Group Testing To Eliminate Efficiently All Defectives in a Binomial Sample

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In group-testing, a set of x units is taken from a total starting set of N units, and the x units ($1 \leqq x \leqq N$) are tested simultaneously as a group with one of two possible outcomes: either all x units are good or at least one defective unit is present (we don't know how many or which ones). Under this type of testing, the problem is to find the best integer x for the first test and to find a rule for choosing the best subsequent test-groups (which may depend on results already observed), in order to minimize the expected total number of group-tests required to classify each of the N units as good or defective. It is assumed that the N units can be treated like independent binomial chance variables with a common, known probability p of any one being defective; the case of unknown p and several generalizations of the problem are also considered.

I. SUMMARY

A finite number, N, of units are to be tested in groups. A "group-test" is a simultaneous test on x units (x to be chosen so that $1 \leqq x \leqq N$) with only two possible outcomes: "success," indicating that all x units are good; and "failure," indicating that at least one of the x units is defective (we don't know how many or which ones). The problem is to define a simple and efficient procedure (or an optimal procedure) for separating all the defective units from the good units — efficiency being defined in the sense of minimizing the expected number of group-tests required. Each unit is assumed to represent an independent observation from a binomial population with a common known *a priori* probability, q, of being good and $p = 1 - q$ of being defective. (The case of q unknown is briefly treated in Section X.)

A procedure (or decision rule), R_1, which describes a mode of action for any given value of q, is proposed and compared with several other procedures applicable to the same problem. The procedure R_1 is simple in the sense that at any time, t, the experimenter must separate the units not yet proven to be good or defective into only two sets; units within either of these two sets need not be distinguishable. If it is given that the identification of units within the group being tested is economically impractical or impossible, then the procedure R_1 is conjectured to be optimal for all values of q.

Explicit instructions for carrying out R_1 are given for $N = 1(1)16$ for all q and for $N = 17(1)100$ for the particular values $q = 0.90$, 0.95 and 0.99. Exact formulae for the expected number of group-tests required under R_1 are given in Table IV B for all q and for $N = 1(1)12$; numerical results for $q = 0.90$, 0.95 and 0.99 are given in Tables V A, V B and V C for $N = 1(1)100$. Other numerical comparisons are made in Tables II A and II B. [Tables II through VIII appear at the end of this paper.]

Another procedure, R_2, which is simpler to compute and compares favorably with R_1, is defined in Appendix A in terms of information theory concepts.

Several different directions for generalization of the problem and corresponding generalizations of the procedure R_1 are considered in Section XI. Industrial applications are mentioned, in addition to the known application to blood testing.

II. INTRODUCTION

A problem which has hitherto been considered only in connection with blood-testing applications[1,2,3] can be shown to have industrial applications, and these have focused interest on a more general treatment of the problem. During World War II, a great saving was accomplished in the field of blood testing by pooling a fixed number of blood samples and testing the pooled sample for some particular disease. If the disease was not present, then several people were passed by a single test; if the disease was present, then there was enough blood remaining in each blood sample to test each one separately. The amount of time, money and effort saved by such a procedure depends on how rare the disease is in the population of people being tested. In this application, the total number of people to be tested was regarded as unknown and very large.

The goal of the problem treated here is the same — namely, to separate the defective units from the good units with a minimal (or approximately minimal) number of group-tests. This problem differs from the blood-testing problem in the following respects:

i. The population size N (number of people to be tested) is known at the outset.

ii. The number of units in each group-test (pooled blood sample) is not necessarily constant.

iii. If a group-test fails (the disease is present) we do not necessarily test each item separately.

In practice, the simplicity of the procedure deserves some consideration. The proposed procedure R_1 defined in Section III, after having been computed and described explicitly in advance of any experimentation, is in some sense no more complicated than the blood-testing procedure described above; this is explained in Section V.

Some typical industrial applications are:

1. It is desired to remove all "leakers" from a set of N devices. One chemical apparatus is available and the devices are tested by putting x of them (where $1 \leqq x \leqq N$) in a bell jar and testing whether any of the gas used in constructing the devices has leaked out into the bell jar. It is assumed that the presence of gas in the bell jar indicates only that there is at least one leaker and that the amount of gas gives no indication of the number of leakers. The *a priori* probability, q, of a unit being good is given by the records of similar units tested in the past.

2. Paper capacitors are tested at most n at a time, and each test indicates by the presence or absence of a current whether or not there is at least one defective present. For given n and q and given cost of unit manufacture, should the operator throw away a whole set of n units if it contains at least one defective? If not, how should he proceed to sort out the defective units to minimize the expected number of tests required? If the cost of a group-test and the cost of producing a unit are known, a related problem is to find a procedure which minimizes the total cost (including testing costs) of producing a good unit.

3. *Christmas tree lighting problem.* A batch of n light bulbs is electrically arranged in series and tested by applying a voltage across the whole batch or any subset thereof. If this is to be done on a routine basis, what procedure should be used to minimize the expected number of tests required to remove all the defective light bulbs, assuming the value of q is given?

4. A test indicates whether or not there is at least one *good* unit present in a batch of n, without indicating which ones or how many are good. Given q, what procedure should be used to remove the good units? This dual problem, which is useful in salvaging good components on a routine basis, is mathematically equivalent to those above, if the definitions of good and defective are interchanged.

A procedure R_1 is defined to solve the above problems, and is compared with several other procedures for the same problem. A procedure R_2, based on maximizing the information in each group-test, is defined in Appendix A. Another procedure, R_3, which does not allow any recombination, is defined in Appendix B. Two "halving procedures," which can be carried out without knowing the true value of q, are defined in Appendix C. Procedures R_7 and R_6 are the best procedures that can be obtained by the methods of Dorfman[1] and Sterrett,[3] respectively, for small population sizes. For $N = 4$, 8 and 12 and various q values, Table II A gives a numerical comparison of the expected number of group-tests required for all these procedures.

Different directions of generalization, some of which are discussed in Section XI, are the following:

1. Two (or more) different kinds of units with (say) known probabilities q_1, q_2 of a unit being good are present, and the two different kinds can be put into the same test group.

2. Two (or more) experimenters may be working on a single set of N units by carrying out simultaneous, parallel group-tests and cooperating in such a way as to minimize the time required to accomplish the task.

3. The restriction is sometimes applied (particularly in blood-testing) that any one unit can be included in at most k group-tests; here the goal is to minimize the expected number of group-tests subject to this restriction. For $k = 2$ the proposed procedure is necessarily based on the method of Dorfman;[1] i.e., if a group-test fails, then the units therein are all tested individually.

4. Various generalizations appear if it is assumed that each test on x units gives three (or more) different possible results. For example, a test could indicate that either (a) all are good or (b) all are defective or (c) there are at least one good unit and at least one defective present.

5. A unit can be defective in either of two ways (e.g., electrical or mechanical) with the two *a priori* probabilities of being defective assumed to be independent but not necessarily equal. If there are two different tests corresponding to the two types of defectives, then, in addition to deciding the next test-group size, it may be necessary to decide which test to use next.

6. For positive continuous chance variables with a known distribution (like weight) the following problem is analagous. It is desired to separate N units into two groups according as the weight per unit is less than or greater than a constant (say, unity). Any number of units can be included in a single weighing. The problem is to accomplish the separation in a minimal number of weighings, assuming that the individual weights are independent observations from the common known distribution.

Many of these generalizations will be omitted from this paper and treated separately.

III. THE PROCEDURE R_1

The procedure R_1 is defined implicitly by a pair of recursion formulae and boundary conditions, but first we shall need some definitions and preliminary results. The units proven to be good and the units proven to be defective are never used in subsequent tests. Aside from such units, this procedure requires that at every stage the remaining units be separated into at most two sets. For one set of size $m \geqq 0$, which we call the *defective set*, it is known that it contains at least one defective unit; for the other set of size $n - m \geqq 0$, which we call the *binomial set*, our *a posteriori* knowledge is, so to speak, in the original binomial state; i.e., given the past history of testing, the units in the binomial set act like independent binomial chance variables with a common probability p of being defective. For the defective set, the conditional probability that Y, the number of defectives present, equals y is

$$\Pr \{Y = y \mid Y \geqq 1\} = \frac{\binom{m}{y} p^y q^{m-y}}{1 - q^m} \qquad (y = 1, 2, \cdots, m). \tag{1}$$

If X denotes the number of defectives present in a subset of size x randomly chosen from the defective set, then

$$\Pr \{X = 0 \mid Y \geqq 1\} = \sum_{y=1}^{m-x} \frac{\binom{m}{y} p^y q^{m-y}}{1 - q^m} \frac{\binom{m - y}{x}}{\binom{m}{x}} = \frac{q^x(1 - q^{m-x})}{1 - q^m}. \tag{2}$$

Before defining the procedure it is convenient to prove a lemma in a more general setting. Let $T(r_i)$ $(i = 1, 2, \cdots, t)$ denote a test on n units $(1 \leqq r_i \leqq n)$ such that there are only two mutually exclusive possible outcomes: a "failure," indicating that there are reast r_i defectives present, and a "success," indicating that at most $r_i - 1$ of the units in the test are defective. In Lemma 1 we consider any integers r_i, r_0 with $1 \leqq r_i \leqq n (i = 0, 1, 2, \cdots, t)$, but the most important application is the case $r_0 = r_1 = \cdots = r_t = 1$. Let $\mathfrak{A}$ be any set of units, and let $\mathfrak{B}_i (i = 1, 2, \cdots, t)$ denote sets not necessarily disjoint from one another but such that each is disjoint from $\mathfrak{A}$; the case $t = 1$ is the one used for procedure R_1. At the outset, all units are independently and binomially distributed with a common probability p of being defective.

Lemma 1: If a test $T(r_i)$ on $\mathcal{A} + \mathcal{B}_i$ produces a failure for $(i = 1, 2, \cdots, t)$ and another test $T(r_0)$ on $\mathcal{A}$ also produces a failure, then for $r_0 \geqq \max r_i$ the conditional distribution associated with all the units in the sets $\mathcal{B}_i (i = 1, 2, \cdots, t)$, given both conditions above, is exactly the same as the original binomial distribution.

Proof: Let A and B_i denote the chance number of defectives present in $\mathcal{A}$ and $\mathcal{B}_i$, respectively. For the jth set $\mathcal{B}_j$ the conditional probability P of interest is

$$P = \Pr\{B_j \leqq b \mid A + B_i \geqq r_i (i = 1, 2, \cdots, t), A \geqq r_0\}. \quad (3)$$

Since $r_0 \geqq \max r_i (i = 1, 2, \cdots, t)$, the condition $A \geqq r_0$ implies that $A + B_i \geqq r_i$, and hence

$$P = \Pr\{B_j \leqq b \mid A \geqq r_0\}. \quad (4)$$

Since $\mathcal{B}_j$ and $\mathcal{A}$ are disjoint, it follows that B_j and A are independent and hence, from (4),

$$P = \Pr\{B_j \leqq b\}, \quad (5)$$

which proves the lemma.

Let $G_1(m, n; q) = G_1(m, n)$ denote the expected number of group-tests remaining to be performed if the defective set is presently of size m, the binomial set is presently of size $n - m$, the *a priori* probability of a good unit is the known constant q and the procedure R_1 is used. For the special case $m = 0$ we use the symbol $H_1(n; q) = H_1(n)$. The values of m and n vary as the procedure is carried out; at the outset, $m = 0$ and $n = N$. It will also be convenient to refer to the *G-situation* or *$G(m, n)$-situation* if $m > 1$ and to the *H-situation* or *$H(n)$-situation* if $m = 0$.

Recursion Formulae Defining Procedure R_1

If x denotes the size of the very next group-test, then we write for any situation with $m = 0$

$$H_1(n) = 1 + \min_{1 \leqq x \leqq n} \{q^x H_1(n - x) + (1 - q^x)\, G_1(x, n)\}, \quad (6)$$

and, with the help of (2) and Lemma 1, we write for $n \geqq m \geqq 2$

$$G_1(m, n) = 1 + \min_{1 \leqq x \leqq m-1} \left\{ \left(\frac{q^x - q^m}{1 - q^m} \right) G_1(m - x, n - x) + \left(\frac{1 - q^x}{1 - q^m} \right) G_1(x, n) \right\}. \quad (7)$$

The *boundary conditions* state that for all q

$$H_1(0) = 0 \quad \text{and} \quad G_1(1, n) = H_1(n - 1) \quad \text{for} \quad n = 1, 2, \cdots . \qquad (8)$$

In (6) and (7) the constant 1 represents the very next group-test of size x and the expression in braces is the conditional expected number of additional group-tests given x. It follows from (6) and (8) that $H_1(1) = 1$ for all q.

Remark 1: To justify writing $G_1(x, n)$ in (7) we make use of Lemma 1 with $t = 2$, $r_2 = 0$, $r_1 = r_0 = 1$; we take $\mathcal{A} + \mathcal{B}_1$ as the defective set of size m and $\mathcal{A}$ as a subset of size $x < m$. Then, by Lemma 1, if the subset $\mathcal{A}$ of size x is shown to contain at least one defective, the *a posteriori* distribution associated with the $m - x$ units in B_1 is exactly binomial. These are then mixed or recombined with the $n - m$ "binomial units," giving a total of $n - x$ binomial units, and this justifies the expression $G_1(x, n)$ in (7).

Remark 2: These two recursion formulae, together with the boundary conditions, allow one to compute successively for any q the functions $G_1(2, 2)$, $H_1(2)$, $G_1(2, 3)$, $G_1(3, 3)$, $H_1(3)$, $G_1(2, 4)$, $G_1(3, 4)$, $G_1(4, 4)$, $H_1(4)$, $\cdots$ to any desired value of m and n.

Remark 3: The integer x which accomplishes the minimization in (6) and (7) for each situation characterized by the integers m and n is particularly important, since this is the size of the next test to be run according to the procedure R_1. These integers $x = x_H(n; q)$ and $x = x_G(m, n; q)$ implicitly define the procedure R_1. An illustration of how the procedure R_1 is to be carried out is given in Section IV.

Remark 4: If $m > 1$, then it is assumed in (7) that a subset of size x with $1 \leqq x < m$ will be taken from the defective set without mixing it with units from the binomial set. It follows from (6), (7) and (8) that any lack of optimality can only arise from this "no mixing" assumption. This assumption was used in the derivation of the algorithm (7) (see Remark 1 above). It will be noted in Section XIII that, when all the units are individually identified, then by dropping this assumption, an improvement to the procedure R_1 for high values of q can be found. A specific example of a modification and improvement of the procedure R_1, which drops the "no mixing" assumption at the expense of more complication will be more thoroughly discussed in a separate paper.

IV. ILLUSTRATION OF THE PROCEDURE R_1

Suppose we start with $N = 12$ units and it is given that $q = 0.98$. Referring to the column headed $H_1(12)$ in Fig. 3, we find that the first test-group is of size $x = 12$; i.e., we start by testing all 12 units. If a success occurs, the experiment is over; if a failure occurs, then according to the column headed $G_1(12, n)$ of Fig. 4 the next test group is of size $x = 4$ chosen at random from the 12. Similarly, we continue along one of the sample paths shown in Fig. 1. The complete "tree" is not shown here, but testing continues in a similar manner and the specific details can be obtained from Figs. 3 and 4, which appear at the end of this paper.

