.

Call $q_1(\cdot)$, $\varphi_1(\cdot)$, and $v_1(\cdot)$ the charge, flux and voltage resulting from the initial state (q_0, φ_0) at time t_0 and the input current i_s. The functions $q_1(\cdot)$ and $\varphi_1(\cdot)$ are the corresponding solutions of (1) and (2); $v_1(t) = \dot{\varphi}_1(t) = \mathfrak{D}(q_1(t))$. We have to show that, starting from the same initial state (q_0, φ_0) at time t_0, the input current resulting from the applied voltage v_1 is precisely i_s.

Let q_2, φ_2 and i_2 be the resulting charge, flux and input current. It is immediate that $v_1(t) = \dot{\varphi}_2(t) = \mathfrak{D}(q_2(t))$. Since $\varphi_2(t_0) = \varphi_0$, we have

TABLE I

Circuit	Highest-Ranked Element	Characteristics That Satisfy Lipschitz Conditions
RL	*R*	$\mathfrak{R}(i)$, $\Gamma(\varphi)$
RC	*C*	$\mathfrak{D}(q)$, $\mathcal{G}(v)$
LC	*C*	$\mathfrak{D}(q)$, $\Gamma(\varphi)$
RLC	*C*	$\mathfrak{D}(q)$, $\mathcal{G}(v)$, $\Gamma(\varphi)$

$\varphi_2 = \varphi_1$. Since the capacitor is one-to-one, q_2 is uniquely defined by the relation above in terms of v_1, hence $q_2 = q_1$. Finally, by Kirchhoff's current law

$$i_2 = \dot{q}_2 + \mathcal{G}(v_2) + \Gamma(\varphi_2)$$
$$= \dot{q}_1 + \mathcal{G}(\mathfrak{D}(q_1)) + \Gamma(\varphi_1).$$

The last expression is precisely i_s by (1). Therefore $i_2 = i_s$. This concludes the proof that the parallel RLC circuit is a one-to-one two-pole. The dual case is covered by the following

Theorem: Consider the following circuits: the series RL, the series RC, the series LC and the series RLC circuit.

(A) If

(a) the lowest-ranked element is voltage-controlled (flux-controlled for inductor),

(b) all other elements are current-controlled (charge-controlled for capacitors), and

(c) the characteristics of all elements satisfy Lipschitz conditions according to Table II,

then the series circuit is voltage-controlled.

(B) If in addition (d) the lowest-ranked element is one-to-one,

then each series circuit is one-to-one.

The proof is similar to that of the previous theorem and is therefore omitted.

III. REDUCTION OF THE NETWORK

Throughout the article we consider networks consisting of nonlinear time-varying resistors, capacitors, inductors (without mutual inductance) and independent sources. We shall label by $\mathfrak{N}$ the network under consideration. Usually, we consider each element and each source as constituting a branch of $\mathfrak{N}$. We denote by E,C,R,L,J the set of branches of $\mathfrak{N}$ which are voltage sources, capacitors, resistors, inductors and cur-

TABLE II

Circuit	Lowest-Ranked Element	Characteristics That Satisfy Lipschitz Conditions
RL	L	$\mathcal{R}(i)$, $\Gamma(\varphi)$
RC	R	$\mathfrak{D}(q)$, $\mathcal{G}(v)$
LC	L	$\mathfrak{D}(q)$, $\Gamma(\varphi)$
RLC	L	$\mathfrak{D}(q)$, $\mathcal{R}(i)$, $\Gamma(\varphi)$

rent sources, respectively. In our discussion, certain networks derived from $\mathfrak{N}$ will play an important role. In order to refer to them conveniently, let us define the following notations: Let "A" be a subset of the set of branches of $\mathfrak{N}$. Let us define[9]

$\mathfrak{N}_A$ to be the network derived from $\mathfrak{N}$ by removing all branches except the ones which are members of A,

$\mathfrak{N}_{(A)}$ to be the network derived from $\mathfrak{N}$ by replacing branches of set A by a short circuit, and

$\mathfrak{N}_{(A)*}$ to be the network derived from $\mathfrak{N}$ by removing the branches of set A.

We shall use these notations as well as combinations of them. For example, $\mathfrak{N}_{(E)C}$ is the network derived from $\mathfrak{N}$ by first replacing branches of set E (the voltage sources) by short circuits and then removing all elements which do not belong to set C. Similarly, $\mathfrak{N}_{(E)(J)*}$ is the network derived from $\mathfrak{N}$ by shorting all voltage sources and removing all current sources.

$S*\mathfrak{N}$ is defined to be the network derived from $\mathfrak{N}$ by separating it into the maximum number of separable subnetworks.

Throughout the article we assume that, first, for any cut set of current sources only, the source currents satisfy Kirchoff's current law, and, second, for any loop of voltage sources only, the source voltages satisfy Kirchoff's voltage law.

Without loss of generality we consider networks that are connected and nonseparable. This assumption does not exclude the possibility that $\mathfrak{N}_{(E)(J)*}$ be both unconnected and/or separable. In the following we shall prove that without a loss of generality we can restrict the discussion to a network $\mathfrak{N}$ such that $\mathfrak{N}_{(E)(J)*}$ is both connected and nonseparable. The proof consists of an algorithm which changes the configuration and reduces $\mathfrak{N}$ into a network $\mathfrak{N}'$ which has the following properties:

(*i*) $\mathfrak{N}'$ consists of connected subgraphs, $\mathfrak{N}_i'$, such that for each one of them, $\mathfrak{N}'_{i(E)(J)*}$ is connected and nonseparable.

(*ii*) For all branches of $\mathfrak{N}$ and $\mathfrak{N}'$ which are not sources, any set of branch currents and voltages is a solution of $\mathfrak{N}$ if and only if it is also a solution of $\mathfrak{N}'$ (when the latter is driven by the corresponding sources).

(*iii*) Current sources of $\mathfrak{N}'$ are linearly related to the current sources of $\mathfrak{N}$. The same is true for voltage sources.

The step-by-step reduction of the network $\mathfrak{N}$ to $\mathfrak{N}'$ is done as follows.

(1) From each loop which consists of voltage sources only, remove one voltage source.

(2) In each cut set which consists of current sources only, replace one of the current sources by a short circuit.

The resulting network is connected; it has a tree which includes *all* the voltage sources as tree branches and *all* the current sources as links.

(3) Each current source J whose fundamental loop includes more that one tree branch is removed from the network and is replaced by a set of current sources identical to J, each one placed in parallel with a tree branch of the fundamental loop.

(4) All current sources that are in parallel with voltage sources are removed.

(5) Any parallel connection of current sources is replaced by one equivalent current source.

(6) Consider each fundamental cut set defined by a voltage source. For each one of them insert in every link a voltage source equal to that which is in the tree branch and, finally, short circuit the tree branch voltage source.

(7) In each link, replace any series connection of voltage sources by one equivalent voltage source.

(8) Separate the network into the maximum number of connected, nonseparable subgraphs.

The resulting network is called $\mathfrak{N}'$. Property (*ii*) follows from the fact that all the steps of the above algorithm do not change the source contribution to any of the fundamental loop equations or the cut set equations. Property (*i*) follows from the fact that all current sources are links of $\mathfrak{N}'$ and all voltage sources are in a link. Property (*iii*) follows from steps (5) and (7). Finally, observe that $S_*\mathfrak{N}_{(E)(J)*}$ is identical with $\mathfrak{N}'_{(E)(J)*}$.

It is well known that the state of the network is completely determined by all the voltages, fluxes, charges, and currents in the branches of the network. In the case of linear networks it is well known that certain proper subsets of these variables may be chosen as the state. For special classes of nonlinear RLC networks similar subsets will be indicated in the sequel.

We call a *solution* of an RLC network any set of voltages and currents of resistors, charges and voltages of capacitors, fluxes and currents of inductors which satisfy the Kirchhoff's laws and the branch characteristics. A network $\mathfrak{N}$ is said to be *determinate* if for any value of the initial state $\mathbf{s}_0$, given at any initial time t_0, and for any value of the sources $\mathbf{E}(\cdot)$, $\mathbf{J}(\cdot)$, there exists one and only one solution for $t \geqq t_0$ on some nonvanishing interval $[t_0, t_\alpha)$.

In the following section we shall describe a broad class of nonlinear RLC networks which are determinate.

IV. ONE-ELEMENT-KIND NETWORK

The purpose of this section is to establish a set of sufficient conditions under which a nonlinear (possibly time-varying) resistor network driven by a set of independent current sources and voltage sources has, for all possible inputs, one and only one set of branch voltages and branch currents that satisfy Kirchhoff's laws. Conditions under which the solution satisfies a Lipschitz condition with respect to the sources are also given.

The analysis of nonlinear resistor networks is almost identical with that of nonlinear capacitor networks or nonlinear inductor networks. Since the nonlinear resistors are the most flexible elements, we shall develop our analysis in terms of resistor networks.

Let us start by making three preliminary remarks:

(*i*) Given a resistor network together with an arbitrary distribution of current sources, it is always legitimate to assume that there are no cut sets of current sources only. (Dually, that there are no loops of voltage sources only.)

(*ii*) Any voltage source in series with a resistor may always be absorbed into a suitably redefined branch characteristic. Refer to Fig. 6, where v_1 and v_2 are the node voltages of nodes 1 and 2 referred to the same datum. Let the current through the resistor be given by its characteristic $g(v,t)$; since $g(v,t) = g(v_1 - v_2 - e,t)$ and since $e(\cdot)$ is a known function of time, we may introduce a new branch characteristic $g_{12}(\cdot,\cdot)$ specified at each instant of time by

$$g_{12}(v_1 - v_2,t) \triangleq g(v_1 - v_2 - e(t),t).$$

In other words, the voltage source e has been absorbed into the time dependence of g_{12}. A similar reasoning applies to a current-controlled resistor in series with a voltage source.

