A Relation Between the Basis Functions of Periodically Varying Nondissipative Circuits

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This note relates to circuits of linear, periodically time-varying, positive capacitors and inductors. More specifically, it notes a property of the basis functions, or natural modes which describe the transient behavior of the circuits. Some other properties of the basis functions are described in Ref. 1 in this issue.

In accordance with Ref. 1

$$I = (S + pCp)\Phi, \qquad E = p\Phi \tag{1}$$

where I and E are column matrices, or vectors representing the excitation currents into and the response voltages at the various nodes, and Φ is an auxiliary vector variable. The matrices C and S are the node matrices of the capacitances and of the reciprocals of the inductances. They are symmetric and at least positive semidefinite. We will assume that C is positive definite. If it is not so originally, a similar equation with a positive definite C can be derived from (1), for example by the transformations described in the Appendix of Ref. 1. The symbol p represents differentiation (*not* frequency) and operates on all quantities following it.

Setting $I = 0$ in (1) makes it a homogeneous, second-order, vector differential equation in, say n dimensions, with periodically varying coefficients. Thus the well-known Floquet-Poincaré theorem requires the solution to be as follows

$$\Phi = \sum_{\sigma=1}^{2n} k_\sigma \varphi_\sigma , \qquad \varphi_\sigma = H_\sigma(t)e^{s_\sigma t}. \tag{2}$$

Here the k_σ's are arbitrary scalar constants and the φ_σ's are the basis functions, or natural modes. The characteristic exponents s_σ are scalar constants. The coefficients H_σ are time-varying vectors. When the s_σ's are all different, the H_σ's vary periodically. Otherwise, they are at most polynomials in t with periodically varying vector coefficients.

The real parts of the s_σ's indicate the damping of the basis functions. When the inductors and capacitors are fixed, all basis functions are undamped. However, when the components vary periodically some of the basis functions may have nonzero damping (as is well known). Some bounds on the damping are derived in Ref. 1. It is also known that the sum of all the characteristic exponents must be zero (when the circuit is nondissipative).

The purpose of this note is to point out the following: *Corresponding to any nondissipative circuit of periodically varying positive inductors and capacitors, any characteristic exponents which have nonzero real parts occur in equal positive and negative pairs.* Since complex exponents must occur in conjugate pairs, it follows that all the s_ν's must fit into pairs or quadruplets of the following sorts:

$$\begin{matrix} +i\omega_k \\ -i\omega_k \end{matrix}, \quad \begin{matrix} +\alpha_j \\ -\alpha_j \end{matrix}, \quad \begin{matrix} +\alpha_h + i\omega_h \\ +\alpha_h - i\omega_h \\ -\alpha_h - i\omega_h \\ -\alpha_h + i\omega_h . \end{matrix} \tag{3}$$

The theorem follows at once from certain general properties of adjointly related differential equations, which are stated below without derivation. The solution of the nonhomogeneous equation (1) may be expressed in terms of functions of the excitation time, τ, and the response time, t. The adjoint equation is the equation whose solution corresponds to interchanging the two times, τ and t. It can be shown that the characteristic exponents of the adjoint equation are the negatives of those of the original equation. It can also be shown that equation (1) is self-adjoint. (The equation is its own adjoint.) Thus the negatives of the characteristic exponents of the original equation are also the characteristic exponents of the original equation.

When the components are fixed, vectors I and E are related by a set of odd rational functions of a frequency variable. Then Φ, whose derivative is E, is related to I by even rational functions. It appears that, more generally, self-adjoint differential equations are useful counterparts, for time-varying circuits, of even rational functions in the theory of fixed circuits. (See also Ref. 2.)

Recall Foster's canonical fixed nondissipative one-ports, comprising series- or parallel-connected subcircuits of one or two components each. The grouping (3) suggests that there *may* be a time-varying counterpart, in which the most complicated subcircuits correspond to quadruplets of $\pm$ complex characteristic exponents. However, although the existence of the grouping (3) is necessary for such a configuration, it does not in itself prove, or even support a strong conjecture, that the configuration is, in fact, a canonical periodically varying nondissipative one-port.

REFERENCES

1. Darlington, S., Linear Time-Varying Circuits — Matrix Manipulations, Power Relations, and Some Bounds on Stability, B.S.T.J., this issue, p. 2575.
2. Darlington, S., Nonstationary Smoothing and Prediction Using Network Theory Concepts, IRE Trans on Circuit Theory, **CT-6**, Special Supplement, May, 1959.