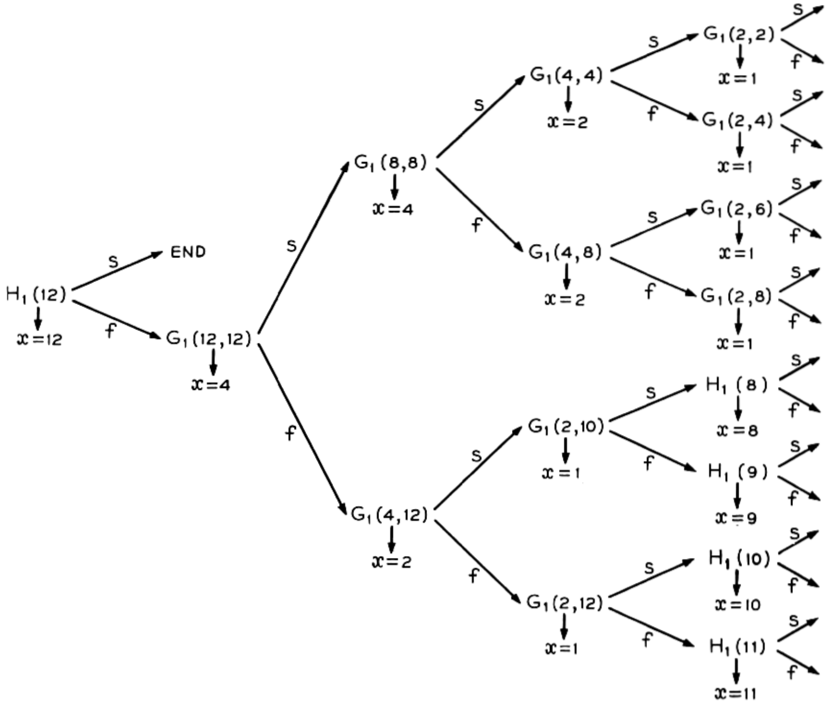

Fig. 1—Initial part of the tree for procedure R_1 for $q = 0.98$ and starting with an $H(12)$-situation.

It is obvious that the above procedure terminates in a finite number of steps. In fact, it can be shown for procedure R_1 (proof is omitted) that the maximum number $M(n)$ $[M(m,n)]$ for any H-situation [G-situation] occurs when q is close to unity and the n unanalyzed units are all defective. It follows easily that

$$\begin{aligned} M(n) &= (n + 1)\,[1 + \alpha(n)] + 1 - 2^{1+\alpha(n)}, \\ M(m,n) &= \alpha(m) + n\,[1 + \alpha(n - 1)] + 1 - 2^{1+\alpha(n-1)}, \ (\mathrm{m} > 1) \end{aligned} \tag{10}$$

where $\alpha(z)$, for any positive integer z, is defined by

$$2^{\alpha(z)} \leqq z < 2^{1+\alpha(z)}. \tag{11}$$

For the example above, $\alpha(12) = 3$ and $M(12) = 37$; it is interesting to note that the constant term in $H_1(n)$ expressed in powers of q in Table IV B is also the length of the longest "chain" (that is, 37 for $n = 12$) in the "tree" used for the interval of q-values ending at unity. Although the maximum number is so large, the expected number of tests is only 2.07. This is explained partly by the fact that the procedure terminates after one group-test with probability 0.7847. A table of such probabilities for the number of tests T required when $q = 0.98$ and $N = 12$ is given below:

T	1	2	3	4	5	6	7	8	9	10–37
Probability	0.7847	0	0	0	0.0801	0.1124	0.0003	0.0016	0.0134	0.0075

If we assign the probability 0.0075 to $T = 10$, we obtain an estimate of $H_1(12)$ (namely, 2.070) which is a lower bound. The exact value, 2.073, can be obtained from the formula for $H_1(12)$ in Table IV B. Similarly, an estimate of the standard deviation is computed to be $\sigma \cong 2.1$, and this also is easily shown to be a lower bound.

It is interesting to note that, if the starting number N is exactly a power of 2 and q is large, the procedure R_1 starts off the same as a "halving" procedure. Such a procedure R_4 is defined in Appendix C for any N, and it has the property that it can be carried out without knowing the true value of q. To compare the means and standard deviations of R_1 and R_4, we consider the case $N = 6$, where $M = 14$ for both R_1 and R_4. For any $q > 0.844$, the expectation under R_1 can be put in the form

$$\begin{aligned} E(T;R_1) = {} & 1(q^6) + 4(3pq^5) + 5(3pq^5 + 2p^2q^4) + 6(2p^2q^4) \\ & + 7(7p^2q^4 + 2p^3q^3) + 8(4p^2q^4 + 5p^3q^3) + 9(8p^3q^3 + 3p^4q^2) \\ & + 10(5p^3q^3 + 5p^4q^2) + 11(3p^4q^2 + 2p^5q) \\ & + 12(3p^4q^2 + p^5q) + 13(p^4q^2 + 2p^5q) + 14(p^5q + p^6), \end{aligned} \tag{12}$$

where, for each term, the expression in parentheses is the probability that T takes on the value of the associated integer coefficient. For any q the corresponding expression under R_4 is

$$\begin{aligned} E(T;R_4) = {} & 1(q^6) + 4(3pq^5) + 5(3pq^5 + 2p^2q^4) + 6(p^2q^4 + p^3q^3) \\ & + 7(7p^2q^4) + 8(5p^2q^4 + 5p^3q^3) + 9(6p^3q^3 + 2p^4q^2) \\ & + 10(7p^3q^3 + 4p^4q^2) + 11(p^3q^3 + 4p^4q^2 + p^5q) \\ & + 12(4p^4q^2 + 2p^5q) + 13(p^4q^2 + 2p^5q) + 14(p^5q + p^6). \end{aligned} \tag{13}$$

The procedure R_1 is better than R_4 (at least for $q > 0.844$) since

$$E(T;R_4) - E(T;R_1) = 2p^2q^4 + 5p^3q^3 + 4p^4q^2 + p^5q \geqq 0. \tag{14}$$

[The fact that each term in (14) is positive indicates that R_1 is better than R_4, even if we know how many defectives are actually present among the N units.] Some numerical comparisons of means and standard deviations are given in Table I. The maximum difference for $N = 6$ between $E(T; R_4)$ and $E(T; R_1)$ occurs at $q = 0.844$, and is equal to 0.0379. For $N = 6$, the procedure R_1 appears to have a variance smaller than that of R_4 for all $q < 1$. A more complete comparison of $E(T; R_1)$ with that of several other procedures is given in Table II A.

TABLE I — COMPARISON OF PROCEDURES R_1 AND R_4 FOR $N = 6$

	$q = 0.85$	$q = 0.90$	$q = 0.95$	$q = 0.98$	$q = 0.99$
$E(T;\, R_1)$	3.8118	2.9434	2.0092	1.4133	1.2083
$E(T;\, R_4)$	3.8472	2.9605	2.0136	1.4141	1.2085
$\sigma(T;\, R_1)$	2.536	2.269	1.762	1.179	0.850
$\sigma(T;\, R_4)$	2.593	2.304	1.776	1.183	0.851

V. THE SIMPLICITY OF R_1

It will be shown in this section that, for any given q and any situation $G(m, n)$, the appropriate x [i.e., the integer which accomplishes the minimization in (7)] does not depend on n. A somewhat simpler method of computing x is given and a new function of m alone is introduced to replace $G_1(m, n)$ in the definition of the procedure R_1. For any m and any pair of integers $(x, x + 1)$ both possible under R_1, there is always a single dividing point $q_G(x) = q_G(x, x + 1; m)$ that separates the interval for x from the interval for $x + 1$. (This property was observed for $m \leqq n \leqq 16$ and is treated as a conjecture for all m and n in Section VII.)

According to Remark 4 in Section III, the procedure R_1 for $m > 1$ is to "break down" the defective set. This "breaking down" is continued until a single unit is established to be defective and removed. Instead of randomizing the order of the units in the defective group again and again before each test group is selected, it will be convenient to assume, without affecting the properties of the procedure R_1, that the order is randomized only once at the outset.† Units or groups of units removed later are then to be taken in that order. If the ith unit (in that order) is the first defective unit, then the "breaking down" mentioned above leads to an H-situation with $n - i$ binomial units, and the converse also holds true.‡

It is convenient to introduce $F_1(m, q) = F_1(m)$ defined as the expected number of group-tests required to "break down" a defective set of size m and for the first time reach an H-situation when q is given and the procedure R_1 is used. Then $F_1(m)$ clearly does not depend on n and the above argument permits us to write

$$G_1(m, n) = F_1(m) + \left(\frac{p}{1 - q^m}\right) \sum_{i=1}^{m} q^{i-1} H_1(n - i). \tag{15}$$

For algebraic simplicity we let

$$G_1{}^*(m, n) = \left(\frac{1 - q^m}{1 - q}\right) G_1\,(m, n) \text{ and } F_1{}^*(m) = \left(\frac{1 - q^m}{1 - q}\right) F_1(m). \tag{16}$$

Then (7) and (15) take on the simpler forms

$$\begin{aligned} G_1{}^*(m, n) = \sum_{i=1}^{m} q^{i-1} \\ + \min_{1 \leqq x \leqq m-1} \{q^x G_1{}^*(m - x, n - x) + G_1{}^*(x, n)\}, \end{aligned} \tag{17}$$

$$G_1{}^*(m, n) = F_1{}^*(m) + \sum_{i=1}^{m} q^{i-1} H_1(n - i). \tag{18}$$

Substituting (18) in (17), the three summations cancel and the result is

$$F_1{}^*(m) = \sum_{i=1}^{m} q^{i-1} + \min_{1 \leqq x \leqq m-1} \{q^x F_1{}^*(m - x) + F_1{}^*(x)\}, \tag{19}$$

which does not depend on n. The boundary condition, $F_1{}^*(1) = 0$ for all q, also does not depend on n. It is clear from this derivation that (19), which does not depend on n, must define the same integer values $x = x_G(m; q)$ as (17) or (7). This proves the following theorem.

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† It should be pointed out that even this single randomization at the outset can be disregarded in carrying out the procedure R_1 if there is no doubt about the assumption of independent chance variables or if the units are already well-mixed in the process of delivery to the experimenter.

‡ It follows from the above that, for any procedure which "breaks down" the defective set in the above-mentioned manner (including a method of testing units from the defective set one at a time until a defective unit is found), the expected number of good units eliminated between a $G(m,n)$-situation and the next H-situation is $q/p - mq^m/(1 - q^m)$, and the number of defective units eliminated is always exactly one.

Theorem 1: For any G-situation with $n \geqq m > 1$ and any q, the size of the next test group, defined implicitly by (7), does not depend on n.

This result simplifies the explicit instructions needed to describe the procedure. Thus the two diagrams, Figs. 3 and 4, describe the procedure R_1 for all values of q and for any $N \leqq 16$.

Equations (15) and (16) can also be substituted in (6), yielding

$$H_1(n) = 1 + \min_{1 \leqq x \leqq n} \left\{ q^x H_1(n - x) + (1 - q) \left[F_1{}^*(x) + \sum_{i=1}^{x} q^{i-1} H_1(n - i) \right] \right\}, \tag{20}$$

which, together with (19), gives a pair of "one-dimensional" recursion formulae for defining R_1 instead of the "two-dimensional" set, (6) and (7).

Remark 5: It should be pointed that, if one were to ask for a procedure that "breaks down" the defective set in as small an expected number of group tests as possible, then one would write (19) as one of the basic recursion formulae defining the procedure. This shows that R_1 "breaks down" the defective set and returns to an H-situation in a minimal number of tests.

VI. SOME PROPERTIES OF R_1 FOR q CLOSE TO UNITY

For any G-situation with $m > 1$, consider the effect of increasing q. It is easy to see by an induction argument that $G_1(m, n)$ is a strictly decreasing function of q. The function $H_1(n)$ is also strictly decreasing unless the value of q is such that the procedure R_1 tests all units one at a time, in which case $H_1(n)$ is constant. The "tree" remains the same in an interval as q increases, and changes only when it becomes more efficient under R_1 to increase the size of some test group in the "tree"; i.e., if we proceed down the "tree" along any path, the first change encountered, if any, will be an increase in some test group size. It therefore seems reasonable to expect (in both G- and H-situations) that the largest value of x assigned by R_1 (say, $x_{\max}$) occurs in an interval of q-values ending at unity. This unproved assertion that under R_1 large values of x are associated with large values of q is an immediate consequence of Conjectures 1 and 2 stated in Section 7.

For fixed $m > 1$, let the integers $\alpha(m)$ and $\beta(m)$

$$[\alpha(m) \geqq 1, \qquad 0 \leqq \beta(m) < 2^{\alpha(m)}]$$

be defined by

$$m = 2^{\alpha(m)} + \beta(m). \tag{20a}$$

Under the assumption that R_1 assigns $x_{\max}$ in an interval of q values ending at unity, it will be shown for any G-situation that in this interval the value of $x = x_{\max}$ is given by

$$x_{\max} = \begin{cases} 2^{\alpha(m)-1} & \text{for} \quad 2^{\alpha(m)} \leqq m \leqq 3 \cdot 2^{\alpha(m)-1} \\ m - 2^{\alpha(m)} & \text{for} \quad 3 \cdot 2^{\alpha(m-1)} \leqq m < 2^{\alpha(m)+1}. \end{cases} \tag{21}$$

As a corollary it then follows that, under R_1 (G-situation),

$$\frac{m}{3} \leqq x_{\max} \leqq \frac{m}{2}. \tag{22}$$

Also in the above-mentioned interval we have

$$F_1{}^*(m) = \alpha(m) \left(\frac{1 - q^m}{1 - q} \right) + q^{m-2\beta(m)} \left[\frac{1 - q^{2\beta(m)}}{1 - q} \right]. \tag{23}$$

Proof of (21), (22) and (23): In (19), since $x \leqq m - 1$ and $m - x \leqq m - 1$, we can use the induction hypothesis (23) to obtain

$$\begin{aligned} F_1{}^*(m) - \sum_{i=1}^{m} q^{i-1} = \min_{1 \leqq x \leqq m-1} \Big\{ q^x \alpha(m - x) \left(\frac{1 - q^{m-x}}{1 - q} \right) \\ + q^{m-2\beta(m-x)} \left(\frac{1 - q^{2\beta(m-x)}}{1 - q} \right) + \alpha(x) \left(\frac{1 - q^x}{1 - q} \right) \\ + q^{x-2\beta(x)} \left(\frac{1 - q^{2\beta(x)}}{1 - q} \right) \Big\}. \end{aligned} \tag{24}$$

For q close to unity, the right side of (24) is equivalent to minimizing

$$Q(x) = [x\alpha(x) + 2\beta(x)] + [(m - x)\alpha(m - x) + 2\beta(m - x)]. \tag{25}$$

For the moment, let us utilize the symmetry of $Q(x)$ for $2 \leqq x \leqq m - 2$ and limit our considerations to $x \leqq m/2$ and $m \geqq 4$.