The dual case can be taken care of in the same manner: in this case, a current source which is in parallel with either a voltage-controlled or a current-controlled resistor can be absorbed into the branch.

Thus, without loss of generality, a network of nonlinear resistors and sources can be thought of as a network of nonlinear time-varying resist-

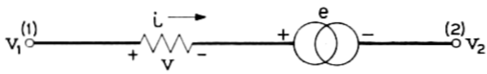

Fig. 6 — Voltage source in series with resistor.

ors with the understanding that the sources have been absorbed in the branch characteristics.

(*iii*) Thus when, as in Theorems I and II below, we consider a network of nonlinear time-varying resistors, we include the case of a network made up of time-varying resistors and of independent sources. There is no loss of generality in considering only connected networks, since it amounts to considering successively each separate part of an unconnected network.

We turn now to the statement of the main theorems.

Theorem I (Existence and Uniqueness): Consider a connected nonseparable network $\mathfrak{N}$ *of nonlinear (possibly time-varying) resistors. In case the resistor joining node* α *to node* β *is voltage-controlled, its characteristic is defined by the function* $g_{\alpha\beta}(\cdot,\cdot)$ *such that* $g_{\alpha\beta}(v_\alpha(t) - v_\beta(t),t)$ *is the current flowing through it at time t from node* α *to node* β*; here* v_α *and* v_β *are the node-to-datum voltages of nodes* α *and* β*. Similarly, if this resistor is current-controlled, its characteristic is defined by the function* $r_{\alpha\beta}(\cdot,\cdot)$ *such that* $r_{\alpha\beta}(i_{\alpha\beta}(t),t)$ *is the voltage difference between node* α *and node* β *at time t; here* $i_{\alpha\beta}$ *is the current through the resistor measured positively if it flows from* α *to* β*.*

If

(*a*) *there exists a tree* $\mathfrak{T}$ *such that all its tree branches are current-controlled and all its links are voltage-controlled,*

(*b*) *for all* α,β*, all t and all x in* $(-\infty,\infty)$

$$g_{\alpha\beta}(x,t) = -g_{\beta\alpha}(-x,t) \text{ if } (\alpha,\beta) \text{ is a link}$$

$$r_{\alpha\beta}(x,t) = -r_{\alpha\beta}(-x,t) \text{ if } (\alpha,\beta) \text{ is a tree branch}$$

(*c*) *for all links and all t,* $g_{\alpha\beta}(\cdot,t)$ *is a monotonically (not necessarily strictly) increasing continuous function defined on* $(-\infty,\infty)$*, and for all tree branches and all t,* $r_{\alpha\beta}(\cdot,t)$ *is a monotonically (not necessarily strictly) increasing continuous function defined on* $(-\infty,\infty)$*.*

Then,

for all current-sources i^s *connected between any pair of nodes and for all voltage sources* e^s *connected in series with network branches there exists one and only one set of branch voltages and branch currents that satisfy the Krichhoff laws and the branch characteristics.*

The conclusion of Theorem I can also be stated as follows: any network $\mathfrak{N}'$, formed from $\mathfrak{N}$ by inserting any set of voltage sources in series with any branch and any set of current sources between any node pair, is *determinate*.

Assumption (*b*) is a consequence of the physical meaning of the functions g and r and of the sign conventions: from a physical point of view they do not restrict the nonlinear resistors in any way. The two corollaries that follow are special cases of Theorem I. Corollary I is an extension of Theorems 2 and 3 of Duffin,[3] and is implied by his 1948 paper.[4] Such an extension has been pointed out by I.W. Sandberg.[15]

Corollary 1: Consider a connected network of nonlinear voltage-controlled (possibly time-varying) resistors.

If

(a) for all branches and all t, $g_{\alpha\beta}(\cdot,t)$ is a monotonically (not necessarily strictly) increasing, continuous function defined on $(-\infty,\infty)$, and

(b) there exists a tree $\mathfrak{T}$ such that all its branches have $g_{\alpha\beta}$'s which are, for all t, monotonically increasing one-to-one mappings of $(-\infty,\infty)$ onto $(-\infty,\infty)$,

then the conclusion of Theorem I holds.

Proof: The conclusion follows directly from Theorem I since the tree branches have $g_{\alpha\beta}$'s that are, for all t, monotonically increasing one-to-one mappings of $(-\infty,\infty)$ onto $(-\infty,\infty)$; hence the tree branches are also current-controlled resistors satisfying assumption (c) of Theorem I.

Corollary 2: Consider a connected network of nonlinear, current-controlled (possibly time-varying) resistors.

If

(a) for all branches and all t, $r_{\alpha\beta}(\cdot,t)$ is a monotonically (not necessarily strictly) increasing, continuous function defined on $(-\infty,\infty)$, and

(b) there exists a tree $\mathfrak{T}$ such that its links have $r_{\beta\alpha}$'s which are, for all t, monotonically increasing one-to-one mappings of $(-\infty,\infty)$ onto $(-\infty,\infty)$, *then* the conclusion of Theorem I holds.

We consider now the extension of the Thévenin and Norton equivalent circuits to nonlinear resistive networks. If we pick an arbitrary node pair of such a network $\mathfrak{N}$, we may regard these nodes as the terminals of a two-terminal network: we shall call the characteristic of this two-terminal network *the input characteristic of* $\mathfrak{N}$ *at these two nodes.* Dually, if we pick a branch and insert two terminals in series with it, we obtain a two-terminal network: we shall call the characteristic of this two-terminal network *the branch-input characteristic of* $\mathfrak{N}$.

Theorem II (Thévenin and Norton equivalent circuits): Consider a network $\mathfrak{N}$ *satisfying the requirements of Theorem I together with the same kind of source distribution.*

Then

(a) the input characteristic of $\mathfrak{N}$ *at any node pair is that of a current-controlled resistor whose characteristic is a continuous, monotonically increasing function defined on* $(-\infty, \infty)$*. This characteristic may be represented by the Thévenin equivalent circuit of Fig. 7(a): a series combination of a voltage source and a monotonically increasing current-controlled resistor whose characteristic passes through the origin.*

(b) The branch-input characteristic of $\mathfrak{N}$ *at any branch is that of a voltage-controlled resistor whose characteristic is a continuous, monotonically increasing function defined on* $(-\infty, \infty)$*. This characteristic may be represented by the Norton equivalent circuit of Fig. 7(b): a parallel combination of a current source and a monotonically increasing voltage-controlled resistor whose characteristic passes through the origin.*

Let us consider some special cases of Theorem II.

Corollary 3: Consider a connected network of nonlinear (possibly time-varying) resistors satisfying assumptions (*a*), (*b*) and (*c*) of Theorem I.

(a) If, in addition, the characteristics of the tree branches of $\mathfrak{T}$ are strictly increasing, then the input characteristic at any node pair is that of a strictly increasing current-controlled resistor. If the characteristics of all tree branches of $\mathfrak{T}$ are continuous, monotonically increasing, one-to-one mappings of $(-\infty, \infty)$ onto $(-\infty, \infty)$ then so is the input characteristic at any node pair.

(b) If the characteristics of the links of the tree $\mathfrak{T}$ are strictly increasing, then the branch-input characteristic is that of a strictly increasing voltage-controlled resistor. If the characteristics of all links of $\mathfrak{T}$ are continuous, monotonically increasing, one-to-one mappings of $(-\infty, \infty)$ onto $(-\infty, \infty)$, then so is any branch-input characteristic.

Proof of Theorems I and II: The proof of these two theorems is divided

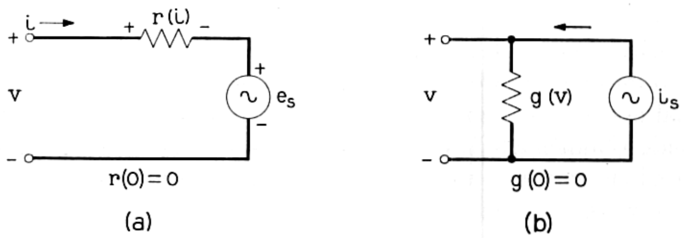

Fig. 7 — (a) Thévenin equivalent circuit: series combination of voltage source and a monotonically increasing current-controlled resistor whose characteristic passes through the origin. (b) Norton equivalent circuit: parallel combination of current source and monotonically increasing voltage-controlled resistor whose characteristic passes through the origin.

into two parts: in part one, we show that if Theorem I holds for a k-node network then Theorem II is true for a k-node network. In part two, we use this implication to prove Theorem I by induction.

The statement of the theorem allows time-varying resistors (and hence includes independent sources); however, in order to have as simple a notation as possible, we write down the proof as if all resistors were time-invariant.

Part One: We show that, for any integer $k \geqq 2$, if Theorem I holds for a k-node network then the input characteristic at any node pair is that of the Thévenin equivalent circuit specified in Theorem II (a). Let us connect the node pair under consideration to a current source i_s (see Fig. 8); this current source is viewed as an additional link, since it is a voltage-controlled resistor. By assumption, to each i_s there is one and only one set of branch currents and voltages that satisfy Kirchhoff's

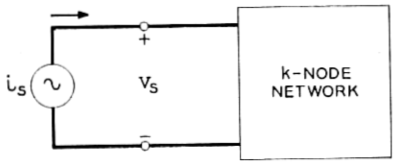

Fig. 8 — Node pair connected to current source.

laws and the branch characteristics. Consider two distinct values of i_s, namely, i_s and i_s'. Let the corresponding branch variables be $\mathbf{v},\mathbf{i}$ and $\mathbf{v}',\mathbf{i}'$. For each current-controlled branch define a number $\tilde{r}$ (which depends on $\mathbf{i}$ and $\mathbf{i}'$) by the relation

$$v - v' \triangleq \Delta v = r(i) - r(i') \triangleq \tilde{r} \cdot (i - i') = \tilde{r} \cdot \Delta i.$$

Since all the current-controlled branches are monotonically increasing, $\tilde{r} \geqq 0$. (If $\Delta i = 0$, $\tilde{r}$ may be taken to be any nonnegative number.) Similarly, we define a $\tilde{g}$ for each voltage-controlled resistor; again $\tilde{g} \geqq 0$. The set of Δv's and Δi's together with Δv_s and Δi_s may be considered as a set of branch voltages and branch currents together with the source voltage and source current of a *linear* resistive network which is obtained by replacing each current-controlled resistor by a linear resistor of resistance $\tilde{r}$, each voltage-controlled resistor by a linear resistor of conductance $\tilde{g}$ and the current source by a current source Δi_s. Since the Δv's and Δi's satisfy Kirchhoff's laws, Tellegen's theorem[12] holds,

$$\Delta v_s \cdot \Delta i_s = \sum \Delta v \Delta i$$

where the sum is over all resistive branches.