Let $[m/2]$ denote the largest integer less than or equal to $m/2$. For $x = [m/2], [m/2] - 1, \cdots$, we consider $Q(x + 1) - Q(x)$ and distinguish several cases according as

$$\begin{aligned} &\text{a.} & \alpha(x - 1) &= \alpha(x) = \alpha(m - x) \leqq \alpha(m - x + 1); \\ &\text{b.} & \alpha(x - 1) + 1 &= \alpha(x) = \alpha(m - x) \leqq \alpha(m - x + 1); \\ &\text{c.} & \alpha(x - 1) &= \alpha(x) = \alpha(m - x - 1) \\ &&&= \alpha(m - x) - 1 = \alpha(m - x + 1) - 1; \\ &\text{d.} & \alpha(x - 1) + 1 &= \alpha(x) = \alpha(m - x - 1) \\ &&&= \alpha(m - x) - 1 = \alpha(m - x + 1) - 1; \\ &\text{e.} & m \geqq 2x + 1 &\text{ and } \quad \alpha(x) < \alpha(m - x - 1). \end{aligned} \tag{26}$$

Using the fact that, for any integer $y > 1$,

$$y[\alpha(y) - \alpha(y-1)] + 2[\beta(y) - \beta(y-1)] = 2, \tag{27}$$

we obtain for all the cases in (26) the result,

$$Q(x+1) - Q(x) = \alpha(x) - \alpha(m-x) \leqq 0. \tag{28}$$

In particular, the value is zero only when (26a) holds and $\alpha(x) = \alpha(m - x)$. Hence, $Q(x)$ is a nonincreasing function of the integer x for $1 \leqq x \leqq m/2$ and is constant for $x_0 \leqq x \leqq m - x_0$, where x_0 is defined by

$$\alpha(x_0 - 1) + 1 = \alpha(x_0) = \alpha(m - x_0 - 1)$$

or (29)

$$\alpha(x_0) = \alpha(m - x_0 - 1) = \alpha(m - x_0) - 1,$$

whichever gives the maximum. These imply that

$$x_0 = 2^{\alpha(x_0)} \leqq m - x_0 - 1 < 2^{\alpha(x_0)+1} = 2x_0$$

or (30)

$$x_0 < m - x_0 = 2^{\alpha(x_0)+1} \leqq 2x_0 .$$

Both lead to the same results; namely,

$$\frac{m}{3} \leqq x_0 < \frac{m}{2} . \tag{31}$$

It follows that

$$\frac{m}{3} \leqq x_{\max} \leqq \frac{2m}{3}$$

and, from (29), for any integer x with $x_0 \leqq x \leqq m - x_0$, the values of $\alpha(x)$ and $\alpha(m - x)$ can differ by at most unity.

For any x with $x_0 \leqq x \leqq m - x_0$ we consider two cases according as $| \alpha(x) - \alpha(m - x) |$ is zero or unity. Let the expression in braces in (24) be denoted $C_1^*(x)$. For integers x with $x_0 \leqq x \leqq m - x_0$ the number of terms in $C_1^*(x)$ is constant and the expression $C_1^*(x)$ is to be minimized by making more powers of q larger.

Case 1. We can write

$$C_1^*(x) = \alpha(x) \left[\frac{1 - q^m}{1 - q}\right] + q^{m-2\beta(m-x)} \left[\frac{1 - q^{2\beta(m-x)}}{1 - q}\right] + q^{x-2\beta(x)} \left[\frac{1 - q^{2\beta(x)}}{1 - q}\right], \tag{32}$$

and the problem is to find the x which minimizes $C_1^*(x)$ for large values of q (i.e., to find $x_{\max}$). Since $\alpha(x) = \alpha(m - x)$, $\beta(m - x) - \beta(x) = m - 2x$ and, using the fact that $x_{\max} \geqq m/3$, we consider only integers $x \geqq m/3$ and obtain

$$m - x \geqq 2(m - 2x) = 2[\beta(m - x) - \beta(x)], \tag{33}$$

and hence

$$m - 2\beta(m - x) \geqq x - 2\beta(x). \tag{34}$$

It follows that the last term in (32) has lower powers of q than the previous term and that (32) is minimized by setting $\beta(x) = 0$, i.e., by taking $x_{\max}$ to be a power of 2. To complete the proof of (21) and (23), we note that

$$\begin{aligned} m = x + (m - x) = 2^{\alpha(x)} + 2^{\alpha(m-x)} + \beta(m - x) \\ = 2^{\alpha(x)+1} + \beta(m - x), \end{aligned} \tag{35}$$

so that $\alpha(m) = \alpha(x) + 1$, $\beta(m) = \beta(m - x)$ and, since $x = x_{\max}$ is a power of 2, we have

$$x_{\max} = 2^{\alpha(m)-1}. \tag{36}$$

Since, by (35), $\beta(m) = \beta(m - x) < \frac{1}{2}2^{\alpha(m)}$, then (36) holds for $2^{\alpha(m)} \leqq m < 3[2^{\alpha(m)-1}]$. Substituting these values of $\alpha(x)$, $\beta(x)$, $\alpha(m - x)$ and $\beta(m - x)$ into (32) and using (24) gives

$$F_1^*(m) = \alpha(m)\left(\frac{1 - q^m}{1 - q}\right) + q^{m-2\beta(m)}\left(\frac{1 - q^{2\beta(m)}}{1 - q}\right), \tag{37}$$

which completes the induction for Case 1. Conversely, if $2^{\alpha(m)} \leqq m < 3[2^{\alpha(m)-1}]$ then $x_0 = 2^{\alpha(m)-1}$ and, for any x with $x_0 \leqq x \leqq m - x_0$, we have $\alpha(x) = \alpha(m - x)$. Hence (32) is minimized for $x_{\max} = x_0 = 2^{\alpha(m)-1}$ by the above argument.

Case 2. Since we are now considering only possibilities outside Case 1, we must have $3[2^{\alpha(m)-1}] \leqq m \leqq 2^{\alpha(m)+1}$. Using (29), and the fact that $x_0 \leqq x \leqq m - x_0$, it follows that either $\alpha(x) = \alpha(m - x - 1) = \alpha(m - x) - 1$, so that $m - x$ is a power of 2 and $x < m/2$, or else $\alpha(m - x) = \alpha(x - 1) = \alpha(x) - 1$, so that x is a power of 2 and $x > m/2$. In the former alternative, we obtain

$$m = x + (m - x) = 3[2^{\alpha(x)}] + \beta(x),$$

$\alpha(m) = \alpha(x) + 1$ and $\beta(m) = 2^{\alpha(x)} + \beta(x) = x$. It follows that

$$x = x_{\max} = m - 2^{\alpha(m)}. \tag{38}$$

In the latter alternative, it can be concluded that $x = 2^{\alpha(m)} > m/2$. In the latter alternative (or in both alternatives), we now compare the values of $C_1^*(x)$, computed from (24), for the two arguments,

$$x_1 = 2^{\alpha(m)} > m/2 \qquad \text{and} \qquad x_2 = m - 2^{\alpha(m)} < m/2.$$

Using the general result that $\beta(m^*) < m^*/2$ for any positive integer m^* and the implied result that $\alpha(m - 2^{\alpha(m)}) = \alpha(m) - 1$, it is easy to show (the details are omitted) that $C_1^*(x_2) < C_1^*(x_1)$ for $0 < q < 1$. Hence, for either alternative, we obtain the same result, (38). As a corollary, (22) holds. In this case $C_1^*(x)$, after algebraic simplification, is given by

$$C_1^*(x) = [\alpha(m) - 1]\left(\frac{1 - q^m}{1 - q}\right) + q^{m-2\beta(m)}\left[\frac{1 - q^{2\beta(m)}}{1 - q}\right], \tag{39}$$

which, using (24), again gives the result (23). The fact that (23) holds for $m = 2$, 3 and 4 is easily shown and the details are omitted. This completes the proof of (23).

It is also possible to get expressions for $G_1^*(m, m)$, $G_1^*(m, n)$ and $H_1(n)$ for large values of q using (23) and the fact that, for any positive n and sufficiently large values of q (depending on n),

$$H_1(n) = 1 + (1 - q^n)G_1(n, n) = 1 + pG_1^*(n, n). \tag{40}$$

Then, by (18), we obtain for large values of q

$$G_1^*(m, m) = m - 1 + qF_1^*(m) + p \sum_{j=2}^{m} F_1^*(j), \tag{41}$$

$$\begin{aligned} G_1^*(m, n) = {} & n - 1 - (n - m - 1)q^m + F_1^*(m) \\ & + p(1 - q^m) \sum_{j=2}^{n-m-1} F_1^*(j) + p \sum_{j=n-m}^{n-1} F_1^*(j), \end{aligned} \tag{42}$$

$$H_1(n) = q + np + pF_1^*(n) + p^2 \sum_{j=2}^{n-1} F_1^*(j), \tag{43}$$

where (42) is to be used only for $m < n$ and the summation from a to b is taken to be zero for $b < a$. Using these and (23), the last equation (i.e., the equation for the last q-interval which ends at unity) can be obtained independently and more simply for $F_1^*(m)$, $G_1^*(m, n)$ and $H_1(n)$.

It is clear from the results above that, as q approaches unity,

$$\lim H_1(n) = 1 \tag{43a}$$

and that, for q in an interval ending at unity, we have $x_H(n; q) = n$. Here $x_H(n; q)$ is the size of the next test group when the procedure R_1

is used for an $H(n)$-situation with *a priori* probability q of a unit being good. In an H-situation the probability that there are no defectives present approaches unity as q approaches unity. In a G-situation with $m > 1$, the probability that there is exactly one defective in the defective set and none in the binomial set approaches unity as q approaches unity. Defining $x_G(m; q)$ in a similar manner, it was shown above that $x_G(m; q) = x_{\max}$ [as given in (21)] in an interval ending at unity, and it follows from (23), (41) and (42) [or (23) and (15)] that, as q approaches unity,

$$\lim F_1(m) = \alpha(m) + 2\frac{\beta(m)}{m} \qquad (m > 1), \quad (43b)$$

$$\lim G_1(m, n) = \begin{cases} 1 + \lim F_1(m) = 1 + \alpha(m) + 2\dfrac{\beta(m)}{m} & (\text{for } n > m) \\ 1 - \dfrac{1}{m} + \alpha(m) + 2\dfrac{\beta(m)}{m} & (\text{for } n = m). \end{cases} \quad (43c)$$

It is interesting to note that the result (43c) depends on m but not on n.

VII. CONJECTURED PROPERTIES OF R$_1$

In this section we shall state some properties which appear to hold for procedure R_1 based on numerical calculations for $N \leqq 16$ but have not been proved for all N.

1. For any G-situation with fixed $m > 1$, if $x_G(m; q)$ denotes the size of the next test group under R_1, then $x_G(m; q)$ is a nondecreasing step function of q with step size unity. That is, for any pair $q \leqq q^+ \leqq 1$,

$$x_G(m; q) \leqq x_G(m; q^+) \quad (44)$$

and, for sufficiently small $\epsilon > 0$,

$$x_G(m; q + \epsilon) \leqq x_G(m; q) + 1. \quad (45)$$

Also for fixed q the value of $x_G(m + 1; q)$ is either the same or one greater than $x_G(m; q)$; i. e.

$$x_G(m; q) \leqq x_G(m + 1; q) \leqq x_G(m; q) + 1. \quad (46)$$

If the dividing point between x and $x + 1$ for any G-situation under R_1 [denoted by $q_G(x; m)$] is shown to exist (and be unique), then the three properties (44), (45) and (46) are equivalent to the two properties that, for any $m > 1$,

$$q_G(x; m) \geqq q_G(x; m + 1) \quad \text{for} \quad 1 \leqq x < x_{\max}, \tag{47}$$

$$q_G(x; m) < q_G(x + 1; m) \quad \text{for} \quad 1 \leqq x < x_{\max} - 1. \tag{48}$$

The assumption used in Section VI that the largest x-values are associated with the largest q-values is a simple consequence of (44).

2. For any H-situtation, we can define $x_H(n; q)$ similar to $x_G(m; q)$ and the property corresponding to (44) still holds: that, for $q \leqq q^+ \leqq 1$ and all positive integers n,

$$x_H(n; q) \leqq x_H(n; q^+). \tag{49}$$

We can also define $q_H(x; n)$ similar to $q_G(x; m)$ with the understanding that, if x does not appear under $H_1(n)$ in Fig. 3, then the interval for x is assumed to have length zero but the endpoints still exist, and, in fact, $q_H(x - 1, n)$ will then be equal to $q_H(x; n)$. Then (49) is equivalent to the property that for all $x \geqq 1$ and all positive integers n

$$q_H(x; n) \leqq q_H(x + 1; n). \tag{50}$$

3. If the experimental situation is such that it is impossible or economically impractical to identify or keep separate the individual units in any test group, then, after each test on a batch of x units, the disposition of the x units must be made on a batch basis. In such a situation it is conjectured that the procedure R_1 is the optimal procedure for all values of q.

4. There are various patterns existing in Table II B both within a column and across columns, none of which have been proved. For example, if $q = q_G(x; m)$ is the dividing point between x and $x + 1$, then the first entry in the appropriate column is $1 - q^{2x} - q^{2x+1}$, and the last entry, for $m = \infty$, is $1 - q^x - q^{x+1}$. In the first column, the entry can be written as $1 - q - q^2 + q^m$ for all m. In the second column of Table II B the pattern shown is to replace the highest power of q (say, q^h) by q^{h-2} or by $q^{h+2} + q^{h+3}$, depending on whether m is odd or even. Thus the pattern displays a cycle of 2. In the third column one can similarly find a pattern with a cycle of 3, starting with $m \geqq 9$. If a general rule for all these patterns were proved then it might be easier to find the dividing points for higher values of m. The conjecture in this case lies in the fact that these patterns exist and can be mathematically established.

VIII. CHARACTER OF R_1 FOR SMALL VALUES OF q

For the procedure R_1 it will now be shown that, when

$$q < q_0 = \tfrac{1}{2}(\sqrt{5} - 1) = 0.618$$

(to three decimal places), then, for both G- and H-situations with any positive integers $m \leqq n$, the units are all tested one at a time. Of course, if we start with the H-situation and test units one at a time, then a G-situation never arises, but in the induction proof that follows it must first be shown for the G-situation and then for the H-situation. This property that units are tested one at a time was recently shown[4] to hold for the optimal procedure (without specifying what the optimal procedure is or whether it exists). Simple formulae for $H_1(n)$, $G_1(m, n)$ and $F_1(m)$ are obtained for $q \leqq q_0$.

Theorem 2: For procedure R_1 with $1 \leqq m \leqq n$ and $0 \leqq q < q_0$,

$$x_G(m; q) = x_H(n; q) = 1, \tag{51}$$

$$H_1(n) = n, \tag{52}$$

$$G_1(m, n) = n - \frac{pq^{m-1}}{1 - q^m}, \tag{53}$$

$$F_1(m) = \frac{q}{p} + \frac{1 - q^{m-1} - mq^m}{1 - q^m}. \tag{54}$$

[Remarks: The last term in (53) results from the possibility of saving one test if the defective set of size m contains exactly one defective unit that is discovered inferentially by showing that all the other $m - 1$ units are good. It is interesting to note that (54) can be obtained by summing the series

$$F_1(m) = \frac{p}{1 - q^m} (1 + 2q + 3q^2 + \cdots + (m - 1)q^{m-2} + (m - 1)q^{m-1}). \tag{55}$$

In the proof below, (54) is shown first and then (52); the proof of these contain the result (51). Then (53) follows from (15), (52) and (54)].

Proof: The proof of (54) is by induction. The result holds for $m = 1$, since $F(1) = 0$. Assuming (54) holds for arguments less than m, we can use (19) with (16) to obtain

$$\begin{aligned} F_1{}^*(m) &= \frac{1 - q^m}{p} + \min_{1 \leqq x \leqq m-1} \left\{ q^x \left[\frac{1 - q^{m-x-1} - (m - x - 1)pq^{m-x}}{p^2} \right] \right. \\ &\quad \left. + \frac{1 - q^{x-1} - (x - 1)pq^x}{p^2} \right\} = \left[\frac{1 - q^{m-1} - (m - 1)pq^m}{p^2} \right] \\ &\quad + \frac{1 - q^m}{p} + \frac{1}{p} \min_{1 \leqq x \leqq m-1} \{x(q^m - q^x) - pq^{x-1}\}. \end{aligned} \tag{56}$$

To prove part of (51) it is now shown that the minimum of the expression above in braces [say, $f_0(x)$] is attained at $x = 1$. Since this is obvious for $m = 2$, it is now assumed that $m \geqq 3$. Then

$$f_0(x) - f_0(1) = (x - 1)q^m + 1 - xq^x - pq^{x-1}, \tag{57}$$

and it suffices to show that, for $q \leqq q_0$ and $x = 2, 3, \cdots, m - 1$,

$$f_1(x) = 1 - xq^x - pq^{x-1} \geqq 0. \tag{58}$$

Similarly, it suffices to show that, for $q \leqq q_0$ and $x = 3, 4, \cdots, m - 1$,

$$f_2(x) = \frac{f_1(x) - f_1(2)}{q} = 1 + q - xq^{x-1} - pq^{x-2} \geqq 0. \tag{59}$$

More generally, if it suffices to show that, for $q \leqq q_0$ and $x = y + 1$, y + 2, $\cdots$, $m - 1$,

$$f_y(x) = 1 + (y - 1)q - xq^{x-y+1} - pq^{x-y} \geqq 0, \tag{60}$$

then it also suffices to show that, for $q \leqq q_0$ and $x = y + 2, y + 3, \cdots$, $m - 1$,

$$f_{y+1}(x) = \frac{f_y(x) - f_y(y + 1)}{q} = 1 + yq - xq^{x-y} - pq^{x-y-1} \geqq 0. \tag{61}$$

Setting $y = m - 3$ in (61), it suffices to show that, for $q \leqq q_0$ and $x = m - 1$ (we can now replace x by $m - 1$),

$$1 + (m - 4)q - (m - 2)q^2 = 1 - q - q^2 + (m - 3)pq \geqq 0. \tag{62}$$

Since q_0 is the root of $1 - q - q^2$ and $m \geqq 3$, the inequality (62) is proved and the minimum of (56) is attained at $x = 1$. Setting $x = 1$ in (56) gives the bracketed expression in (56) and proves the result (54).