Since all branches have monotonically increasing characteristics, this is a sum of nonnegative terms and $\Delta v_s \Delta i_s \geqq 0$. In other words, $\Delta i_s > 0$ implies that $\Delta v_s \geqq 0$: that is, the Thévenin equivalent circuit has a current-controlled monotonically increasing characteristic. The continuity of the characteristic follows from the following considerations: irrespective of the values of the $\tilde{r}$'s and $\tilde{g}$'s, the fact that assumption (*c*) of Theorem I requires them to be nonnegative implies that the current transfer ratio from the current source to any branch has a magnitude no larger than unity;[11] hence $\Delta i_s \to 0$ implies $\Delta i \to 0$ for all branches. Since the tree branches have continuous characteristics, it follows that, for them, $\Delta v \to 0$ and, by Kirchhoff's voltage law, the same holds for the links. Hence $\Delta i_s \to 0$ implies $\Delta v_s \to 0$, i.e., the current-controlled characteristic of the Thévenin equivalent circuit is continuous. The proof of part (*b*) of Theorem II follows exactly the dual of the above argument.

Part Two: Let us prove Theorem I for a two-node network (see Fig. 9). Let us plot on the (v,i) plane of Fig. 10 the characteristics of the current-controlled tree branch and that of the voltage-controlled link, taking into account the sign conventions defined on Fig. 9. By assumption, the functions g and r are both continuous and have $(-\infty, \infty)$ as domains; therefore their representative curves intersect at least at one point (v,i). We assert that it is the only one: indeed, suppose there were a second one, (v',i'); then the monotonicity of r and g imply, respectively

$$(v' - v)(i' - i) \geqq 0 \quad \text{and} \quad (v' - v)(i' - i) \leqq 0.$$

Hence

$$(v' - v)(i' - i) = 0.$$

Suppose $v' = v$; then since g is a function

$$i = g(-v) = g(-v') = i'.$$

Similarly, if $i' = i$, the fact that r is a function implies $v = v'$. Hence for all possible sources, there is one and only one set of branch voltages and currents that satisfies Kirchhoff's laws and the branch characteristics.

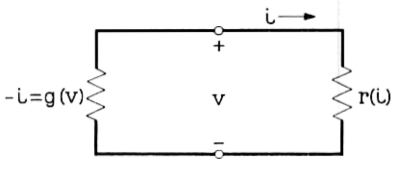

Fig. 9 — Two-node network.

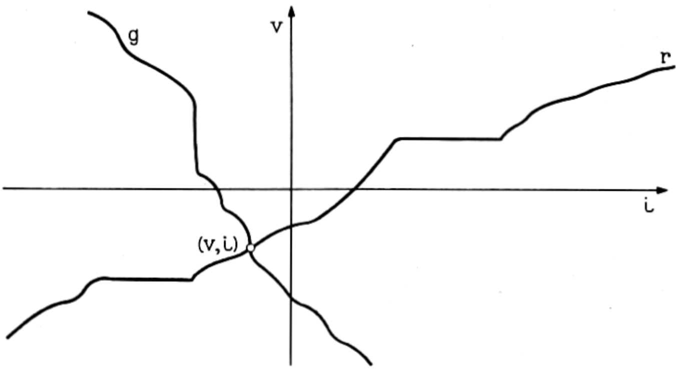

Fig. 10 — Characteristics of current-controlled tree branch and of voltage-controlled link as function of the tree branch current and voltage.

Thus Theorem I is established for a two-node network. The next step in the proof of Theorem I is to show that if it is true for an n-node network it is true for an $(n + 1)$-node network. Consider the n-node network shown in Fig. 11. We shall build out this network into an $(n + 1)$-node network.

Let us first connect the tree branch between node n and node $(n + 1)$, i.e., a current-controlled resistor. (There is no loss of generality in assuming that the numbering of the nodes is such that the branch $(n,n + 1)$ is a tree branch.) It is obvious that, for this network, the existence and uniqueness of the solution holds for all sources. Consequently, from part one of the proof, the input characteristic at any two nodes of this particular $(n + 1)$-node network has the equivalent circuits specified by Theorem II. The next step is to add a link, say between node k and node $(n + 1)$. Since the input characteristic at the node pair $(k,n + 1)$ is as specified in Theorem II (a), the voltage and current in the link are uniquely determined by the reasoning given for the case $n = 2$, and consequently the distribution of voltages and currents in all branches of the network is uniquely determined.

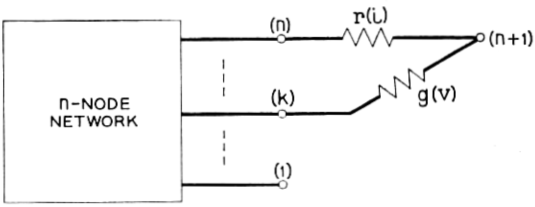

Fig. 11 — N-node network built out to $(n + 1)$-node network.

The process of constructing the $(n + 1)$-node network from the n-node network can be carried out step by step, adding a link at a time. Thus at the end of the process there is one and only one set of branch voltages and currents in the $(n + 1)$-node network that satisfies Kirchhoff's laws and the branch characteristics. Q.E.D.

For the purpose of solving the network differential equations of a general nonlinear *RLC* network it is important to know, for the resistive network case, under what conditions the function which maps the sources, $(\mathbf{E},\mathbf{J})$, into the branch voltages and currents, $(\mathbf{v},\mathbf{i})$, satisfies a Lipschitz condition. It is immediately clear that additional assumptions are required: consider Fig. 12, which shows the characteristic of a current-controlled resistor which fulfills the conditions of Theorem I. In the neighborhood of the operating point A, this resistor may appear to small signals either as an open circuit or a short circuit. Note that the same statement would apply if the resistor were voltage-controlled. It is obvious that under such conditions, the mapping $(\mathbf{E},\mathbf{J}) \rightarrow (\mathbf{v},\mathbf{i})$ will not satisfy a Lipschitz condition. As shown in the following theorem, only weak additional assumptions are required.

Theorem III: Consider a connected network of nonlinear (possibly time-varying) resistors which satisfies conditions (a), (b) and (c) of Theorem I. If, in addition, the following Lipschitz conditions are satisfied: there is a real-valued function $h(R,t)$, defined and positive for $R > 0$ and all t, such that

$$| g_{\alpha\beta}(x,t) - g_{\alpha\beta}(x',t) | \leqq h(R,t) | x - x' |$$

for all links of $\mathfrak{J}$, for all x, x' in $(-R,R)$ and all t and

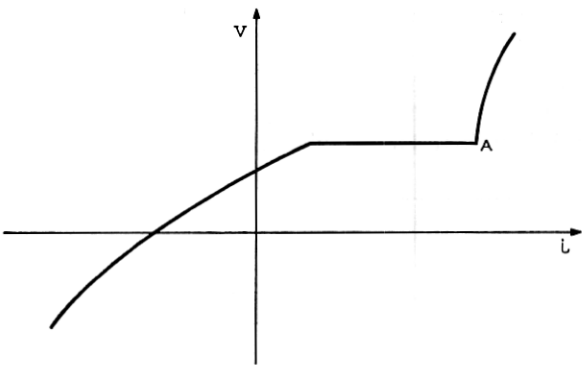

Fig. 12 — Characteristic of current-controlled resistor fulfilling conditions of Theorem I; note unbounded slope at point A.

$$| r_{\alpha\beta}(x,t) - r_{\alpha\beta}(x',t) | \leqq h(R,t) | x - x' |$$

for all branches of $\mathfrak{Z}$, *for all* x,x' *in* $(-R,R)$ *and all* t, *then the mapping which maps* $(\mathbf{E},\mathbf{J})$ *into* $(\mathbf{v},\mathbf{i})$ *satisfies a Lipschitz condition.**

Proof: Consider the effect of a change in the voltage sources $\mathbf{E}$ on the branch voltages $\mathbf{v}$ and branch currents $\mathbf{i}$. $\mathbf{E}$ is the vector whose ith component is the output voltage e_i of the source located in the ith branch. In the present $(n + 1)$ node network there are at most $n(n + 1)/2$ branches, hence $\mathbf{E}$ has at most that many components. Suppose that the change from $\mathbf{E}$ to $\mathbf{E} + \Delta\mathbf{E}$ is obtained by changing e_i to $e_i + \Delta e_i$ successively with $i = 1, 2, \cdots$. Call α,β the terminals of the first branch and call N_1 the remainder of the network (see Fig. 13). Since the input characteristic of N_1 is monotonically increasing, and since an increase of the voltage across the nonlinear resistance R increases the current through it or keeps it constant, the change in the input voltage of N_1,

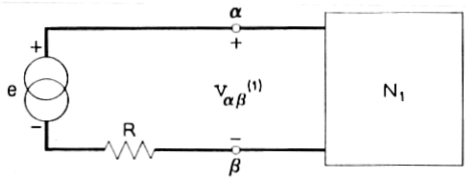

Fig. 13 — Nonlinear resistor R and voltage source in one branch of $(n + 1)$-node network; N_1 represents remainder of network.