To prove (52), we substitute (54) in (20) to obtain

$$\begin{aligned} H_1(n) &= 1 + \min_{1 \leqq x \leqq n} \left\{ q^x(n - x) - pq^{x-1} + p \sum_{i=1}^{x} iq^{i-1} \right. \\ &\qquad \left. + p \sum_{i=1}^{x} (n - i)q^{i-1} \right\} \\ &= n + 1 - \max_{1 \leqq x \leqq n} \{xq^x + pq^{x-1}\}, \end{aligned} \tag{63}$$

where the value of q^{x-1} for $q = 0$ and $x = 1$ is taken to be unity.

To prove the rest of (51), it is now shown that the maximum of the expression above in braces [say, $h(x)$] is attained at $x = 1$ for $q \leqq q_0$.

Then, for $q \leqq q_0$ and $x \geqq 1$,

$$\begin{aligned} h(x) - h(x+1) &= xpq^x - 2q^x + q^{x-1} \\ &= q^{x-1}[1 - q - q^2 + (x-1)pq] \geqq 0. \end{aligned} \tag{64}$$

Clearly, the maximum of $h(x)$ is attained at $x = 1$ not only for $0 < q \leqq q_0$ but also for $q = 0$. Setting $x = 1$ in (63) gives $H_1(n) = n$, and this completes the proof of (52). The fact that the minimum is attained only at $x = 1$ for $q < q_0$ in both (56) and (63) proves (51) and shows that, under R_1 with $q < q_0$, units are tested one at a time.

IX. CONSTRUCTION OF TABLES FOR R_1

Figs. 3 and 4 describe the procedure R_1 for $n = 2(1)16$ and $m \leqq n$ in the form of two diagrams that are easy to use in a practical situation. Tables III A and III B give the polynomials, the roots of which are the dividing points in Figs. 3 and 4. Table IV A gives the polynomial equation for $F_1^*(m)$ for $m = 2(1)16$. Tables IV B and IV C give the polynomial equations for $H_1(n)$ and $G_1^*(m, n)$, respectively, for $n = 2(1)12$ and $2 \leqq m \leqq n$. These can be obtained from (6), (7) and (8), or from (18), (19) and (20) and the boundary conditions, $H_1(0) = F_1^*(1) = 0$. For the sake of brevity, Table IV C has been reduced so that it gives the results only for $q \geqq 0.85$, and only for pairs m,n which can arise starting from an $H(n)$-situation with $n \leqq 12$.

Having computed $H_1(n)$ and $F_1^*(m)$ for $2 \leqq m \leqq n \leqq 12$, we can make the procedure R_1 explicit for $12 \leqq n \leqq 16$ by a different method, which will now be explained. Let $H_1(n \mid x)$ denote the value of $H_1(n)$ if (i.e., for those q-values for which) the next sample size is x; let $F_1^*(m \mid x)$ be defined similarly. Then (20) can be written as

$$\begin{aligned} H_1(n \mid x) - H_1(n-1) &= 1 + pF_1^*(x) \\ &\quad - \sum_{i=1}^{x-1} q^i[H_1(n-i) - H_1(n-i-1)]. \end{aligned} \tag{65}$$

Writing a similar equation for $H_1(n \mid y)$ for $y > x$ and subtracting gives

$$\begin{aligned} H_1(n \mid x) - H_1(n \mid y) &= -p[F_1^*(y) - F_1^*(x)] \\ &\quad + \sum_{j=x}^{y-1} q^j[H_1(n-j) - H_1(n-j-1)]. \end{aligned} \tag{66}$$

If $4 \leqq x < y \leqq 16$ and $12 < n \leqq 16$, then the right member of (66) involves H-function arguments only up to 12. In particular, for $y = x + 1$ we set the right member of (66) equal to zero and obtain a polynomial whose root (between zero and one) is the dividing point, $q_H(x, n)$, between x and $x + 1$. Table II A shows that, for $n > 12$ and $x < 4$, the pattern for the dividing points is well stabilized; for example, the value of $q_H(1, n) = 0.618$ (to three decimal places) is shown to hold for all n in Section VIII. By considering various pairs x, y (most often of the type x, $x + 1$) in (66) it is possible to determine the procedure R_1 for the $H(n)$-situation for $12 < n \leqq 16$, without explicitly computing the formula for $H_1(n)$.

Similarly, we can do the same for the G-situation by using

$$\begin{aligned} F_1^*(m \mid x) - F_1^*(m \mid y) = F_1^*(y) - F_1^*(x) \\ + q^y F_1^*(m - y) - q^x F_1^*(m - x), \end{aligned} \tag{67}$$

but in this case (according to Section VI) we need only consider values $y = x + 1$ up to and including $m/2$.

Table II A gives a numerical comparison for $N = 4$, 8 and 12 of R_1 and several other procedures, two of which are based on the work of Dorfman[1] and Sterrett;[3] the others are defined in the Appendices to this paper. Table II B gives a brief numerical comparison of $H_1(n)$ and $H_3(n)$ (corresponding to procedure R_3 defined in Appendix B) for large values of n [viz., $n = 10(10)100$] and $q = 0.90$, 0.95 and 0.99; these entries were computed on the IBM-704.

Tables V A, V B and V C give the numerical values of $H_1(n)$ and $G_1(m, n)$ as well as the values of $x_G(m; q)$ and $x_H(n; q)$ for $q = 0.90$, 0.95 0.99 for $2 \leqq n \leqq 100$, and for appropriate values of m; these entries were computed on the IBM-704. For $q = 0.90$ the value of x is always at most 9 and hence, if we start with an H-situation, there is no need to consider values of $m > 9$; similarly, for $q = 0.95$ we disregard values of $m > 19$. For $q = 0.99$ we should consider all values of m up to and including $m = 100$ but many of these were omitted for the sake of brevity. It is interesting to note that $G_1(m, n)$ is strictly monotonic in the second argument (and hence also in the argument $n - m$) for fixed m, but it is curious and difficult to explain why it is not monotonic in the first argument for fixed n.

X. A SUGGESTED PROCEDURE FOR THE CASE OF UNKNOWN q

It is reasonable to expect that a knowledge of good procedures for the case of known q will suggest good procedures for the case of unknown q. From this point of view we consider modifications of the basic procedure R_1 that make it adaptable when q is unknown. It is suggested that after each test we form a new estimate of q and that the procedure R_1 be used with the estimated value in place of the true value. At the outset we can start with an estimate based on past experience or we can start by testing one unit at a time. A thorough investigation of the relative merit of this procedure has not been carried out. Some discussion on the maximum likelihood method of estimating q is given below.

Let d and s denote the number of units proven defective and proven good, respectively, so that at any stage of experimentation we have

$$N = d + s + m + (n - m) = d + s + n, \tag{68}$$

where N is the total number of units at the outset, m is the size of the defective set (which is known to contain at least one defective) and $n - m$ is the size of the binomial set. The likelihood L of the observed result (68) is given by

$$L = \binom{N - n}{d} p^d q^{N-n-d}(1 - q^m). \tag{69}$$

Then it is easily shown that

$$\frac{d}{dq}(\log L) = -\frac{d}{dp}(\log L) = -\frac{1}{pq}\left[d - (N - n)p + \frac{mpq^m}{1 - q^m}\right]. \tag{70}$$

Setting the latter equal to zero, we find that, for $m \neq 0$, the maximum likelihood estimate $\hat{q}$ of q is the root of

$$(N - n + m)(1 - \hat{q}^m)(1 - \hat{q}) - d(1 - \hat{q}^m) - m(1 - \hat{q}) = 0 \tag{71}$$

or, equivalently,

$$s - d \sum_{i=1}^{m} \hat{q}^i - (m + s)\hat{q}^m = 0 \tag{72}$$

and, for $m = 0$, we have $\hat{q} = s/(d + s)$, the usual estimate. For $s = 0$ and $m + d \geqq 1$, we get $\hat{q} = 0$ and, for $s = 1$, it is easily seen, using the Descartes Rule of Signs, that (72) has exactly one root $\hat{q}$ (allowing multiplicities) in the unit interval and hence $\hat{q}$ is uniquely defined. The remaining case, $s = m = d = 0$, can only occur at the outset when there is no observations on which to base an estimate. It is interesting to note that the same result (71) or (72) can also be obtained by computing the conditional expected proportion of defectives among the N units, given the observed s, d, m and n, and setting it equal to $1 - q$. The equation thus obtained is the same as (71), and its root is $\hat{q}$.

The above method of getting an estimate is being suggested in connection with procedure R_1, but it can also be used in connection with the procedures R_2 and R_4 (see Appendices) without any change. For procedures R_3 and R_5 we can have several defective sets and several binomial sets at any one time, and (71) then becomes

$$d + (1 - \hat{q}) \left[\sum_{i=1}^{I} \frac{m_i}{1 - \hat{q}^{m_i}} - \left(N - \sum_{j=1}^{J} n_j' \right) \right] = 0, \tag{73}$$

where $m_1, m_2, \cdots, m_I$ are the sizes of the defective sets, $n_1', n_2', \cdots, n_J'$ are the sizes of the binomial sets, and

$$N = d + s + \sum_{i=1}^{I} m_i + \sum_{j=1}^{J} n_j'. \tag{74}$$

If the number of tests carried out is large (and hence $N - n$ must be large), the maximum likelihood estimate is approximately normally distributed with expectation equal to the true value of q and variance given by

$$\sigma^2(\hat{q}) \cong \left[E \left(\frac{d \log L}{dq} \right)^2 \right]^{-1}, \tag{75}$$

where m and n in (69) are to be regarded as chance variables. Taking expectation first for fixed m and n and then with respect to m and n, gives

$$\sigma^2(\hat{q}) \cong \left[E \left(\frac{N - n}{pq} \right) + E \left(\frac{mq^{m-1}}{1 - q^m} \right)^2 \right]^{-1}. \tag{76}$$

Since $mpq^{m-1} < 1 - q^m$ for all m and all $q < 1$, and since the expectation of a square is nonnegative, for asymptotically large $N - n$

$$\frac{p^2 q}{p + q[N - E(n)]} \leqq \sigma^2(\hat{q}) \leqq \frac{pq}{N - E(n)}. \tag{77}$$

In particular, if we continue to test until a fixed proportion $\theta > 0$ of the N units are determined to be good or bad (i.e., until $n/N = 1 - \theta$, approximately) then, for asymptotically large N (so that $N\theta$ is also large), we obtain

$$\frac{p^2 q}{N\theta p + q} \leqq \sigma^2(\hat{q}) \leqq \frac{pq}{N\theta}. \tag{78}$$

For $\theta = 1$ and large N, the two bounds are essentially equal and the common value is the same as for ordinary binomial sampling. In general, at any stage of experimentation it appears to be conservative to estimate $\sigma^2(\hat{q})$ in the same way as for ordinary binomial sampling based on $N - n$ observations, using the value of n that is actually realized at that time.

In regard to the procedure, if the size of the very first test group is based on past experience, the question arises as to whether this past experience should also enter into the second, third and other early estimates of q. If it does not enter, then in the early tests we may find sudden jumps from testing very small numbers to testing very large numbers and *vice versa*, both of which are undesirable. This makes it useful to find a method to continue to use past experience until the estimate of q (without using past experience) is stabilized. In the absence of past experience, this same feature may make it desirable to test several units one at a time before starting to use any group-testing procedure.

XI. SOME GENERALIZATIONS OF R_1

Returning to the case of known probabilities q, we consider some generalizations of the same basic problem and, in each case, the appropriate generalization of the procedure R_1. The appropriate formulae will be given, but only a few simple computations will be carried out.

1. Two (or more) different kinds of units with known probabilities (say, $q_1 \leqq q_2$) of a good unit are present and both can be put into the same test group.

Let $H_{11}(n_1, n_2)$ denote the expected number of tests required under the proposed procedure R_{11} if there are n_i units of type i with $q = q_i (i = 1, 2)$ and the binomial chance variables associated with the units are mutually independent. Let $G_{11}(m_1, m_2; n_1, n_2)$ denote the expected number of tests required under R_{11} if there is a defective set containing m_1 units of type 1 and m_2 units of type 2 (known to contain at least one defective among the $m_1 + m_2$ units) and a binomial set containing $n_1 - m_1 \geqq 0$ of type 1 and $n_2 - m_2 \geqq 0$ of type 2. The recursion formulae corresponding to (6), (7) and (8) are

$$H_{11}(n_1, n_2) = 1 + \min \{q_1^x q_2^y H_{11}(n_1 - x, n_2 - y) + (1 - q_1^x q_2^y) G_{11}(x, y; n_1, n_2)\}, \tag{79}$$

where the minimum is over pairs (x, y) with $0 \leqq x \leqq n_1$, $0 \leqq y \leqq n_2$ and $x + y \geqq 1$, and

$$G_{11}(m_1, m_2; n_1, n_2) = 1 + \min \left\{ \left(\frac{q_1^x q_2^y - q_1^{m_1} q_2^{m_2}}{1 - q_1^{m_1} q_2^{m_2}} \right) \cdot G_{11}(m_1 - x, m_2 - y; n_1 - x, n_2 - y) + \left(\frac{1 - q_1^x q_2^y}{1 - q_1^{m_1} q_2^{m_2}} \right) G_{11}(x, y; n_1, n_2) \right\}, \tag{80}$$

where the minimum is over pairs (x, y) with $0 \leqq x \leqq m_1$, $0 \leqq y \leqq m_2$ and $1 \leqq x + y \leqq m_1 + m_2 - 1$. The boundary conditions state that, for all $q_1 \leqq q_2$,

$$G_{11}(1, 0; n_1, n_2) = H_{11}(n_1 - 1, n_2) \quad \text{for all} \quad n_1 \geqq 1, n_2 \geqq 0, \tag{81}$$

$$G_{11}(0, 1; n_1, n_2) = H_{11}(n_1, n_2 - 1) \quad \text{for all} \quad n_1 \geqq 0, n_2 \geqq 1, \tag{82}$$

$$\begin{aligned} G_{11}(m_1, 0; n_1, 0) &= G_1(m_1, n_1; q_1); \\ G_{11}(0, m_2; 0, n_2) &= G_1(m_2, n_2; q_2), \end{aligned} \tag{83}$$

$$H_{11}(n_1, 0) = H_1(n_1; q_1); \qquad H_{11}(0, n_2) = H_1(n_2; q_2), \tag{84}$$

where the right-hand member of each equality in (83) and (84) refers to the basic procedure R_1 defined by (6), (7) and (8).

It is clear that $H_{11}(1, 0) = H_{11}(0, 1) = 1$ and $H_{11}(0, 0) = 0$. It follows from (80) that, for $q_1 \leqq q_2$,

$$G_{11}(1, 1; 1, 1) = \frac{2 - q_2 - q_1 q_2}{1 - q_1 q_2}, \tag{85}$$

and the rule is to test first the unit of type 2. Using this result, we can compute

$$H_{11}(1, 1) = 1 + \min(1, 2 - q_2 - q_1 q_2), \tag{86}$$

and the rule is to test either unit separately if $1 - q_2 - q_1 q_2 > 0$ and to test both simultaneously if $1 - q_2 - q_1 q_2 \leqq 0$. The latter inequality is a direct generalization of the inequality $1 - q - q^2 \leqq 0$, which played a prominent role in the basic procedure R_1. We state (without proof) that, if $q_1 \leqq q_2 < \frac{1}{2}(\sqrt{5} - 1) = 0.618$ (to three decimals), all testing is carried out one unit at a time.

2. Two (or more) experimenters may be working on a single set of N units by carrying out simultaneous, parallel group-tests and cooperating in such a way as to minimize the time required to accomplish the task.

It is clear that no saving can be effected in the expected total number of tests by having more than one experimenter. However, if the simultaneous tests are regarded as a stage, each of which lasts the same amount of time, then minimizing the expected number of stages is equivalent to minimizing the expected time required to accomplish the task. These remarks indicate that there may be some conflict in these two aims of reducing the expected time and the expected total number of tests. For this reason, it should be stated that our primary emphasis in this problem is to reduce the expected time.

Let m and m' denote the sizes of defective sets and let $n - (m + m') \geqq 0$ denote the size of the binomial set. Let $H_{12}(n)$ denote the expected number of stages required for $m = m' = 0$ by the proposed procedure R_{12}. Let $G_{12}(m, m', n)$ denote the expected number of stages required by R_{12} if we have two defective sets of size m, m' and one binomial set of size $n - m - m' \geqq 0$. Let $G_{12}(m, 0, n)$ and $G_{12}(0, m, n)$ be denoted by $G_{12}(m, n)$, so that $G_{12}(0, n) = H_{12}(n)$. The recursion formulae for R_{12} are

$$\begin{aligned} H_{12}(n) = 1 + \min_{\substack{x,\, y \geqq 1 \\ x + y \leqq n}} [&q^{x+y} H_{12}(n - x - y) + q^{x}(1 - q^{y}) G_{12}(y, n - x) \\ &+ q^{y}(1 - q^{x}) G_{12}(x, n - y) + (1 - q^{x})(1 - q^{y})\, G_{12}(x, y, n)], \end{aligned} \tag{87}$$

$$\begin{aligned} G_{12}(m, m', n) = 1 + \min_{\substack{1 \leqq x \leqq m-1 \\ 1 \leqq y \leqq m'-1}} &\left\{ \left(\frac{q^{x} - q^{y}}{1 - q^{m}}\right)\left(\frac{q^{y} - q^{m'}}{1 - q^{m'}}\right) \right. \\ &\cdot G_{12}(m - x, m' - y, n - x - y) \\ &+ \left(\frac{q^{x} - q^{m}}{1 - q^{m}}\right)\left(\frac{1 - q^{y}}{1 - q^{m'}}\right) G_{12}(m - x, y, n - x) \\ &+ \left(\frac{1 - q^{x}}{1 - q^{m}}\right)\left(\frac{q^{y} - q^{m'}}{1 - q^{m'}}\right) G_{12}(x, m' - y, n - y) \\ &\left. + \left(\frac{1 - q^{x}}{1 - q^{m}}\right)\left(\frac{1 - q^{y}}{1 - q^{m'}}\right) G_{12}(x, y, n) \right\} \end{aligned} \tag{88}$$

and

$$G_{12}(m, n) = 1 + \min \left\{ \min_{\substack{x,\, y \geqq 1 \\ x + y \leqq m}} G_{12}'(x, y), \min_{\substack{1 \leqq x \leqq m-1 \\ 1 \leqq y \leqq n-m}} G_{12}''(x, y) \right\}, \tag{89}$$

where $G_{12}'(x, y)$ and $G_{12}''(x, y)$ are defined by

$$\begin{aligned} G_{12}'(x, y) = &\left(\frac{q^{x+y} - q^{m}}{1 - q^{m}}\right) G_{12}(m - x - y, n - x - y) \\ &+ \frac{q^{x}(1 - q^{y})}{1 - q^{m}}\, G_{12}(y, n - x) \\ &+ \frac{q^{y}(1 - q^{x})}{1 - q^{m}}\, G_{12}(x, n - y) \\ &+ \frac{(1 - q^{x})(1 - q^{y})}{1 - q^{m}}\, G_{12}(x, y, n) \end{aligned} \tag{90}$$

and

$$\begin{aligned} G_{12}''(x, y) = {} & \frac{q^y(q^x - q^m)}{1 - q^m} G_{12}(m - x, n - x - y) \\ & + \frac{q^y(1 - q^x)}{1 - q^m} G_{12}(x, n - y) \\ & + \frac{(1 - q^y)(q^x - q^m)}{1 - q^m} G_{12}(m - x, y, n - x) \\ & + \frac{(1 - q^y)(1 - q^x)}{1 - q^m} G_{12}(x, y, n) . \end{aligned} \tag{91}$$

The boundary conditions state that, for all q,

$$\begin{aligned} G_{12}(m, 1, n) = G_{12}(1, m, n) = G_{12}(m, n - 1) \\ \text{for} \quad 0 \leqq m \leqq n - 1, \end{aligned} \tag{92}$$

$$G_{12}(1, n) = H_{12}(n - 1) \quad \text{for} \quad n \geqq 1, \tag{93}$$

$$H_{12}(0) = 0. \tag{94}$$

It is easy to see that $H_{12}(1) = G_{12}(2, 2) = H_{12}(2) = 1$ and $G_{12}(3, 4) = G_{12}(4, 4) = 2$ for all q.

Remark 6: The extra complication in (89) insures that, for $n \geqq 2$, one experimenter will not be idle while another is carrying out a test.

Remark 7: It is conjectured that in (89) the possibility $x + y = m$ can be omitted, with the exception of the single case $m = n = 2$ (and $m' = 0$).

Remark 8: It is also conjectured that G_{12}'', which is needed for the cases $n > m = 2$, can be disregarded when $m > 2$; i.e., that G_{12}' always gives a smaller minimum for $m > 2$.

Remark 9: It is conjectured that, at any stage in which $m = m' \geqq 2$ or in which we have both $m = m' = 0$ and n even, the two test group sizes, x and y, will be equal. If either m or $m' = 0$, it is conjectured that the two test group sizes will differ by at most unity.

Further calculations yield

Test Group Sizes

x y

$$G_{12}(2, 3) = \frac{2 + q}{1 + q} \qquad \text{for all } q \qquad 1D\ 1B, \tag{95}$$

$$G_{12}(3, 3) = \frac{2 + 2q + q^2}{1 + q + q^2} \qquad \text{for all } q \qquad 1D\ 1D, \tag{96}$$

$$G_{12}(2, 4) = \begin{cases} 2 & \text{for } 0 \leqq q < 0.682 \quad 1D\ 1B, \\ \dfrac{3 + q - q^3}{1 + q} & \text{for } 0.682 \leqq q \leqq 1.000\ 1D\ 2B, \end{cases} \tag{97}$$

$$G_{12}(2, 2, 4) = \frac{2 + 4q + q^2}{(1 + q)^2} \qquad \text{for all } q \qquad 1D\ 1D', \tag{98}$$

$$H_{12}(4) = \begin{cases} 2 & \text{for } 0 \leqq q < 0.691 \quad 1B\ 1B, \\ 3 - 3q^2 + 2q^3 - q^4 & \text{for } 0.691 \leqq q \leqq 1.000\ 2B\ 2B, \end{cases} \tag{99}$$

where $1B$ indicates that 1 unit is taken from the binomial set to form one of the two test-groups and D, D' denote different defective sets.

It is interesting to compare the above result for $H_{12}(4)$ for $q \geqq 0.691$ with the procedure R_{12}^* of giving each experimenter two units to analyze independently of each other and without any mutual cooperation. Let T denote the total number of tests and S denote the number of stages required. Then, for $q \geqq 0.691$ (letting T_1, T_2 denote the number of tests in two independent experiments with $n = 2$ under R_1), it is easily shown that

$$E\{S \mid R_{12}^*\} = E\{\max(T_1, T_2) \mid R_1\} = 3 - q^2 - q^4, \tag{100}$$

$$E\{T \mid R_{12}\} = 2H_{12}(4) - 2q - 2q^2 + 2q^3 - 2q^4. \tag{101}$$

Hence we find that, for $q \geqq 0.691$,

$$\begin{aligned} &E\{S \mid R_{12}\} - E\{S \mid R_{12}^*\} \\ &\quad = H_{12}(4) - (3 - q^2 - q^4) = -2q^2(1 - q) \leqq 0, \end{aligned} \tag{102}$$

$$\begin{aligned} &E\{T \mid R_{12}\} - E\{T \mid R_{12}^*\} \\ &\quad = 6 - 2q - 2q^2 + 2q^3 - 2q^4 - 2H_1(2) = 2q^3(1 - q) \geqq 0, \end{aligned} \tag{103}$$

which illustrates the fact that R_{12} effects an improvement in the expected number of stages at the expense of a slight increase in the expected total number of tests.

3. In this generalization we apply the restriction that any one unit can be included in at most K group-tests. This is particularly appropriate in the blood testing application, where a single blood sample can be used in a small number K of blood tests and the patient does not want to be annoyed by having more than one blood sample taken.

In this problem there is again only one defective set but it is now denoted by a vector $\mathbf{m} = \{m_0, m_1, \cdots, m_{K-1}\}$, where $m_j \geqq 0$ is the number of units that have already been included in j group-tests. Similarly, the union of the binomial and defective sets is denoted by

$$\mathbf{n} = \{n_0, n_1, \cdots, n_{K-1}\}$$

and the binomial set alone is the difference $\mathbf{n} - \mathbf{m}$. The symbol for the size of the next test group will be $\mathbf{x} = \{x_0, x_1, \cdots, x_{K-1}\}$, where x_j is the number of units taken from n_j in an H-situation (from m_j in a G-situation). The symbols x, m, n will be used for the sum of the components in the vectors $\mathbf{x}$, $\mathbf{m}$, $\mathbf{n}$, respectively. Let $G_{13}^K(\mathbf{m}; \mathbf{n})$ denote the expected number of group-tests required to remove all defective units if the defective set is $\mathbf{m}$ and the binomial set is $\mathbf{n} - \mathbf{m}$. If $m = 0$, we denote this expectation by $H_{13}^K(\mathbf{n})$. For the special case in which $\mathbf{n}$ has all except one component (say, n_j) equal to zero, we will drop the zeros and write $H_{13}^K(n_j)$, with a scalar argument. The recursion formulae for this procedure R_{13}^K are given by

$$H_{13}^K(\mathbf{m}) = 1 + \min_{\substack{\text{all } \mathbf{x} \text{ with} \\ 1 \leqq x \leqq n \text{ and} \\ 0 \leqq x_j \leqq n_j \\ (j = 0, 1, \cdots, K-1)}} \{q^x H_{13}^K(\mathbf{n} - \mathbf{x}) + (1 - q^x) G_{13}^K(\mathbf{x}; \mathbf{n})\}, \tag{104}$$

$$G_{13}^K(\mathbf{m}; \mathbf{n}) = 1 + \min_{\substack{\text{all } \mathbf{x} \text{ with} \\ 1 \leqq x \leqq m-1 \\ 0 \leqq x_j \leqq m_j \\ (j = 0, \cdots, K-1)}} \left\{ \left(\frac{q^x - q^m}{1 - q^m} \right) G_{13}^K(\mathbf{m} - \mathbf{x}; \mathbf{n} - \mathbf{x}) + \left(\frac{1 - q^x}{1 - q^m} \right) G_{13}^K(\mathbf{x}; \mathbf{n}) \right\}, \tag{105}$$

where, as usual, $m > 1$. The boundary conditions state that, for all q, we have $H_{13}^K(\mathbf{O}) = 0$ and, for $m = 1$, we can write $G_{13}^K(\mathbf{m}; \mathbf{n}) = H_{13}^K(\mathbf{n} - \mathbf{m})$. It is easy to see that $H_{13}^K(\mathbf{n}) = 1$ for $n = 1$ and all q. Some further computations for $n = 2$ and $n = 3$ give, for any K:

$$\begin{array}{llll} & \underline{q-\text{interval}} & \underline{x-\text{value}} & \underline{j-\text{value}} \end{array}$$

$$H_{13}^K(2_j) = \left\{\begin{array}{llll} H_1(2) & 0 \leqq q \leqq 1.000 & (\text{see } R_1) & 0 \leqq j \leqq K - 2 \\ 2 & 0 \leqq q \leqq 1.000 & (0, \cdots, 0, 0, 1) & j = K - 1 \end{array}\right. \tag{106}$$

$$H_{13}^K(3_j) = \left\{\begin{array}{ll} H_1(3) \qquad 0 \leqq q \leqq 1.000 \qquad (\text{see } R_1) & 0 \leqq j \leqq K - 3 \\ \left\{\begin{array}{lll} 3 & 0 \leqq q \leqq 0.618 & (0, \cdots, 0, 1, 0) \\ 4 - q - q^2 & 0.618 \leqq q \leqq 0.707 & (0, \cdots, 0, 1, 0) \\ 4 - q^2 - 2q^3 & 0.707 \leqq q \leqq 1.000 & (0, \cdots, 0, 3, 0) \end{array}\right\} & j = K - 2 \\ 3 \qquad 0 \leqq q \leqq 1.000 \qquad (0, \cdots, 0, 0, 1) & j = K - 1. \end{array}\right. \tag{107}$$

If $K = 1$, all units are tested individually. If $K = 2$, then, after a set of size $m > 1$ is shown to be defective, all the units in that set are tested individually; this is the procedure recommended by Dorfman.[1] A more thorough investigation for particular values of $K \geqq 3$ will be given in a separate paper.

Under R_1 for any $H(n)$-situation the maximum number of additional tests $M^*(n)$ in which any particular unit will be included before experimentation is concluded (allowing a random reordering of units in the binomial and defective sets after every test) occurs when q is close to unity and the sample has all units defective. It is easily seen that, under R_1,

$$M^*(n) = M(n) - (n - 1) = (n + 1)\alpha(n) + 3 - 2^{1+\alpha(n)}, \quad (108)$$

where $\alpha(n)$ and $M(n)$ are defined by (10) and (11). Under $R_{13}{}^K$, let $j(\mathbf{n})$ denote the largest subscript associated with a nonzero component of $\mathbf{n}$. For $K \geqq M^*(n) + j(\mathbf{n})$, the restriction that no unit should be included in more than K group-tests does not affect the procedure R_1, and hence we have for procedure $R_{13}{}^K$

$$H_{13}{}^K(\mathbf{n}) = H_1(n) \quad \text{for} \quad 0 \leqq j(\mathbf{n}) \leqq K - M^*(n), \qquad (109)$$

which generalizes some results in (106) and (107) and shows that $R_{13}{}^K$ is a generalization of R_1.