$\Delta v_{\alpha\beta}^{(1)}$ due to the change of e_1 to $e_1 + \Delta e_1$ is such that $| \Delta e_1 | \geqq | \Delta v_{\alpha\beta}^{(1)} |$. (The superscript 1 indicates that only the source voltage in the first branch has been changed.) Call $\Delta v_k^{(1)}$ the corresponding change in the kth branch voltage. We assert that

$$| \Delta v_k^{(1)} | \leqq | \Delta v_{\alpha\beta}^{(1)} | \leqq | \Delta e_1 |.$$

For the particular change in the sources under consideration, we may define, as in the proof of Theorem I, for each tree branch a suitable $\tilde{r}$ and for each link a suitable $\tilde{g}$. Observe now that Δe_1 and the $\Delta v_k^{(1)}$ may be interpreted as being the source voltage and the resulting branch voltage of a linear network which has the same configuration as the given nonlinear network but in which each nonlinear resistor is replaced by $\tilde{r}$ or $\tilde{g}$ as required. By assumption (c) of Theorem I, all the $\tilde{r}$'s and $\tilde{g}$'s are nonnegative, hence all the voltage transfer ratios $| \Delta v_k^{(1)}/\Delta e^1 |$ of

* Incidentally, if the network $\mathfrak{N}$ was derived from another network $\mathfrak{N}_A$, by applying to $\mathfrak{N}_A$ the algorithm of Section III, then the mapping $(\mathbf{E},\mathbf{J}) \rightarrow (\mathbf{v},\mathbf{i})$ is one-to-one.

the linear network cannot[11] exceed 1 and the inequality asserted above follows. Thus, for all i's and k's,

$$| \Delta v_k^{(i)} | \leqq | \Delta e_i |.$$

Let Δv_k be the change in the voltage across the kth branch when $\mathbf{E}$ becomes $\mathbf{E} + \Delta\mathbf{E}$. Summing over i, using the triangle inequality, and defining the norm of a vector as the sum of the magnitude of its components, we get

$$| \Delta v_k | \leqq \| \Delta\mathbf{E} \|. \tag{3}$$

Since there are at most $n(n + 1)/2$ branches, we get finally

$$\| \Delta\mathbf{v} \| \leqq [n(n + 1)/2] \| \Delta\mathbf{E} \| \tag{4}$$

where $\Delta\mathbf{v}$ is the change in the branch voltages corresponding to the change of the voltage sources from $\mathbf{E}$ to $\mathbf{E} + \Delta\mathbf{E}$. We next bound the change in the branch currents. Applying (3) to a link and using the Lipschitz condition we find that

$$| \Delta i_k | \leqq h(R,t) \| \Delta\mathbf{E} \| \qquad \text{(for all links)}$$

and since there are at most $n(n - 1)/2$ links and the change in a tree branch current is equal to the change in the sum of currents of the links which belong to its fundamental cut set,

$$| \Delta i_k | \leqq h(R,t)[n(n - 1)/2] \| \Delta\mathbf{E} \| \qquad \text{(for all branches)}.$$

Thus

$$\| \Delta\mathbf{i} \| \leqq h(R,t)[n^2(n^2 - 1)/4] \| \Delta\mathbf{E} \|. \tag{5}$$

The effect of a change in the current sources from $\mathbf{J}$ to $\mathbf{J} + \Delta\mathbf{J}$ is obtained in a dual manner. Since the current transfer ratio may not exceed unity[11] we get

$$| \Delta i_k | \leqq \| \Delta\mathbf{J} \|$$

and

$$\| \Delta\mathbf{i} \| \leqq [n(n + 1)/2] \| \Delta\mathbf{J} \|. \tag{6}$$

This implies

$$| \Delta v_k | \leqq h(R,t) \| \Delta\mathbf{J} \| \qquad \text{(for all tree branches)}$$

and, by Kirchhoff's voltage law,

$$| \Delta v_k | \leqq h(R,t)n \| \Delta\mathbf{J} \| \qquad \text{(for all branches)}.$$

Finally

$$\| \Delta \mathbf{v} \| \leqq h(R,t)[n^2(n+1)/2] \| \Delta \mathbf{J} \|. \tag{7}$$

Using the usual product topology[17] for both the product spaces of voltage sources and current sources on the one hand and branch voltages and branch currents on the other, and invoking (4) to (7), we conclude that the mapping $(\mathbf{E},\mathbf{J}) \rightarrow (\mathbf{v},\mathbf{i})$ is Lipschitz.

V. NONLINEAR RLC NETWORKS

The previous section required all elements of the network to be of the same kind and to have a monotonically increasing characteristic. In this section both requirements are removed. In addition to independent sources, the network consists of nonlinear (possibly time-varying) resistors, capacitors and inductors and some of the elements are allowed to have characteristics with negative slope.

As a first step let us make one remark. Theorems I, II, and III would still hold if all resistors were montonically decreasing instead of monotonically increasing. In the more complicated situation considered here the same possible choice exists. For example, separable subnetworks of $\mathfrak{N}_{(E)C}$ which contain more than one capacitor could just as well contain monotonically decreasing capacitors. For simplicity, we shall assume that all monotonic elements are increasing.

In order to state the following theorem we need two definitions. A network (or subnetwork) is called a *self-loop* if it consists of a single branch whose end-points are identified: it consists of one branch and one node. A network (or subnetwork) is called an *open branch* if it consists of a single branch whose end-points are not identified: it consists of one branch and two nodes.

Theorem IV: Let $\mathfrak{N}$ be a network of independent sources and nonlinear (possibly time-varying) resistors, capacitors and inductors (without mutual inductance) such that: capacitors of $\mathfrak{N}$ are either charge-controlled or monotonically increasing voltage-controlled; resistors are either voltage-controlled or current-controlled; inductors are either flux-controlled or monotonically increasing current-controlled. It is further assumed that $\mathfrak{N}$ and $\mathfrak{N}_{(E)(J)^*}$ are nonseparable and connected. The network $\mathfrak{N}$ is determinate if:

(1) The capacitor network $S{*}\mathfrak{N}_{(E)C}$ satisfies the following requirements:

(a) Open branches of $S{*}\mathfrak{N}_{(E)C}$ are charge-controlled and contain all charge-controlled capacitors which are not monotonically increasing.

(b) Each subnetwork of $S{*}\mathfrak{N}_{(E)C}$ which contains more than one element

has a tree with monotonically increasing charge-controlled tree branches and monotonically increasing voltage-controlled links.

(*c*) *Self-loops of* $S{*}\mathfrak{N}_{(E)C}$ *are voltage-controlled.*

(2) *The resistive network* $S{*}\mathfrak{N}_{(EC)R}$ *satisfies the following requirements:*

(*a*) *Open branches are current-controlled and contain all current-controlled resistors which are not monotonically increasing.*

(*b*) *Each subnetwork which contains more than one element has a tree with monotonically increasing current-controlled tree branches and monotonically increasing voltage-controlled links.*

(*c*) *Self-loops are voltage-controlled and contain all voltage-controlled resistors which are not monotonically increasing.*

(3) *The inductive network* $S{*}\mathfrak{N}_{(ECR)L}$ *satisfies the following requirements:*

(*a*) *Open branches are current-controlled.*

(*b*) *Each subnetwork which contains more than one element has a tree with monotonically increasing current-controlled tree branches and monotonically increasing flux-controlled branches.*

(*c*) *Self-loops are flux-controlled and contain all flux-controlled inductors that are not monotonically increasing.*

(4) *In any finite interval, and for all time, the characteristics of the network resistors, capacitors and inductors satisfy a Lipschitz condition with respect to the following variables:*

tree branches: capacitors, with respect to q
resistors and inductors, with respect to i
links: capacitors and resistors, with respect to v
inductors, with respect to φ.

Remarks: Note that nonmonotonic voltage-controlled capacitors and current-controlled inductors were excluded from the discussion. Such capacitors and inductors may be included in the discussion provided they fall into the following trivial cases: each nonmonotonic voltage-controlled capacitor is in parallel with a voltage source and each nonmonotonic current-controlled inductor is in series with a current source. In such cases, $\mathfrak{N}_{(E)(J)*}$ is separable unless $\mathfrak{N}$ contains one element only.

The above conditions insure the existence and uniqueness[18] of the solution on some nonvanishing interval $[t_0, t_\alpha)$, where $t_\alpha > t_0$. The length of this interval cannot be specified without further assumptions on the Lipschitz constants $h(R,t)$. This is the well known problem of finite escape time. In particular, if for all branch characteristics the same Lipschitz constant can be used and holds over the whole domain of the characteristic, then the solution exists and is unique on $[t_0, \infty)$ for all regulated **E**'s and **J**'s.

Proof of Theorem IV: Let us denote the voltages and charges of the capacitive branches by $(\mathbf{e}_c, \mathbf{q}_c)$. Similarly, denote the voltages and currents of the resistive branches by $(\mathbf{e}_r, \mathbf{i}_r)$ and fluxes and currents of inductive branches by $(\varphi_L, \mathbf{i}_L)$. Voltage sources and current sources will denoted as usual by $\mathbf{E}$ and $\mathbf{J}$.

We assert that conditions (2) and (4) of Theorem IV imply, first, that the currents and the voltages of the resistive branches at time t, $(\mathbf{e}_r(t),\mathbf{i}_r(t))$ are uniquely determined by the values, *at the same time* t, of the capacitor voltages, the voltage sources, the inductor currents and the current sources, $(\mathbf{e}_c(t),\mathbf{E}(t),\mathbf{i}_L(t),\mathbf{J}(t))$; and, second, that the mappings

$$\mathbf{e}_r(t) = \mathbf{f}_{e_r}(\mathbf{e}_c(t),\mathbf{E}(t),\mathbf{i}_L(t),\mathbf{J}(t),t) \tag{8}$$

$$\mathbf{i}_r(t) = \mathbf{f}_{i_r}(\mathbf{e}_c(t),\mathbf{E}(t),\mathbf{i}_L(t),\mathbf{J}(t),t) \tag{9}$$

satisfy Lipschitz conditions.