Since units in the last component of the defective or the binomial sets cannot be tested in groups, we can remove at any time for individual testing all units in the last component of the binomial set and all but one of the units in the last component of the defective set without affecting the expected number of tests under $R_{13}{}^K$. It is easy to show that this leads to the two reduction formulae

$$H_{13}{}^K(\mathbf{n}) = n_{K-1} + H_{13}{}^K(n_0, \cdots, n_{K-2}, 0), \qquad (110)$$

$$\begin{aligned} G_{13}{}^K(\mathbf{m}; \mathbf{n}) = n_{K-1} &+ \left(\frac{1 - q^{-1+m_{K-1}}}{1 - q^m}\right) H_{13}{}^K(n_0, \cdots, n_{K-2}, 0) \\ &+ q^{-1+m_{K-1}} \left(\frac{1 - q^{+1-m_{K-1}}}{1 - q^m}\right) \qquad (111) \\ &\cdot [G_{13}{}^K(m_0, \cdots, m_{K-2}, 1; n_0, \cdots n_{K-2}, 0) - 1], \end{aligned}$$

which are useful in computations and for checking.

It is conjectured that, for $m = 0$ or $m > 1$, the procedure $R_{13}{}^K$ can always be carried out by putting in the next test-group only units that have been included in the same number of group tests (i.e., units in the same subset); the only possible exception to this is that, in any H-situation, if q is sufficiently close to unity, then $R_{13}{}^{K}$ will call for a test of all the remaining units; i.e., $\mathbf{x} = \mathbf{n}$.

Under the above conjecture, it is possible to carry out a simplification as in Section V and show, in direct analogy with (19), that

$$F_{13}{}^{*K}(\mathbf{m}) = \sum_{i=1}^{m} q^{i-1} + \min_{1 \leqq x \leqq m_i - 1} \{q^x F_{13}{}^{*K}(\mathbf{m} - \mathbf{x}) + F_{13}{}^{*K}(\mathbf{x})\}, \quad (112)$$

where $\mathbf{x}$ has x_i in the $(i + 1)$th position and zeros elsewhere (so that $x = x_i$), and i is defined as the subscript associated with the first non-zero component of $\mathbf{m}$. The function $F_{13}{}^{K}(\mathbf{m})$ is defined as the expected number of group-tests required to reach the next H-situation, and then we define, as in Section V,

$$F_{13}{}^{*K}(\mathbf{m}) = \left(\frac{1 - q^m}{1 - q}\right) F_{13}{}^{K}(\mathbf{m}). \quad (113)$$

Hence under the above conjecture it is again seen that, for any G-situation with $m > 1$, the next test-group $\mathbf{x}$ depends on $\mathbf{m}$ but is independent of $\mathbf{n} - \mathbf{m}$.

It appears to be true (but has not been rigorously proved) that, in this case also, for $q < q_0 = 0.618$ (to three decimal places) all units are tested one at a time.

XII. AN ASYMPTOTIC FORMULA FOR $H_1(n)$

In this section we shall use results obtained by considering an information procedure R_2, which is defined in Appendix A. The procedure R_2 appears to be a best test in the sense that it maximizes the information in the very next test but does not take into account the exact finite number of units present and the possible ways of distributing them among subsequent tests. It is therefore intuitively reasonable to expect that the procedure R_1 tends toward R_2 in the H-situation as $n \to \infty$ and also in the G-situation as $m \to \infty$. A more rigorous proof of this assertion would be desirable. It should also be pointed out that there is considerable numerical evidence in Tables IIIA and IIIB of the above assertion, which explains the reason for putting opposite $m = \infty$ and $n = \infty$ in these tables the polynomial equations

$$1 - q^x - q^{x+1} = 0 \qquad (x = 1, 2, \cdots), \quad (114)$$

which are derived for procedure R_2 in Appendix A.

We shall now derive an asymptotic formula for $H_1(n)$ for large n based on the assumption that the above reasoning is correct. For large values of n and fixed q, the expected number of tests required under procedure R_1 is approximately given by

$$H_1(n) \cong n \left(\frac{\begin{array}{l}\text{expected number of tests needed to reach} \\ \hfill \text{the next } H\text{-situation under } R_1\end{array}}{\begin{array}{l}\text{expected number of units analyzed between} \\ \hfill H\text{-situations under } R_1\end{array}} \right). \qquad (115)$$

The ratio of n to the denominator in (115) is the approximate number of H-situations reached if we start with n units, and this is clearly to be multiplied by the expected number of tests required to proceed from one H-situation to the next. Let T and U denote the chance variables in the numerator and demoninator, respectively, of (115). For a fixed q we find from the limiting procedure R_2 that, for an $H(n)$-situation with n large, we will, under R_1, "almost always" be using the same test-group size x, where x is that positive integer for which q^x is closer to one-half than is either q^{x-1} or q^{x+1}. Then, for this fixed integer x, which depends on the given q, we have

$$E\{U \mid R_1\} = xq^x + p \sum_{j=1}^{x} jq^{j-1} = \frac{1 - q^x}{1 - q}, \qquad (116)$$

which is obtained by assuming a single randomization of the order of the units at the outset and considering the different possible positions of the first defective.

Since $F_1(x)$ is the expected number of tests required under R_1 to get from a $G(m, n)$-situation to the next $H(n)$-situation, we have

$$E\{T \mid R_1\} = q^x + (1 - q^x)[1 + F_1(x)] = 1 + pF_1^*(x), \qquad (117)$$

where $F_1^*(x)$ is tabulated in Table IV A for $x = 2(1)16$ and all values of q. Hence, we obtain from (115), (116) and (117)

$$H_1(n) \cong \frac{np[1 + pF_1^*(x)]}{1 - q^x}, \qquad (118)$$

where x is defined above in terms of q. This is the main result of this section; we now consider some special cases.

For values of q close to unity we can use (23) to replace $F_1^*(x)$ by an explicit expression. If we also replace $(1 - q^z)/(1 - q)$ by z for q close to unity, we obtain for q close to unity

$$H_1(n) \cong \frac{n}{x} \{1 + p[x\alpha(x) + 2\beta(x)]\}, \qquad (119)$$

where $\alpha(x)$ and $\beta(x)$ are defined in (20a). If q approaches unity, x becomes large; if $\alpha(x) \geqq 2$, then $2\beta(x) < 2^{\alpha(x)+1} \leqq x\alpha(x)$ and, since $\alpha(x) \rightarrow \infty$, it follows that we can disregard $2\beta(x)$ in (119). For q close to unity and x large, the dividing points get closer and closer, and we obtain

$$0 \cong 1 - q^x - q^{x+1} \cong 1 - 2q^x, \tag{120}$$

so that $x \cong \left[\log_2\left(\frac{1}{q}\right)\right]^{-1}$. Also, from the definition of $\alpha(x)$,

$$2^{\alpha(x)} \leqq x \leqq 2 \cdot 2^{\alpha(x)}, \tag{121}$$

so that

$$\log_2\left(\frac{x}{2}\right) \leqq \alpha(x) \leqq \log_2(x). \tag{122}$$

Using the upper value in (122) gives

$$H_1(n) \cong n \log_2\left(\frac{1}{q}\right) + np \log_2\left[\log_2\left(\frac{1}{q}\right)\right]^{-1}. \tag{123}$$

It can be shown that the first term in (123) goes to zero faster than the second as q approaches unity, and hence we drop the first term and rewrite the second in the form

$$H_1(n) \cong -np \log_2 p. \tag{124}$$

In particular, if $p = 1/n$ so that $q = 1 - (1/n)$, we obtain from (124)

$$H_1(n) \cong \log_2 n. \tag{125}$$

In this case, we also have $x = n = 1/p$, and a better estimate is obtained by setting $q^x = e^{-1}$ in (23) and (118). Assuming that (23) is either equal to $F_1{}^*(x)$ or is a good approximation to it, we obtain

$$H_1(n) \cong \frac{n}{x}\left[\frac{1 + pF_1{}^*(x)}{1 - e^{-1}}\right] = \frac{n}{x}\left[1 + \alpha(x) + \frac{e^{2\beta(x)/x}}{e - 1}\right]. \tag{126}$$

For the case $n = N = 100$ and $q = 0.99$, we obtain 8.32 as the exact value from Table V C; $\log_2 100 = 6.64$, from (125); 7.79, from (119); and 8.20 from (126).

A rough lower bound on $H(n)$ for any procedure can be easily obtained from information theory. The total information in n units is

$$-n(p \log_2 p + q \log_2 q),$$

and this is to be equated with the product of the expected number of tests $H(n)$ and the average information obtained per group-test. Since the maximum information per group-test (in bits) is unity, we obtain for any procedure

$$H(n) \geqq -n(p \log_2 p + q \log_2 q). \tag{127}$$

For $n = N = 100$ and $q = 0.99$, this gives 8.09 as a lower bound. Since a result better than 8.09 is impossible, the smallness of the difference $8.32 - 8.09 = 0.23$ is an indication of how far R_1 can possibly be from an optimal solution. However, it should not be inferred that the lower bound (127) can be reached for any value of q (except possibly for $p = q = \frac{1}{2}$) by any procedure. In fact, for $q < \frac{1}{2}$ and q decreasing towards zero, it has been shown[4] that an optimal procedure must have $H(n) = n$, whereas the right member of (127) approaches zero.

It has been pointed out to the authors by S. W. Roberts that a lower bound for $G(m, n)$ for any procedure is easily shown to be

$$\begin{aligned} G(m,n) \geqq & \sum_{i=1}^{m} \frac{pq^{i-1}}{1-q^m}\left[\log_2\left(\frac{pq^{i-1}}{1-q^m}\right)\right. \\ & \left. + (n-i)(p \log_2 p + q \log_2 q)\right] \\ = & \frac{1}{1-q^m}\left[q^m \log_2 q^m + (1-q^m)\log_2(1-q^m)\right] \\ & - \left(n + \frac{mq^m}{1-q^m}\right)(p \log_2 p + q \log_2 q). \end{aligned} \tag{128}$$

XIII. LACK OF OPTIMALITY OF PROCEDURE R_1

To illustrate the fact that R_1 is not optimal in the general case when units are identifiable and "mixing" of units from the binomial and defective sets is allowed, we shall describe a method of obtaining an improvement on R_1. It is sufficient to consider the case $N = 3$, but the case $N = 4$ is more typical, and we shall use the latter. Let R_0^* denote a procedure for $N = 4$, part of which is described by Fig. 2 and the remaining part of which is arbitrary. (We can therefore also regard R_0^* as a set of procedures, with the common part shown in Fig. 2.) Let a_1, a_2, b_1, b_2 denote individual units; it will be assumed that the a-units are distinguishable from the b-units. The part of Fig. 2 enclosed by dashed lines is different from R_1, since it includes mixing; the rest of the procedure agrees with R_1 for q close to unity. For q close to unity and $m = 2$, after the first two group-tests result in failure, we should act as if there was exactly one defective present until it is proved otherwise. Then, for q close to unity, the above procedure R_0^* terminates in one or two ad-

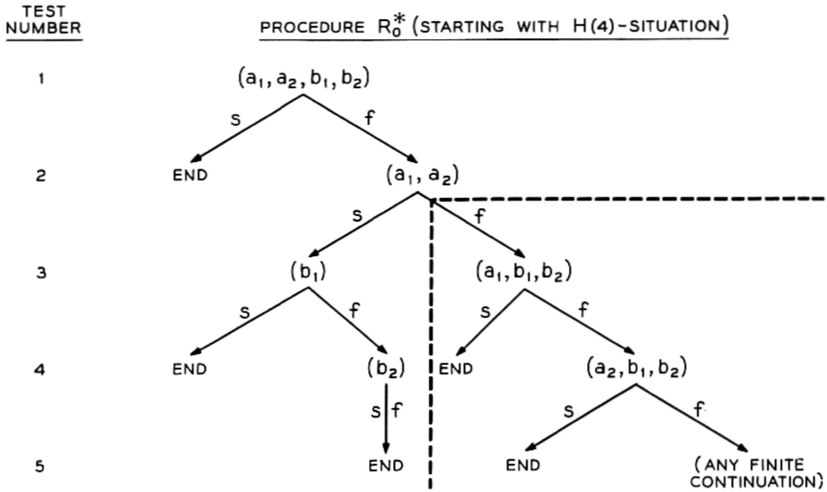

Fig. 2 — Procedure $R_0{}^*$.

ditional tests with probability close to $\frac{1}{2}$ for each. More precisely, we obtain for the conditional expected number of additional tests required under $R_0{}^*$, given that the first two tests result in a failure,

$$G(2,4 \mid R_0{}^*) = \frac{pq^3}{1-q^2} + \frac{2pq^3}{1-q^2} + \frac{p^2f(q)}{1-q^2} = \frac{3q^3 + pf(q)}{1+q}, \tag{129}$$

where $f(q)$ is a polynomial in q. In comparison, we have under R_1 for $0.707 < q < 1.000$

$$\begin{aligned} G_1(2,4) &= 1 + \frac{pH_1(3)}{1-q^2} + \frac{pqH_1(2)}{1-q^2} \\ &= \frac{1}{1+q}(6 + 2q - 2q^2 - 2q^3). \end{aligned} \tag{130}$$

For q approaching unity, the value in (129) approaches $\frac{3}{2}$, while that in (130) approaches 2. This proves that any finite continuation in Fig. 2 will be better than R_1 for q sufficiently close to unity. In a particular procedure to be discussed in a separate paper, the dividing point for the $G(2, 4)$-situation between "no-mixing" and "mixing" is

$$q = (1 + \sqrt{33})/8 = 0.843$$

(to three decimal places). The maximum improvement over R_1 for $n = 4$ in the expected number of tests required for the H-situation is a decrease of 0.04. The price to be paid for this improvement will be an increase in the complexity of the procedure.

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APPENDIX A

The Information Procedure

Another procedure which was investigated is based on choosing that value of x which maximizes the "amount of information" that the next test will give. The amount of information in a test with two outcomes is $p \log_2(1/p) + q \log_2(1/q)$ if p is the probability of either outcome. Hence, equating the information in an H-situation obtained by taking x and $x + 1$ in the next test gives

$$\begin{aligned} q^x \log_2 q^x + (1 - q^x) \log_2 (1 - q^x) \\ = q^{x+1} \log_2 q^{x+1} + (1 - q^{x+1}) \log_2 (1 - q^{x+1}), \end{aligned} \tag{131}$$

whose root will be used as a dividing point between x and $x + 1$. It is easy to verify that, for any integer x and any integer $n \geqq x + 1$, the unique positive root of (131) is also the unique positive root of

$$1 - q^x - q^{x+1} = 0. \tag{132}$$

It is interesting to note that the solution implicit in (132) is easily seen to be equivalent to finding that positive integer x (for fixed, known q) for which q^x is closer to $\frac{1}{2}$ than is either q^{x-1} or q^{x+1}. In fact, by (132), the right endpoint of the interval for x is such that q^x and q^{x+1} are centered about $\frac{1}{2}$ and the left endpoint of the interval for x is such that q^x and q^{x-1} are centered about $\frac{1}{2}$. Similarly, for the G-situation with $m > 1$, we equate

$$\left(\frac{q^x - q^m}{1 - q^m}\right) \log_2 \left(\frac{q^x - q^m}{1 - q^m}\right) + \left(\frac{1 - q^x}{1 - q^m}\right) \log_2 \left(\frac{1 - q^x}{1 - q^m}\right) \tag{133}$$

with the same expression, except that x is replaced by $x + 1$, and find that, for any $n \geqq m > 1$, the dividing point between x and $x + 1$ is the unique root in the interior of the unit interval (if it exists) of

$$1 - q^x - q^{x+1} + q^m = 0. \qquad (134)$$

If we remove the root $q = 1$ in (134), the dividing point is the unique positive root (if it exists) of

$$1 + q + q^2 \cdots + q^{x-1} - q^{x+1} - q^{x+2} - \cdots - q^{m-1} = 0. \qquad (135)$$

If the root does not exist for some $m > 1$, then $x + 1$ will never be used for that m. It should be noted that the left member of (134) is a strictly increasing function of x and, for $x \geqq (m - 1)/2$, $m > 1$ and any fixed q with $0 \leqq q < 1$, we have

$$1 - q^x - q^{x+1} + q^m \geqq (1 - q^{(m-1)/2})(1 - q^{(m+1)/2}) > 0. \qquad (136)$$

It follows that the highest value of x for which a nondegenerate root exists is such that $x + 1 < (m + 1)/2$ and hence, under this procedure, we never take a test group of size greater than $m/2$. It is interesting to note that the dividing points for any G-situation do not depend on n.

These equations define a new procedure R_2, which we shall also call the *information procedure*. For this, let $F_2(m)$ denote the expected number of group-tests required to "break up" a defective set of size m, i.e., to reach an H-situation. Let $F_2{}^*(m) = (1 - q^m)p^{-1}F_2(m)$. Then we can write as in (20), for any n and for the appropriate interval where the next test group is of size x,

$$F_2{}^*(m \mid x) = \sum_{i=1}^{m} q^{i-1} + q^x F_2{}^*(m - x) + F_2{}^*(x) \qquad (m > 1), \quad (137)$$

$$H_2(n \mid x) = 1 + q^x H_2(n - x) + pF_2{}^*(x) + p\sum_{i=1}^{x} q^{i-1} H_2(n - i). \quad (138)$$

The boundary conditions state that $F_2{}^*(1) = H_2(0) = 0$ for all q. Those expressions for $F_2{}^*(m)$ which are used to generate expressions for $H_2(n)$ for $2 \leqq m \leqq n \leqq 12$ are given in Table VIII; the resulting expressions for $H_2(n)$ (with x-values) are given in Table VI. Table VII gives the dividing points for $n = 1(1)100$ and for $m = 1(1)16$, $20(5)100$ for procedure R_2.

It should be noted that the $F_2{}^*(m)$ as well as the $H_2(n)$-function are not all continuous. At the point of discontinuity the x corresponding to the smaller expectation should be used.

It is interesting to observe in the numerical comparisons of Table III A that the procedure R_2 compares quite favorably with the procedure R_1. In addition, the fact that the dividing points are easier to compute makes it better for practical applications, since the dividing points for R_1 are only known exactly up to $n = 16$. It is also interesting to note that the limiting expressions in Table III A as $n \to \infty$ and in Table III B as $m \to \infty$ are the same as (132).

It is interesting to note that a succession of modifications $R_2^{(j)}(j = 1, 2, \ldots)$ of the information procedure, R_2, are possible such that $R_2^{(1)} = R_2$ and $R_2^{(j)} = R_1$ for $j \geqq M(m,n)$. Here $M(m,n)$, as defined in (10), is the maximum number of group-tests required if we start with a $G(m,n)$-situation [where $G(0,n)$ corresponds to $H(n)$]. Under the procedure $R_2^{(j)}$ we find and use that x which maximizes the ratio of the information expected from at most j group-tests to the conditional expected number of tests required given that we will stop after at most j group-tests. In the special case when there is no possibility of stopping before j tests, we can disregard the denominator and simply maximize the information. For the case $j = 1$ this is clearly equivalent to R_2. For $j \geqq M(m,n)$ the information expected from at most j tests is the same regardless of what x is used next and of what sample path is taken, since all units are then analyzed. Hence, the numerator above can be disregarded and the problem is to minimize the denominator or expected number of tests. Under the assumption of "no-mixing" of units from the binomial and defective sets, this gives the procedure R_1.

For any $H(n)$-situation with $n \geqq 4$ and $j \geqq 2$, these procedures appear to eliminate the possibility of taking $n - 1$ units in the next test-group. For example, if $n = 4, j = 2$ and $q > 0.618$, then we will want to compare $x = 2$ and $x = 3$. For $j = 1$, the dividing point between $x = 2$ and $x = 3$ is $q = 0.755$. Since neither $x = 2$ nor $x = 3$ can result in termination after one test, we can disregard the denominator and compare for $x = 2$ and $x = 3$ the information expected from two group-tests. After simplification, the difference between the results expected after $x = 2$ and $x = 3$ can be written as

$$p^2q[(1 + q) \log_2 (1 + q) - q \log_2 q] \geqq 0, \tag{139}$$

which shows that $x = 2$ is preferable to $x = 3$ for all $q > 0.618$. The same result holds for all $j \geqq 2$. Then we find that the dividing point between $x = 2$ and $x = 4$ for $j = 2$ is the nondegenerate root between zero and unity of

$$\begin{aligned}(2 - 2q - q^4 - q^5 - 2q^7)q \log_2 q - (2 + q^6)& \\ (1 - q^2) \log_2 (1 + q) - q^4(1 - q^4) \log_2 p = 0,&\end{aligned} \tag{140}$$

which is 0.789 to three decimal places. For $j \geqq 8 = M(0,4)$ the corresponding dividing point for $R_2^{(j)} = R_1$ is the root of $1 - q^2 - q^4 = 0$, or 0.786 to three decimal places. Curiously enough, the same result 0.786 is also the dividing point between $x = 2$ and $x = 4$ for $j = 1$.

For the special case $x = 1$, we state without proof that, for any $H(n)$-situation, the dividing point between $x = 1$ and $x = 2$ is again $\frac{1}{2}(\sqrt{5} - 1) = 0.618$ to three decimal places.

Formulae for the expected number of tests under $R_2^{(j)}$ for $1 < j < M(m,n)$ have not been derived in this paper.

APPENDIX B

Definition of Procedure R_3

It may happen in some problems that recombination is undesirable or impossible, or it may be that we are interested in finding out just how much is saved by allowing recombinations. Both are good reasons for considering a procedure R_3 that is similar to R_1 except that recombinations are not allowed. This simply means that any two operationally formed sets cannot be combined to form a new set from which subsequent test-groups are to be taken. In procedure R_1 the possibility of mixing proper subsets of two different sets was never used, and the same will be true for R_3. If both recombinations and mixing are not used, then, as the experiment continues, the operationally formed sets can only be broken down further into smaller and smaller sets, yielding a nested set of partitions; i.e., any two units separated at some stage remain separated in subsequent stages. It follows that any defective set present is not affected by the number or size or nature of other sets present. Hence, we define $G_3(m)$, with a single argument, as the conditional expected number of group-tests required to remove all the defectives from a set of size m which is known to have at least one defective.

The recursion formulae for R_3 are, for $n \geqq 1$ and $m > 1$,

$$H_3(n) = 1 + \min_{1 \leqq x \leqq n} \{q^x H_3(n - x) + (1 - q^x)[G_3(x) + H_3(n - x)]\}, \tag{141}$$

$$G_3(m) = 1 + \min_{1 \leqq x \leqq m-1} \left\{\left(\frac{q^x - q^m}{1 - q^m}\right) G_3(m - x) + \left(\frac{1 - q^x}{1 - q^m}\right) [G_3(x) + H_3(n - x)]\right\}, \tag{142}$$

with boundary conditions $H_3(0) = G_3(1) = 0$ for all q. If we let $G_3^*(m)$ denote $(1 - q^m)p^{-1}G_3(m)$ and simplify, we obtain

$$H_3(n) = 1 + \min_{1 \leqq x \leqq n} \{H_3(n - x) + pG_3^*(x)\}, \tag{143}$$

$$\begin{aligned} G_3^*(m) = \sum_{i=1}^{m} q^{i-1} + \min_{1 \leqq x \leqq m-1} \Bigg\{ & G_3^*(x) + q^x G_3^*(m - x) \\ & + H_3(n - x)\left(\sum_{i=1}^{x} q^{i-1}\right)\Bigg\}, \end{aligned} \tag{144}$$

with boundary conditions $H_3(0) = G_3^*(1) = 0$ for all q.

Numerical comparisons of the results for R_1 and R_3 are given in Tables II A and II B.

APPENDIX C

Definition of Two Halving Procedures

Two "halving" procedures R_4 and R_5 are defined below, the principal purpose being to compare the results on the expected number of tests required with comparable results for R_1 and R_3. The procedure R_4 allows recombinations exactly as in R_1, while R_5 is the same as R_4 except that recombinations are not allowed. Both R_4 and R_5 are of particular interest, since they can be carried out without knowing the true value of q.

The procedure R_4 is carried out like R_1 except that, if the defective set is of size $m > 1$, the next test group is a subset of size $m' = [m/2]$ (i.e., the largest integer contained in $m/2$) randomly selected from the defective set and, if $m = 0$ or 1, the entire binomial set is used in the next test-group. In particular, we start with all N units in the first test-group. The recursion formulae for R_4 are

$$H_4(n) = q^n + (1 - q^n)[1 + G_4(n, n)] = 1 + (1 - q^n)G_4(n, n), \tag{145}$$

$$\begin{aligned} G_4(m,n) = 1 & + \left(\frac{1 - q^{m'}}{1 - q^m}\right) G_4(m', n) \\ & + q^{m'} \left(\frac{1 - q^{m-m'}}{1 - q^m}\right) G_4(m - m', n - m') \qquad (m > 1), \end{aligned} \tag{146}$$

with the same boundary conditions as in R_1. If we let $F_4(m)$ denote the expected number of tests required to break up a defective set of size m, it can be shown as in the case of R_1 that

$$G_4(m, n) = F_4(m) + \left(\frac{p}{1 - q^m}\right) \sum_{i=1}^{m} q^{i-1} H_4(n - i). \tag{147}$$

If we let

$$G_4{}^*(m,n) = \left(\frac{1 - q^m}{1 - q}\right) G_4(m,n) \quad \text{and} \quad F_4{}^*(m) = \left(\frac{1 - q^m}{1 - q}\right) F_4(m), \tag{148}$$

the recursion formulae for R_4 reduce to

$$F_4{}^*(m) = \sum_{i=1}^{m} q^{i-1} + F_4{}^*(m') + q^{m'} F_4{}^*(m - m') \qquad (m > 1), \tag{149}$$

$$\begin{aligned} H_4(n) &= 1 + pG_4{}^*(m, n) \\ &= 1 + p\{F_4{}^*(n) + \sum_{i=1}^{n} q^{i-1} H_4(n - i)\}, \end{aligned} \tag{150}$$

with boundary conditions $F_4{}^*(1) = H_4(0) = 0$ for all q. For $n \leqq 5$, the results are the same as those for R_1, if we take q close to unity in the formulae for R_1. The results for $H_4(n)$ for $n = 6(1)12$ are, for all q:

$$\begin{aligned} H_4(6) &= 14 - 9q - 2q^2 - q^3 - q^5, \\ H_4(7) &= 17 - 11q - 3q^2 - q^3 + q^4 - 2q^5, \\ H_4(8) &= 21 - 14q - 4q^2 - q^3 + q^4 - 2q^5, \\ H_4(9) &= 25 - 18q - 4q^2 - q^3 + q^4 - 2q^5 + q^7 - q^9, \\ H_4(10) &= 29 - 22q - 4q^2 + q^4 - 3q^5 + q^7 - q^9, \\ H_4(11) &= 33 - 26q - 4q^2 + q^3 - 4q^5 + 2q^6 + q^7 - q^8 - q^9, \\ H_4(12) &= 37 - 29q - 4q^2 - 4q^5 + 2q^6 + q^7 - q^8 - q^9. \end{aligned} \tag{151}$$

The recursion formulae for the halving procedure R_5, which continues to separate sets into smaller and smaller subdivision, are

$$H_5(n) = 1 + (1 - q^n) G_5(n), \tag{152}$$

$$\begin{aligned} G_5(m) = 1 &+ \left(\frac{1 - q^{m'}}{1 - q^m}\right) [G_5(m') + H_4(m - m')] \\ &+ q^{m'} \left(\frac{1 - q^{m-m'}}{1 - q^m}\right) G_5(m - m') \quad (m > 1), \end{aligned} \tag{153}$$

where m' is defined as above. Here it was not necessary to use a double argument with G because there is no recombination allowed. The boundary condition is $G_5(1) = 0$ for all q. If we define $G_4^*(m)$ as in (153) and use (152), the recursion formulae reduce to

$$H_5(n) = 1 + pG_5^*(n), \tag{154}$$

$$G_5^*(m) = \sum_{i=1}^{m'} q^{i-1} + \sum_{i=1}^{m} q^{i-1} + G_5^*(m') + G_5^*(m - m') \qquad (\mathrm{m} > 1), \tag{155}$$

with the boundary condition $G_5^*(1) = 0$ for all q. For $n \leqq 3$, the results are the same as for R_1 if we take q close to unity in the formulae for R_1. The results for $H_5(n)$ for $n = 4(1)12$ are:

$$\begin{aligned}
H_5(4) &= 7 - 2q - 3q^2 - q^4,\\
H_5(5) &= 9 - 3q - 3q^2 - q^3 - q^5,\\
H_5(6) &= 11 - 4q - 2q^2 - 3q^3 - \quad - q^6,\\
H_5(7) &= 13 - 4q - 4q^2 - 2q^3 - q^4 - q^7,\\
H_5(8) &= 15 - 4q - 6q^2 - 3q^4 - q^8,\\
H_5(9) &= 17 - 5q - 6q^2 - q^3 - 2q^4 - q^5 - q^9,\\
H_5(10) &= 19 - 6q - 6q^2 - 2q^3 - 3q^5 - q^{10},\\
H_5(11) &= 21 - 7q - 5q^2 - 4q^3 \quad 2q^5 - q^6 - q^{11},\\
H_5(12) &= 23 - 8q - 4q^2 - 6q^3 - 3q^6 - q^{12}.
\end{aligned} \tag{156}$$

APPENDIX D

Known Procedures

An attempt has been made to put the Dorfman procedure[1] and the Sterrett procedure[3] in the best form that is comparable with the other procedures treated here. For each of $N = 4$, 8 and 12, we have found the division into equal (or approximately equal) subsets such that the Dorfman plan of testing defective sets one at a time gives the smallest possible expected number of tests required. It should not be inferred that these results would be the same if a straightforward application of the tables published by Dorfman and Sterrett, respectively, were made, since their tables are only concerned with very large N. In the Dorfman plan we use a common test-group size for binomial sets and, for defective sets, the units are all tested one at a time. In the Sterrett plan, there is a common test group size for binomial sets at the outset and, for defective sets, the units are tested one at a time only until a defective unit is found. Then the remaining units, from that defective set only, are pooled and tested. This is continued until that particular defective set is completely analyzed before we start with other sets. We have also assumed that logical inference would be used whenever possible in the Dorfman and Sterrett procedures.

For the Dorfman procedure R_7, if the common group size is c, then, for any binomial set of size n (where $n \leqq c$), we obtain

$$\begin{aligned} H_7(n) &= q^n + (1 - q^n)\left(n + 1 - \frac{pq^{n-1}}{1 - q^n}\right) \\ &= n + 1 - q^{n-1} - (n - 1)q^n. \end{aligned} \tag{157}$$

For example, for $N = 12$ and $q = 0.90$, we find that $c_1 = c_2 = c_3 = 4$ gives the best results and, using (157), with $c = 4$, we obtain

$$H_7(12) = 3H_7(4) = 3(5 - q^3 - 3q^4) = 6.908. \tag{158}$$

For the Sterrett procedure R_6, if the group size at the outset is c, then, for each binomial set of size n (where $n \leqq c$), we obtain

$$\begin{aligned} H_6(n) &= q^n + p[2 + H_6(n-1)] + qp[3 + H_6(n-2)] \\ &\qquad + \cdots + q^{n-2}p[n + H_6(1)] + q^{n-1}pn \\ &= q^n + npq^{n-1} + p\sum_{i=2}^{n} iq^{i-2} + p\sum_{i=1}^{n-1} q^{i-1}H_6(n - i), \end{aligned} \tag{159}$$

with boundary condition $H_6(1) = 1$ for all q. It can be verified (we omit the details) that the solution of this system is given by

$$H_6(n) = (2n - 1) - (n - 1)q - \sum_{i=2}^{n} q^i. \tag{160}$$

For example, for $N = 12$ and $q = 0.90$, we find that $c_1 = c_2 = c_3 = 4$ gives the best results and, using (160), with $c = 4$, we obtain

$$H_6(12) = 3H_6(4) = 6.315. \tag{161}$$

APPENDIX E

Cost Considerations

In this Appendix we introduce another procedure R_8, which brings into play the cost of throwing away a good unit and balances it against the cost of conducting another group-test. It is interesting to note that R_8 was the solution given when the problem was first brought to the authors' attention in a practical application.