Given any set of capacitor voltages $\mathbf{e}_c$ and inductor currents $\mathbf{i}_L$ such that Kirchoff's voltage law is satisfied in each loop formed by capacitors and voltage sources, and such that Kirchoff's current law is satisfied in each cut set formed by inductors and current sources, let us replace each capacitor with a voltage source whose voltage is equal to the voltage of the replaced capacitor and replace each inductor with a current source whose current is equal to the current of the replaced inductor. The network consists now of resistors, current sources and voltage sources only.

Let us use the algorithm of Section III to change the configuration of the sources and to separate the network into its separable parts. Let us denote the resulting network by $\mathfrak{N}^R$ and its sources by $(\mathbf{E}^R,\mathbf{J}^R)$.

The network $\mathfrak{N}^R$ has three sets of subnetworks: (a) connected nonseparable subnetworks which contain sources and two or more resistive branches, (b) subnetworks containing one resistive branch in parallel with a voltage source, and (c) subnetworks containing one resistive branch in parallel with a current source.

Consider the first set of subnetworks. Denote the branch voltages and branch currents of these subnetworks by $(\mathbf{e}_r, \mathbf{i}_r)_1$, and their sources by $(\mathbf{E}_1^R,\mathbf{J}_1^R)$. From conditions (2) and (4) of Theorem IV it follows that each subnetwork contains a tree whose resistive branches are monotonically increasing current-controlled resistors and whose links are monotonically increasing voltage-controlled resistors, and that all elements satisfy Lipschitz conditions. Therefore, from Theorems I and III of Section IV it follows that $(\mathbf{e}_r, \mathbf{i}_r)_1$ are uniquely denfied by $(\mathbf{E}_1^R,\mathbf{J}_1^R)$ and that the mapping $(\mathbf{e}_r(t),\mathbf{i}_r(t)) = \mathbf{f}_r(\mathbf{E}_1^R(t),\mathbf{J}_1^R(t),t)$ satisfies Lipschitz conditions.

The second set of networks corresponds to self-loops of $S*\mathfrak{N}_{(EC)R}$. In each subnetwork, the resistor is voltage-controlled, and hence the current through it is uniquely determined in terms of the voltage source. Since the characteristics satisfy a Lipschitz condition, the mapping from space $\mathbf{E}_2^R$ (the voltage sources) to $(\mathbf{e}_r, \mathbf{i}_r)_2$ (the branch voltages and currents in the subnetworks of the second set) satisfies Lipschitz conditions.

Subnetworks of the third set correspond to open branches of $S*\mathfrak{N}_{(EC)R}$. As each subnetwork contains only one resistor and a current source, the requirement that the branch be current-controlled is enough to insure uniqueness of $(\mathbf{e}_r, \mathbf{i}_r)_3$, the branch voltages and currents of this set, in terms of the corresponding sources $\mathbf{J}_3^R$. From condition (4) it follows that the mapping from $\mathbf{J}_3^R$ to $(\mathbf{e}_r, \mathbf{i}_r)_3$ satisfies the Lipschitz condition.

The voltages of sources $\mathbf{E}^R$ are linear combinations of the voltages $\mathbf{E}$ and $\mathbf{e}_c$, and the currents $\mathbf{J}^R$ are linear combinations of $\mathbf{J}$ and $\mathbf{i}_L$ (see Section IV). From this linearity property and from the properties of the above relations between the voltages and currents of the resistive branches and $(\mathbf{E}^R,\mathbf{J}^R)$ it follows that $(\mathbf{e}_r(t),\mathbf{i}_r(t))$ are uniquely defined by $(\mathbf{e}_c(t),\mathbf{E}(t),\mathbf{i}_L(t),\mathbf{J}(t))$ and that the mappings in (8) and (9) satisfy Lipschitz conditions.

Let us now consider the capacitors of the network $\mathfrak{N}$. Given any set of resistor currents $\mathbf{i}_r$ and inductor currents $\mathbf{i}_L$ such that Kirchoff's current law is satisfied in each cut set formed by resistors, inductors and current sources, let us replace the inductive and resistive branches of $\mathfrak{N}$ by current sources with currents equal to the corresponding currents $\mathbf{i}_L$, $\mathbf{i}_r$. The network consists now of sources and capacitors only. Let us use the algorithm of Section III to change the configuration of the sources and separate the network into its separable parts. The resulting network is denoted by $\mathfrak{N}^C$ and its sources by $\mathbf{E}^C,\mathbf{J}^C$. We are going to establish an analogy between $\mathfrak{N}^C$ and its sources by $\mathbf{E}^C,\mathbf{J}^C$. We are going to establish analogy between $\mathfrak{N}^C$ and $\mathfrak{N}^R$ and use the result just proved for $\mathfrak{N}^R$ to deduce a similar result for $\mathfrak{N}^C$.

$\mathfrak{N}^C$ consists of the three sets of subnetworks which were described in connections with $\mathfrak{N}^R$. Consider the second set of subnetworks of $\mathfrak{N}^C$, which consists of single capacitors in parallel with a voltage source. Except for the trivial case where $\mathfrak{N}$ consists only of a single capacitor in parallel with a voltage source, this set is empty, for otherwise $\mathfrak{N}_{(E)(J)*}$ would be separable. Condition (1) implies that each subnetwork of the first set has a tree, say τ_C, whose tree branches are monotonically increasing and charge-controlled, and whose links are monotonically increasing and voltage-controlled. For each subnetwork of this set, with each fundamental cut set of τ_C defined by a capacitive tree branch, we assign a variable q_i equal to the sum of the charges on all the capacitors of that cut set. For each subnetwork of the third set, we assign a q_i equal to the charge on the capacitor. $\mathbf{q}$ will denote the vector whose components are the q_i's.

The analogy between $\mathfrak{N}^C$ and $\mathfrak{N}^R$ is established in four steps:

(*i*) For $\mathfrak{N}^R$, the Kirchhoff current law applied to the ith cut set associated with a resistive tree branch reads

$$J_i^R = \sum_k i_{ki}$$

where i_{ki} is the current in the kth branch of the ith cut set. The i_{ki} are components of $\mathbf{i}_r$. For $\mathfrak{N}^C$, we have by definition of q_i,

$$q_i = \sum_k q_{ki}$$

where q_{ki} is the charge in the kth branch of the ith cut set. The q_{ki} are components of $\mathbf{q}$.

(*ii*) For both $\mathfrak{N}^R$ and $\mathfrak{N}^C$, the Kirchhoff voltage law holds.

(*iii*) Condition (1) imposes requirements on the topology and element characteristics of $\mathfrak{N}^C$ which are entirely similar to those imposed on $\mathfrak{N}^R$ by condition (2).

(*iv*) Finally, the elements of $\mathfrak{N}^R$ and $\mathfrak{N}^C$ satisfy analogous Lipschitz conditions by condition (4).

Therefore, the variables $(\mathbf{e}_c, \mathbf{q}_c)$ and $(\mathbf{E}^C, \mathbf{q})$ of $\mathfrak{N}^C$ are analogous to the variables $(\mathbf{e}_r, \mathbf{i}_r)$ and $(\mathbf{E}^R, \mathbf{J}^R)$ of $\mathfrak{N}^R$.

Remembering that $\mathbf{E}^C$ is linearly related to $\mathbf{E}$, we conclude that the voltages and charges of the capacitors at time t are uniquely determined by the values, *at the same time* t, of the voltage sources $\mathbf{E}(t)$ and $\mathbf{q}(t)$, and that the mapping

$$\mathbf{e}_c(t) = \mathbf{f}_{e_c}(\mathbf{E}(t), \mathbf{q}(t), t) \tag{10}$$

$$\mathbf{q}_c(t) = \mathbf{f}_{q_c}(\mathbf{E}(t), \mathbf{q}(t), t) \tag{11}$$

satisfies Lipschitz conditions.

Since q_i in any fundamental cut set is equal to the sum of the capacitor charges

$$\frac{dq_i(t)}{dt} = J_i^C(t)$$

where $J_i^C(t)$ is the contribution of the current sources to the ith cut set. As $\mathbf{J}^C$ is a linear combination of $\mathbf{i}_r$, $\mathbf{i}_L$, and $\mathbf{J}$ it follows that

$$\frac{d}{dt}\mathbf{q}(t) = \mathbf{f}_q(\mathbf{i}_r(t), \mathbf{i}_L(t), \mathbf{J}(t)) \tag{12}$$

where $\mathbf{f}_q$ is linear and does not depend explicitly on time.

Let us now consider the inductors of the network $\mathfrak{N}$. Given any set of resistor voltages $\mathbf{e}_r$ and capacitor voltages $\mathbf{e}_c$ such that Kirchoff's voltage law is satisfied in each loop formed by capacitors, resistors and voltage sources, let us replace the capacitor and resistor branches of $\mathfrak{N}$ with voltage sources equal to the corresponding voltages $\mathbf{e}_c$, $\mathbf{e}_r$. The network now consists of sources and inductors only. Let us again use the algorithm of Section III to change the configuration of the sources and separate the network into its separable parts. The resulting network is denoted by $\mathfrak{N}^L$ and its sources by $\mathbf{E}^L$,$\mathbf{J}^L$. As in the case of the resistive network, we are going to use the result previously proved for $\mathfrak{N}^R$ to deduce a similar result for $\mathfrak{N}^L$.

$\mathfrak{N}^L$ consists of the three sets of subnetworks which were described in connection with $\mathfrak{N}^R$. Consider the third set of subnetworks of $\mathfrak{N}^L$, which consists of single inductors in parallel with a current source. Except for the trivial case where $\mathfrak{N}$ consists only of a single inductor in parallel with a current source, this set is empty, for otherwise $\mathfrak{N}_{(E)(J)^*}$ would be separable. Condition (3) implies that each subnetwork of the first set has a tree, say τ_L, whose tree branches are monotonically increasing and current-controlled and whose links are monotonically increasing and flux-controlled. For each subnetwork of the first set, with each fundamental loop of τ_L defined by an inductive link, we assign a variable φ_i equal to the sum of the fluxes of all the inductors of that loop. For each subnetwork of the second set we assign a φ_i equal to the flux of the inductor. $\boldsymbol{\varphi}$ will denote the vector whose components are the φ_i's.

The analogy between $\mathfrak{N}^L$ and $\mathfrak{N}^R$ is established in four steps:

(*i*) For $\mathfrak{N}^R$ the Kirchhoff voltage law applied to the *i*th loop associated with a resistive link reads

$$E_i^R = \sum_k e_{ki}$$

where e_{ki} is the voltage across the *k*th branch of the *i*th loop. The e_{ki} are components of $\mathbf{e}_r$. For $\mathfrak{N}^L$ we have by definition of φ_i,

$$\varphi_i = \sum_k \varphi_{ki}$$

where φ_{ki} is the flux in the *k*th branch of the *i*th loop. The φ_{ki} are components of $\boldsymbol{\varphi}_L$.

(*ii*) For both $\mathfrak{N}^R$ and $\mathfrak{N}^L$ the Kirchhoff current law holds.

(*iii*) Condition (3) imposes requirements on the topology and element characteristics of $\mathfrak{N}^L$ which are entirely similar to those imposed on $\mathfrak{N}^R$ by condition (2).

(*iv*) Finally, the elements of $\mathfrak{N}^R$ and $\mathfrak{N}^L$ satisfy analogous Lipschitz conditions by condition (4).

Therefore, the variables $(\boldsymbol{\varphi}_L, \mathbf{i}_L)$ and $(\boldsymbol{\varphi}, \mathbf{J}^L)$ of $\mathfrak{N}^L$ are analogous to the variables $(\mathbf{e}_r, \mathbf{i}_r)$ and $(\mathbf{E}^R, \mathbf{J}^R)$.

As $\mathbf{J}^L$ is linearly related to $\mathbf{J}$, we conclude that fluxes and currents of the inductors at time t are uniquely determined by the values, at the same time t, of $(\boldsymbol{\varphi}(t), \mathbf{J}(t))$, and that the mappings

$$\boldsymbol{\varphi}_L(t) = \mathbf{f}_{\varphi_L}(\boldsymbol{\varphi}(t), \mathbf{J}(t), t) \tag{13}$$

$$\mathbf{i}_L(t) = \mathbf{f}_{i_L}(\boldsymbol{\varphi}(t), \mathbf{J}(t), t) \tag{14}$$

satisfy Lipschitz conditions.

Since φ_i in each loop is equal to the sum of the fluxes in the loop, it follows from the Kirchhoff voltage law that

$$\frac{d}{dt}\varphi_i(t) = E_i^L(t)$$

where $E_i^R(t)$ is the contribution of the voltage sources in the ith loop. As $\mathbf{E}^L$ is a linear combination of $\mathbf{e}_c$, $\mathbf{e}_r$, and $\mathbf{E}$, it follows that

$$\frac{d}{dt}\boldsymbol{\varphi}(t) = \mathbf{f}_\varphi(\mathbf{e}_c(t), \mathbf{e}_r(t), \mathbf{E}(t)) \tag{15}$$

where $\mathbf{f}_\varphi$ is linear and does not depend explicitly on time.

Any solution of the network requires that (8), (9), (10), (11), (12), (13), (14) and (15) be satisfied simultaneously. It is shown in the following that these equations determine a unique solution.

In (12) and (15) substitute values of $\mathbf{e}_c$, $\mathbf{e}_r$, $\mathbf{i}_r$ and $\mathbf{i}_L$ from (10), (11), (9) and (14). The results are

$$\frac{d}{dt}\mathbf{q} = \mathbf{f}_q[\mathbf{f}_{i_r}(\mathbf{f}_{e_c}(\mathbf{E},\mathbf{q},t),\mathbf{E},\mathbf{f}_{i_L}(\boldsymbol{\varphi},\mathbf{J},t),\mathbf{J},t),\mathbf{f}_{i_L}(\boldsymbol{\varphi},\mathbf{J},t),\mathbf{J}] \tag{16}$$

$$\frac{d}{dt}\boldsymbol{\varphi} = \mathbf{f}_\varphi[\mathbf{f}_{e_c}(\mathbf{E},\mathbf{q},t),\mathbf{f}_{e_r}(\mathbf{f}_{e_c}(\mathbf{E},\mathbf{q},t),\mathbf{E},\mathbf{f}_{i_L}(\boldsymbol{\varphi},\mathbf{J},t),\mathbf{J},t),\mathbf{E}]. \tag{17}$$

Since the right-hand sides of (16) and (17) are compositions of functions satisfying Lipschitz conditions, these equations may be rewritten as

$$\frac{d}{dt}\mathbf{q}(t) = \mathbf{F}_q(\mathbf{E}(t),\mathbf{q}(t),\boldsymbol{\varphi}(t),\mathbf{J}(t),t) \tag{18}$$

$$\frac{d}{dt}\varphi(t) = \mathbf{F}_\varphi(\mathbf{E}(t),\mathbf{q}(t),\varphi(t),\mathbf{J}(t),t) \tag{19}$$

where $\mathbf{F}_q$ and $\mathbf{F}_\varphi$ satisfy Lipschitz conditions in $\mathbf{q}$ and φ. Therefore, for any $\mathbf{E}(\cdot)$ and $\mathbf{J}(\cdot)$ that are regulated functions of time[13] and for any initial values of φ and $\mathbf{q}$, the differential equations (18) and (19) determine uniquely $\varphi(\cdot)$ and $\mathbf{q}(\cdot)$, and the solutions are continuous functions of time.[18] In terms of $\mathbf{E}(\cdot)$, $\mathbf{J}(\cdot)$, $\varphi(\cdot)$ and $\mathbf{q}(\cdot)$, equations (8), (9), (10), (11), (13) and (14) determine uniquely the currents and voltages of the resistive branches, the voltages and charges of the capacitive branches, and the fluxes and currents of the inductive branches. Therefore the network $\mathfrak{N}$ is determinate. (Incidentally, the proof shows that the state of the network may be represented by $(\mathbf{q}, \varphi)$.)

It is worth indicating an immediate consequence of (18) and (19) and the other circuit relations.

Corollary: If the conditions of Theorem IV are satisfied, $\mathbf{E}$ and $\mathbf{J}$ are continuous functions of time, and all elements depend continuously on time, then $\mathbf{e}_c$, $\mathbf{q}_c$, $\mathbf{e}_r$, $\mathbf{i}_r$, φ_L, $\mathbf{i}_L$ are continuous functions of time; in other words, jump phenomena[1] are excluded.

Corollary: Let the network $\mathfrak{N}$ consist of independent sources, nonlinear (possibly time-dependent) monotonically increasing one-to-one resistors, capacitors and inductors. If the characteristics of all elements Lipschitz conditions as described in condition (4) of Theorem IV, the network $\mathfrak{N}$ is determinate.

Corollary: If branches of a network $\mathfrak{N}$ consist of: (a) voltage sources, current sources; (b) one-to-one monotonically increasing resistors, capacitors, and inductors whose characteristics satisfy condition (4) of Theorem IV; (c) one-to-one two-poles of the types described by the theorems of Section II and which satisfy conditions (*a*), (*b*), (*c*) and (*d*) of these theorems: *then* network $\mathfrak{N}$ is determinate.

Given a physical circuit or device, it may happen that a particular model of the circuit does not satisfy the conditions of Theorem IV. For example, this model $\mathfrak{N}'$ might be such that $S{*}\mathfrak{N}'_{(E)C}$ includes a parallel connection of two charge-controlled capacitors, $D_1(q)$ and $D_2(q)$, with only D_1 monotonically increasing. Under these conditions, it may happen that the current through the parallel combination does not determine uniquely the voltage across it. If, however, the model is changed (call it $\mathfrak{N}''$) and a resistor (or inductor) is inserted in series with D_2, then $S{*}\mathfrak{N}''_{(E)C}$ now includes an open branch D_2, condition (1) of Theorem IV is no longer violated, and $\mathfrak{N}''$ is determinate. Obviously, this idea may be used in the case of inductors and resistors.

Finally, let us conclude this section by a discussion which draws attention to some consequences of the conditions of Theorem IV. Some of the properties considered here will be used in the next section for writing in detail the network equations.

Let $\mathfrak{N}$ be a network which satisfies the conditions of Theorem IV. Denote by r, r_{1-1}, g the sets of resistors which are current-controlled (but not voltage-controlled), one-to-one, and voltage-controlled (and not current-controlled), respectively. Similarly, denote respectively by d, d_{1-1}, c the charge-controlled (and not voltage-controlled), one-to-one, and voltage-controlled capacitors, and by l, Γ_{1-1}, Γ the current-controlled (and not flux-controlled), one-to-one, and flux-controlled inductors.

Let us carry out the following operations:

(a) choose a forest of $\mathfrak{N}_E$

(b) choose a forest of $\mathfrak{N}_{(E)d}$

(c) choose a forest of $\mathfrak{N}_{(Ed)d_{1-1}}$

(d) choose a forest of $\mathfrak{N}_{(Edd_{1-1})r}$

(e) choose a forest of $\mathfrak{N}_{(Edd_{1-1}r)r_{1-1}}$

(f) choose a forest of $\mathfrak{N}_{(Edd_{1-1}rr_{1-1})l}$

(g) choose a forest of $\mathfrak{N}_{(Edd_{1-1}rr_{1-1}l)\Gamma_{1-1}}$.

Since the conditions of Theorem IV are satisfied by $\mathfrak{N}$, it follows that *the union of these forests forms a tree of* $\mathfrak{N}$ which we denote by τ. The construction of this tree is an extension of Bryant's procedure.[9]

This can be proved in the following way: From the conditions of Theorem IV it follows that the union of the forests chosen by (b) and (c) [by (d) and (e), by (f) and (g)] are forests of $\mathfrak{N}_{(E)C}$, [$\mathfrak{N}_{(EC)R}$, and $\mathfrak{N}_{(ECR)L}$, respectively]. Let us add the network's resistors to $\mathfrak{N}_{(ECR)L}$. This is done by splitting nodes and adding the new branches between them. Consider a node which was split, say, to three nodes and a resistor subnetwork connected between these nodes, It is clear that the subtree of this resistive subnetwork completes the forest of $\mathfrak{N}_{(ECR)L}$ for a forest of the $\mathfrak{N}_{(EC)RL}$. We can use the same argument to show that by adding the capacitors and voltage sources we get a forest of the network which includes all branches but the current sources. However, the current sources do not form any cut set and therefore are links of this forest. Thus τ, the union of these forests, is a tree of $\mathfrak{N}$.

From the construction of the tree, the conditions of Theorem IV, and the above discussion it follows that τ *contains all current and charge-controlled elements which are not one-to-one, and all voltage and flux-controlled elements which are not one-to-one are links of this tree.*

Consider the fundamental cut set of $\mathfrak{N}$ defined by an element of set d, a charge-controlled capacitor whose characteristic is not monotonically increasing. By assumption, this capacitor is an open branch of $S*\mathfrak{N}_{(E)C}$;

this cut set may not contain capacitors or voltage sources and therefore consists solely of resistors, inductors and current sources. Similar properties exist for fundamental loops defined by the various links. One can exhibit these properties by making a table in which links and tree branches are partitioned according to the types of their elements and properties of their characteristics; for each link and branch the table specifies the type of elements that are allowed to be in the corresponding loop or cut set. As the table is complicated, it is omitted and only some of the more interesting properties are listed below. Here we are going to make use of the rank-order of the elements, *ECRLJ*, defined in Section II:

(*i*) Tree branches with characteristics which are not monotonically increasing are the *highest* ranked elements in their own fundamental cut set. Thus, for example, a charged-controlled nonmonotonically increasing capacitor has a fundamental cut set which may include links which are resistors, inductors and current sources but no other capacitors.

(*ii*) Links with characteristics which are not monotonically increasing are the *lowest* ranked elements in their own fundamental loop. Thus, a fundamental loop defined by a nonmonotonically increasing resistor may have only capacitors or voltage sources in its tree branches.

VI. EQUATIONS FOR RLC NETWORKS

The purpose of this section is to write explicitly the equations of a nonlinear *RLC* circuit of the type considered in the previous section. Another purpose is to exhibit the similarities and differences between the equations that describe linear networks and those that describe the nonlinear networks under consideration.

To simplify the exposition consider the resistive network of Fig. 14. Call τ_1 the tree formed by the branches 1,2,3, and the voltage source E. If the network were linear, the fundamental cut set equations would read

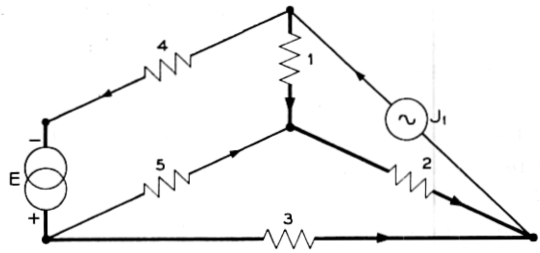

Fig. 14 — Resistive network.

$$\begin{aligned} g_1e_1 + g_4(e_1 + e_2 - e_3 - E) &= J_1 \\ g_4(e_1 + e_2 - e_3 - E) + g_2e_2 - g_5(-e_2 + e_3) &= J_1 \\ -g_4(e_1 + e_2 - e_3 - E) + g_3e_3 + g_5(-e_2 + e_3) &= 0 \end{aligned} \tag{20}$$

where e_i is the voltage across the ith branch and g_i is the conductance of this branch. In the well known matrix form, the equations become

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} g_1 & 0 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 & 0 \\ 0 & 0 & g_3 & 0 & 0 \\ 0 & 0 & 0 & g_4 & 0 \\ 0 & 0 & 0 & 0 & g_5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ E \end{bmatrix} = \begin{bmatrix} J_1 \\ J_1 \\ 0 \end{bmatrix} \tag{21}$$

or, more generally,

$$\mathbf{\Delta}_{T(R),R} \boldsymbol{G}_R \mathbf{\Delta}'_{T(RE),R} \begin{bmatrix} \mathbf{e} \\ \mathbf{E} \end{bmatrix} + \mathbf{\Delta}_{T(R),L(J)} \mathbf{J} = \mathbf{0} \tag{22}$$

where $\mathbf{e}$, $\mathbf{E}$ and $\mathbf{J}$ are column vectors whose components are the tree branch voltages, $\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$, the voltage sources, $[E]$, and the current sources, $[J]$, of the network; $\boldsymbol{G}_R$ is the branch admittance matrix. The $\mathbf{\Delta}$'s are appropriate submatrices of the fundamental cut set matrix $\mathbf{Q}$. The first subscript of $\mathbf{\Delta}$ denotes the rows and the second subscript denotes the columns of $\mathbf{Q}$ whose intersection forms the submatrix. Thus $\mathbf{\Delta}_{T(RE),L(R)}$ is a submatrix formed by the intersection of rows corresponding to resistive and voltage source tree branches and columns corresponding to resistive links; $\mathbf{\Delta}_{T(R),R}$ is formed by the intersection of rows corresponding to resistive tree branches and columns corresponding to resistive branches. $\mathbf{\Delta}_{T(R),J}$ is defined similarly. The prime over a matrix indicates transposition. Now, let the resistors become monotonically increasing one-to-one nonlinear resistors. Without loss of generality we can assume these new resistors to be time invariant. Let $\bar{g}_1(\cdot)$, $\bar{g}_2(\cdot)$, $\bar{g}_3(\cdot)$, $\bar{g}_4(\cdot)$ and $\bar{g}_5(\cdot)$ be their characteristics.

The cut set equations are:

$$\begin{aligned} \bar{g}_1(e_1) + \bar{g}_4(e_1 + e_2 - e_3 - E) &= J_1 \\ \bar{g}_4(e_1 + e_2 - e_3 - E) + \bar{g}_2(e_2) - \bar{g}_5(e_3 - e_2) &= J_1 \\ -\bar{g}_4(e_1 + e_2 - e_3 - E) + \bar{g}_5(e_3 - e_2) + \bar{g}_3(e_3) &= 0 \end{aligned} \tag{23}$$

where, for example, $\bar{g}_1(e_1)$ is now the value of the function $\bar{g}_1$ evaluated at e_1.

The similarity between (20) and (23) suggests a shorthand notation for writing the equations of nonlinear networks. By the product $\boldsymbol{A}*\mathbf{x}$ (where $\boldsymbol{A}$ is a diagonal matrix whose elements are functions $a_i(\cdot)$ and $\mathbf{x}$ is a column vector whose components are $x_1, x_2, \cdots, x_n$), we denote the column vector whose ith component is $a_i(x_i)$, that is, the ith diagonal element of $\boldsymbol{A}$ evaluated at the ith component of $\mathbf{x}$. With this symbolic notation the equations of the network of Fig. 14 can be written (for the nonlinear case) in a form analogous to (22).

$$\boldsymbol{\Delta}_{T(R),R}\boldsymbol{G}_R*\left(\boldsymbol{\Delta}'_{T(RE),R}\begin{bmatrix}\mathbf{e}\\ \mathbf{E}\end{bmatrix}\right) + \boldsymbol{\Delta}_{T(R),L(J)}\mathbf{J} = \mathbf{0}$$

where $\boldsymbol{G}_R$ is the diagonal matrix whose elements are the characteristics $\bar{g}_1, \bar{g}_2, \cdots, \bar{g}_5$ and the $*$ operation must be interpreted as indicated above. $\boldsymbol{G}_R$ will be referred to the branch characteristic matrix. With this symbolic notation, cut set matrices, loop matrices and branch resistance matrices may be used to writing equations of nonlinear networks in the same way as for linear networks.

Let us now assume that the elements of the tree τ_1 of the network of Fig. 14 are monotonically increasing current-controlled but not voltage-controlled and the links are monotonically increasing voltage-controlled but not current-controlled. Since the tree branches are not voltage-controlled, the equations cannot be written in the form of (22). Let $\bar{r}_1(\cdot)$, $\bar{r}_2(\cdot)$ and $\bar{r}_3(\cdot)$ represent the characteristics of the tree branches and i_1, i_2 and i_3 be the currents of the corresponding tree branches. In terms of the tree branch voltages and currents the cut set equations become:

$$\begin{aligned} i_1 + \bar{g}_4(e_1 + e_2 - e_3 - E) &= J_1 \\ i_2 + \bar{g}_4(e_1 + e_2 + e_3 - E) - \bar{g}_5(e_3 - e_2) &= J_1 \\ i_3 - \bar{g}_4(e_1 + e_2 - e_3 - E) + \bar{g}_5(e_2 - e_3) &= 0. \end{aligned} \tag{24}$$

The other set of equations is

$$\begin{aligned} e_1 &= \bar{r}_1(i_1) \\ e_2 &= \bar{r}_2(i_2) \\ e_3 &= \bar{r}_3(i_3) \end{aligned} \tag{25}$$

or symbolically

$$\mathbf{i}_{T(R)} + \boldsymbol{\Delta}_{T(R),L(R)}\boldsymbol{G}_{L(R)}*\left(\boldsymbol{\Delta}'_{T(RE),L(R)}\begin{bmatrix}\mathbf{e}_{T(R)}\\ \mathbf{E}\end{bmatrix}\right) + \boldsymbol{\Delta}_{T(R),L(J)}\mathbf{J} = \mathbf{0}. \tag{26}$$

$$\mathbf{e}_{T(R)} = \boldsymbol{R}_{T(R)}*\mathbf{i}_{T(R)} \tag{27}$$

where $\mathbf{i}_{T(R)}$, $\mathbf{e}_{T(R)}$, $\mathbf{E}$ and $\mathbf{J}$ are the tree currents and voltages, and voltage and current sources respectively. G and R are the link and branch characteristic matrices, and in our example

$$G_{L(R)} = \begin{bmatrix} g_4 & 0 \\ 0 & \bar{g}_5 \end{bmatrix}, \qquad R_{T(R)} = \begin{bmatrix} \mathcal{R}_1 & 0 & 0 \\ 0 & \mathcal{R}_2 & 0 \\ 0 & 0 & \mathcal{R}_3 \end{bmatrix}.$$

The $\mathbf{\Delta}$'s are appropriate submatrices of the fundamental cut set matrix $\mathbf{Q}$. The first subscript denotes the rows and the second subscript denotes the columns of $\mathbf{Q}$ whose intersection forms the submatrix. Thus $\mathbf{\Delta}_{T(RE),L(R)}$ is a submatrix formed from the intersection of rows corresponding to resistive and voltage source tree branches and columns corresponding to resistive links. $\mathbf{\Delta}_{T(R),L(R)}$ and $\mathbf{\Delta}_{T(R),L(J)}$ are defined similarly. A comparison of (26), (27) and (22) shows that in the case of current-controlled tree branches and voltage-controlled links which are not one-to-one, we need *both* $\mathbf{i}_{T(R)}$ and $\mathbf{e}_{T(R)}$ for a straightforward writing of the cut set equations and the branch characteristic equations. Either $\mathbf{i}_{T(R)}$ or $\mathbf{e}_{T(R)}$ can be eliminated from the equations. The resulting equations are:

$$\mathbf{i}_{T(R)} + \mathbf{\Delta}_{T(R),L(R)} G_{L(R)} * \left(\mathbf{\Delta}'_{T(RE),L(R)} \begin{bmatrix} R_{T(R)} * \mathbf{i}_{T(R)} \\ \mathbf{E} \end{bmatrix} \right) + \mathbf{\Delta}_{T(R),L(J)} \mathbf{J} = \mathbf{0} \tag{28}$$

Or

$$\mathbf{e}_{T(R)} + R_{T(R)} * \left\{ \mathbf{\Delta}_{T(R),L(R)} G_{L(R)} * \left(\mathbf{\Delta}'_{T(RE),L(R)} \begin{bmatrix} \mathbf{e}_{T(R)} \\ \mathbf{E} \end{bmatrix} \right) \right\} + R_{T(R)} * (\mathbf{\Delta}_{T(R),L(J)} \mathbf{J} = \mathbf{0}. \tag{29}$$

Fundamental loop equations can be written in a similar way using both the voltages and currents of the links $\mathbf{i}_{L(R)}$ and $\mathbf{e}_{L(R)}$. The equations are

$$\mathbf{e}_{L(R)} + \mathbf{1}_{L(R),T(R)} R_{T(R)} * \left(\mathbf{1}'_{L(RJ),T(R)} \begin{bmatrix} \mathbf{i}_{L(R)} \\ \mathbf{J} \end{bmatrix} \right) + \mathbf{1}_{L(R),T(E)} \mathbf{E} = \mathbf{0}$$

$$\mathbf{i}_{L(R)} = G_{L(R)} * \mathbf{e}_{L(R)}$$

where the $\mathbf{1}$'s are appropriate submatrixes of the fundamental tie set matrix $\mathbf{B}$. Similarly to (26) and (27), either $\mathbf{e}_{L(R)}$ or $\mathbf{i}_{L(R)}$ can be eliminated.

We now write the equations for a general *RLC* network (which satisfies the requirements of Theorem IV) by performing the following steps†

† Other systems of variables are possible. For example, one can choose charges and fluxes as above and voltages of resistive links whose loop does not consist of capacitors and voltage sources only.

(*i*) A tree is chosen as explained in Section V.

(*ii*) Variables are chosen. We choose here the charges on the capacitive tree branches, $\mathbf{q}_D$, the currents of the resistive tree branches whose fundamental cut set *does not* consist of inductors and current sources only, $\mathbf{i}_R$, and the fluxes of the inductive links, φ_Γ.

The equations make use of the following characteristic branch matrixes: $\boldsymbol{C}$ and $\boldsymbol{D}$ are diagonal matrixes whose elements are the characteristics of voltage-controlled and charged-controlled capacitors, respectively; $\boldsymbol{G}$ and $\boldsymbol{R}$, those of voltage-controlled and current-controlled resistors, respectively; $\boldsymbol{L}$ and $\boldsymbol{\Gamma}$, those of current-controlled and flux-controlled inductors, respectively. Without loss of generality we can assume that the elements are time-invariant. The equations are

$$\frac{d}{dt}\left\{\mathbf{q}_D + \boldsymbol{\Delta}_{T(C),L(C)}\boldsymbol{C}_{L(C)}*\left(\boldsymbol{\Delta}'_{T(CE),L(C)}\begin{bmatrix}\boldsymbol{D}_{T(C)}*\mathbf{q}_D\\ \mathbf{E}\end{bmatrix}\right)\right\}$$
$$+\boldsymbol{\Delta}_{T(C),L(\bar{R})}\boldsymbol{G}_{L(\bar{R})}*\left(\boldsymbol{\Delta}'_{T(C\bar{R}E),L(\bar{R})}\begin{bmatrix}\boldsymbol{D}_{T(L)}*\mathbf{q}_D\\ \boldsymbol{R}_{T(\bar{R})}*\mathbf{i}_R\\ \mathbf{E}\end{bmatrix}\right) \tag{30}$$
$$+\boldsymbol{\Delta}_{T(C),L(L)}\boldsymbol{\Gamma}_{L(L)}*\varphi_\Gamma + \boldsymbol{\Delta}_{T(C),L(J)}\mathbf{J} = \mathbf{0}$$

$$\mathbf{i}_R + \boldsymbol{\Delta}_{T(\bar{R}),L(\bar{R})}\boldsymbol{G}_{L(\bar{R})}*\left(\boldsymbol{\Delta}'_{T(C\bar{R}E),L(\bar{R})}\begin{bmatrix}\boldsymbol{D}_{T(C)}*\mathbf{q}_D\\ \boldsymbol{R}_{T(\bar{R})}*\mathbf{i}_R\\ \mathbf{E}\end{bmatrix}\right) \tag{31}$$
$$+\boldsymbol{\Delta}_{T(\bar{R}),L(L)}\boldsymbol{\Gamma}_{L(L)}*\varphi_\Gamma + \boldsymbol{\Delta}_{T(\bar{R}),L(J)}\mathbf{J} = \mathbf{0}$$

$$\frac{d}{dt}\left\{\varphi_\Gamma + \mathbf{1}_{L(L),T(L)}\boldsymbol{L}_{T(L)}*\left(\mathbf{1}'_{L(LJ),T(L)}\begin{bmatrix}\boldsymbol{\Gamma}_{L(L)}*\varphi_\Gamma\\ \mathbf{J}\end{bmatrix}\right)\right\}$$
$$+\mathbf{1}_{L(L),T(R_1)}\boldsymbol{R}_{T(R_1)}*\left(\mathbf{1}'_{L(LJ),T(R_1)}\begin{bmatrix}\boldsymbol{\Gamma}_{L(L)}*\varphi_\Gamma\\ \mathbf{J}\end{bmatrix}\right) \tag{32}$$
$$+\mathbf{1}_{L(L),T(C)}\boldsymbol{D}_{T(C)}*\mathbf{q}_D + \mathbf{1}_{L(L),T(\bar{R})}\boldsymbol{R}_{T(\bar{R})}*\mathbf{i}_R$$
$$+\mathbf{1}_{L(L),T(E)}\mathbf{E} = \mathbf{0}$$

where R_1 is the set of resistive tree branches whose fundamental cut set contains inductive links and current sources only; and $\bar{R}$ is the set which contains all other resistive branches.

The terms in the brackets in (30) and (32) are equal to our state variables $\mathbf{q}$ and φ of Section V. One can write the equations in terms of these variables: the relations between $\mathbf{q}_D$ and $\mathbf{q}$ and φ_Γ and $\boldsymbol{\phi}$ are given by

$$\mathbf{q}_D + \mathbf{\Delta}_{T(C),L(C)}\mathbf{C}_{L(C)}* \left(\mathbf{\Delta}'_{T(CE),L(C)} \begin{bmatrix} \mathbf{D}_{T(C)}*\mathbf{q}_D \\ \mathbf{E} \end{bmatrix}\right) = \mathbf{q}$$

$$\boldsymbol{\varphi}_\Gamma + \mathbf{1}_{L(L),T(L)}\mathbf{L}_{T(L)}* \left(\mathbf{1}'_{L(LJ),T(L)} \begin{bmatrix} \mathbf{\Gamma}_{L(L)}*\boldsymbol{\varphi}_\Gamma \\ \mathbf{J} \end{bmatrix}\right) = \boldsymbol{\varphi}.$$

In summary, the equations of the *RLC* nonlinear network are written in a way which is a generalization of the methods used in linear networks. However, great care must be taken of the fact that some characteristics are representable by functions which do not have inverses. This section indicated a method for tackling the problem. In this section, the equations are written in terms of three sets of variables: $\mathbf{q}_D$, the charges on the capacitive tree branches; $\boldsymbol{\varphi}_L$, the fluxes in the inductive links and $\mathbf{i}_R$, the currents in the resistive tree branches whose fundamental cut sets do not consist of only inductors and current sources. It is interesting to note that (except for the trivial case where $\mathfrak{N}$ consists of a single capacitor in parallel with a voltage source or a single inductor in parallel with a current source) the dimension of the state vector $(\mathbf{q},\boldsymbol{\varphi})$ used above is the same as in the linear case: [number of independent initial conditions] = [number of reactive elements] − [number of independent capacitor-only tie sets] − [number of independent inductor-only cut sets].[7,8,9]

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