For procedure R_8 we divide all the N units into approximately equal subsets of size x', where x' is the nearest positive integer to the solution in x of

$$(1 - p)^x = \tfrac{1}{2}, \tag{162}$$

where p is the known *a priori* probability of a unit being defective. It is assumed here that $p \ll \frac{1}{2}$. Each subgroup is tested either simultaneously or in sequence, and good subgroups are removed. Assuming $x' > 1$, subsets shown to contain at least one defective are pooled. Since the same number of defectives has now been put into a pooled set of approximate size $N/2$, it follows that the probability of drawing a defective from the pooled set is approximately double the original *a priori* probability p. Then the pooled subset is again divided into approximately equal subsets of size x'', where x'' is the nearest positive integer to the solution of

$$(1 - 2p)^x = \tfrac{1}{2}. \tag{163}$$

The process is repeated (say) a total of t times. If tp gets larger than $\frac{1}{2}$, x is taken to be unity and the units would then be tested one at a time. However, it may be more economical to stop the procedure before tp reaches $\frac{1}{2}$ and scrap the pooled defective subgroup. The amount of saving may be substantial if the cost of manufacturing a unit, say c_0, is small compared to the cost of each group-test, say c_1.

Suppose, for example, that we start with $N = 8$ units and q is given to be 0.90. The approximate solution of (162) is $x' = 7$ but, since this leaves a subset of size 1, we make our first test on all 8 units. If the test is a success we are through; otherwise we look for a solution of (163) and find that $x'' = 3$. The 8 units are divided into subsets of size 3, 3 and 2, and each is tested. Good subsets are removed. We purposely avoid the next stage which requires testing one at a time. Hence, any of these three sets of size 3, 3, and 2 that proves to contain at least one defective is scrapped.

Let T denote the number of tests required in the above example and let D denote the total number of scrapped units (i.e., good units and defectives that are discarded). Then, for this procedure R_8 with $N = 8$ and $q = 0.90$, we obtain

$$E\{T \mid R_8\} = 4 - 3q^8 = 2.709, \tag{164}$$

$$E\{D \mid R_8\} = 8 - 2q^2 - 6q^3 = 2.006. \tag{165}$$

If we define the expected loss $E\{L \mid R_i\}$ for any procedure R_i by

$$E\{L \mid R_i\} = c_0 E\{D \mid R_i\} + c_1 E\{T \mid R_i\}, \tag{166}$$

we find for procedures R_1 and R_8, respectively,

$$E\{L \mid R_1\} = 0.800\, c_0 + 3.904\, c_1, \tag{167}$$

$$E\{L \mid R_8\} = 2.006\, c_0 + 2.709\, c_1. \tag{168}$$

A comparison of these two expressions shows that R_8 will be more economical in this case if the ratio

$$\frac{c_1}{c_0} \geqq \frac{2.006 - 0.800}{3.904 - 2.709} = 1.009 = 1, \text{ approximately.} \qquad (169)$$

Hence, R_8 is more economical in this case if the total cost of a single test is greater than the total cost of manufacturing a single unit.

Similarly, it can be shown that it would be more economical to stop after the first test (and scrap all 8 units if there is at least one defective present) when the ratio of the two costs in (169) is greater than 1.492 (or approximately 1.5).

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3. Sterrett, A., On the Detection of Defective Members of Large Populations, Ann. Math. Stat., **28,** 1957, p. 1033.
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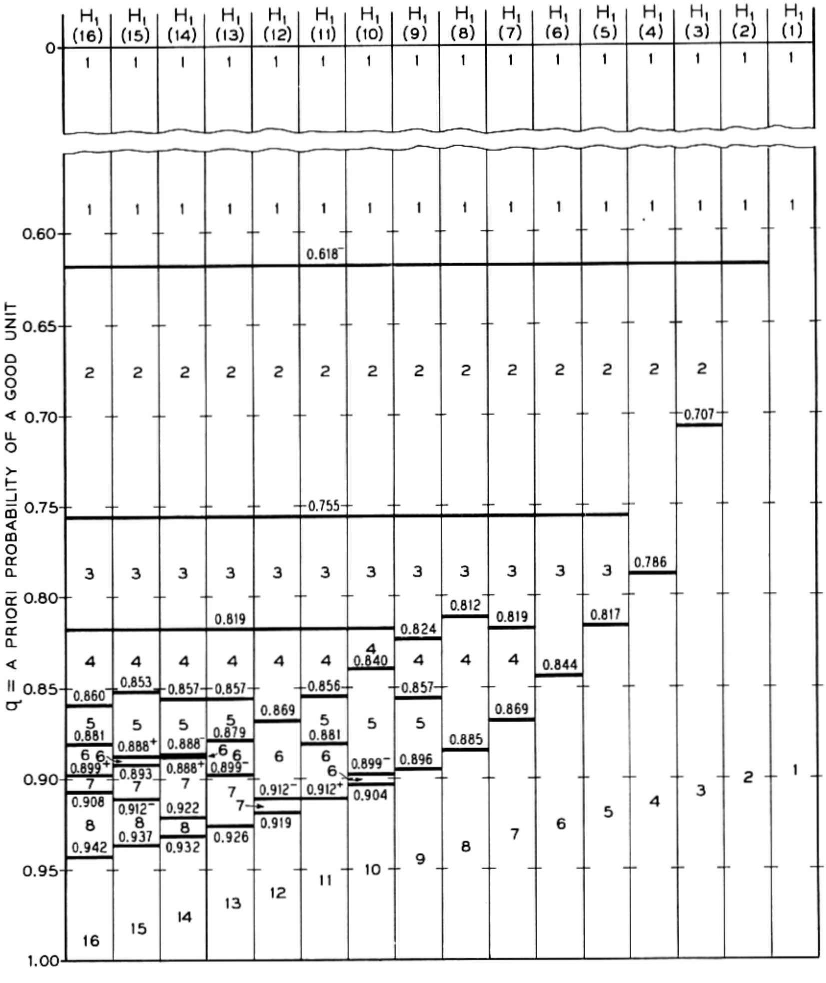

Fig. 3 — Diagram showing number of observations to be taken in any H-situation for $n = 1$ through 16 — for procedure R_1.

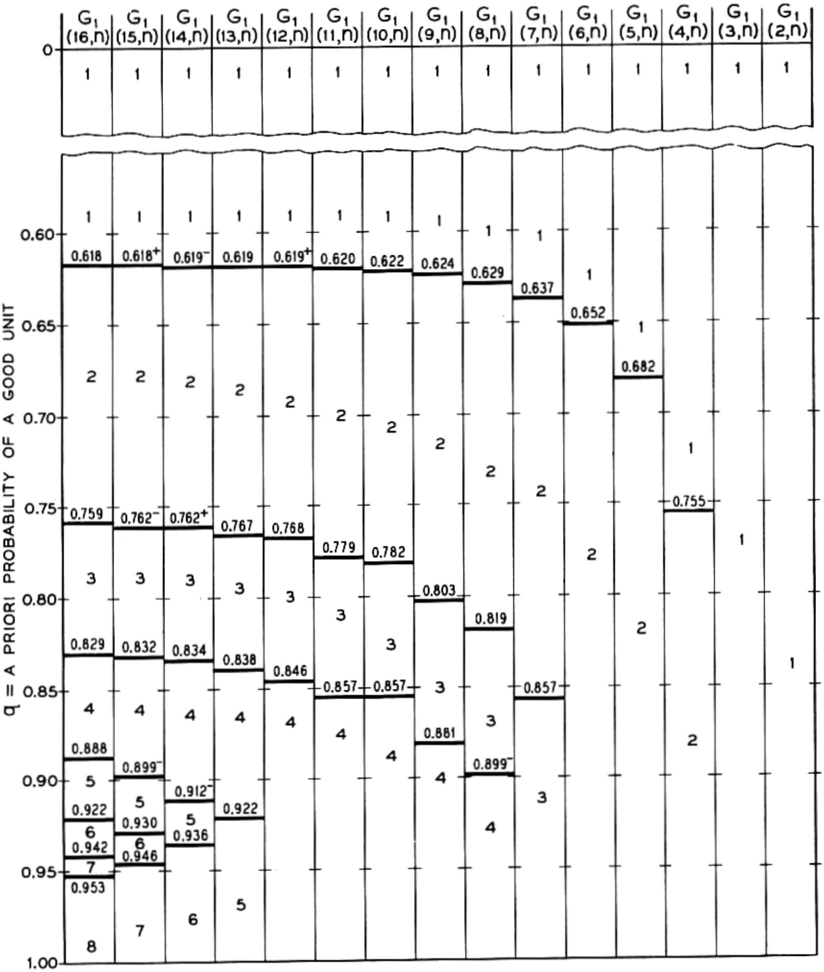

Fig. 4 — Diagram showing number of observations to be taken in any G-situation for $m = 2$ through 16 and any $n \geqq m$ — for procedure R_1 .

TABLE II A — COMPARISON OF THE EXPECTED NUMBER OF GROUP-TESTS REQUIRED FOR DIFFERENT PROCEDURES FOR $N = 4, 8, 12$ AND SELECTED VALUES OF q

Procedure	q							
	0.00	0.50	0.75	0.90	0.95	0.98	0.99	1.00
R_1 (Proposed Procedure)	4.000	4.000	3.332	2.051	1.538	1.218	1.110	1.000
	8.000	8.000	6.619	3.904	2.499	1.612	1.308	1.000
	12.000	12.000	9.905	5.790	3.594	2.073	1.543	1.000
R_2 (Information Procedure)	4.000	4.000	3.332	2.051	1.538	1.218	1.110	1.000
	8.000	8.000	6.663	4.141	2.500	1.612	1.308	1.000
	12.000	12.000	9.956	5.839	3.599	2.078	1.544	1.000
R_3 (Proposed without Recombinations)	4.000	4.000	3.375	2.105	1.576	1.236	1.119	1.000
	8.000	8.000	6.719	4.052	2.631	1.680	1.345	1.000
	12.000	12.000	10.062	6.103	3.851	2.209	1.617	1.000
R_4 (Halving with Recombinations)	4.000	4.000	3.332	2.051	1.538	1.218	1.110	1.000
	21.000	12.875	7.670	3.906	2.500	1.612	1.308	1.000
	37.000	21.408	12.365	6.021	3.620	2.078	1.544	1.000
R_5 (Halving without Recombinations)	7.000	5.188	3.496	2.114	1.578	1.236	1.119	1.000
	15.000	11.309	7.576	4.141	2.678	1.700	1.355	1.000
	23.000	17.203	11.653	6.309	3.900	2.229	1.627	1.000
R_6 (Sterrett)	4.000	4.000	3.375	2.105	1.576	1.236	1.119	1.000
	8.000	8.000	6.750	4.210	3.151	1.807	1.412	1.000
	12.000	12.000	10.125	6.315	4.333	2.973	1.852	1.000
R_7 (Dorfman)	4.000	4.000	3.375	2.303	1.699	1.292	1.148	1.000
	8.000	8.000	6.750	4.605	3.398	2.176	1.609	1.000
	12.000	12.000	10.125	6.908	5.097	3.334	2.354	1.000

TABLE II B — COMPARISON OF $H_1(n)$ AND $H_3(n)$ FOR THREE VALUES OF q AND LARGER n-VALUES

n	$q = 0.90$		$q = 0.95$		$q = 0.99$	
	$H_1(n)$	$H_3(n)$	$H_1(n)$	$H_3(n)$	$H_1(n)$	$H_3(n)$
10	4.872	5.101	3.039	3.242	1.425	1.481
20	9.572	10.155	5.940	6.456	2.051	2.221
30	14.301	15.209	8.791	9.626	2.738	3.057
40	19.024	20.260	11.671	12.798	3.478	3.943
50	23.750	25.361	14.555	16.009	4.243	4.853
60	28.475	30.415	17.438	19.246	5.026	5.792
70	33.200	35.469	20.316	22.415	5.830	6.754
80	37.925	40.520	23.197	25.591	6.647	7.717
90	42.650	45.621	26.078	28.781	7.477	8.683
100	47.375	50.675	28.959	32.019	8.230	9.687

TABLE III A — TO BE USED WITH PROCEDURE R_1 IN ANY $H(n)$ SITUATION WITH $n \leqq 16$ OR n VERY LARGE

Each polynomial shown with its unique real root in the unit interval $0 < q < 1$; these roots determining the region where the next test group is of size x.

The exponential symbols $+$, $-$ indicate only the relative magnitude of two different roots that are equal to three decimal places (i.e., $a^- < a^+$).

n	The unique real root in the unit interval of the polynomial shown is the dividing point between:															
	$x = 1$ and $x = 2$		$x = 2$ and $x = 3$		$x = 3$ and $x = 4$		$x = 4$ and $x = 5$		$x = 5$ and $x = 6$		$x = 6$ and $x = 7$		$x = 7$ and $x = 8$		(maximum possible $x < n$) and $x = n$	
2	$1-q-q^2$	0.618													(see first column)	
3	$1-q-q^2$	0.618	(see last col.)												$1-2q^2$	0.707
4	$1-q-q^2$	0.618													$1-q^2-q^4$	0.786
5	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755											$1-q+q^3-2q^5$	0.817
6	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755											$1-q+q^2-q^4-q^6$	0.844
7	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819									$1-q^4-q^6$	0.869
8	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q-q^8$	0.812									$1-q^3+q^5-2q^7$	0.885
9	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q-q^9$	0.824	$1-q^4-q^5$	0.857							$1-q^3+q^7-2q^9$	0.896
10	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819	$1-q^2-q^7$	0.840	$1-q^6-q^7$	0.899^-					$1-q^2+q^6-2q^{10}$	0.904
11	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819	$1-q+q^4-q^6-q^8$	0.856	$1-q^5-q^6$	0.881					$1-q^2+q^5-q^9-q^{11}$	0.912^+
12	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819	$1-q^4-q^6$ ($x = 4$ and $x = 6$)*			0.869	$1-q^7-q^8$	0.912^-			$1-q+q^4-q^{10}-q^{12}$	0.919
13	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819	$1-q^4-q^5$	0.857	$1-q+q^5-q^7-q^{11}$	0.879	$1-q^6-q^7$	0.899^-			$1-q+q^3-q^9-q^{13}$	0.926
14	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819	$1-q^4-q^5$	0.857	$1-q+q^5-q^6+q^{10}-2q^{12}$	0.888^-	$1-q+q^2-q^3+q^4-q^5-q^{11}$	0.888^+	$1-q^8-q^9$	0.922	$1-q^8-q^{12}$	0.932
15	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819	$1-q-q^{12}$	0.853	$1-q+q^2-q^3+q^4-q^5-q^{11}$	0.888^+	$1-q+q^7-q^8+q^{12}-2q^{14}$	0.893	$1-q^7-q^8$	0.912^-	$1-q^8-q^{14}$	0.937
16	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819	$1-q-q^{13}$	0.860	$1-q^5-q^6$	0.881	$1-q+q^6-q^7+q^{13}-2q^{15}$	0.899^+	$1-q^7-q^{14}-q^{15}$	0.908	$1-q^8-q^{16}$	0.942
∞	$1-q-q^2$	0.618	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819	$1-q^4-q^5$	0.857	$1-q^5-q^6$	0.881	$1-q^6-q^7$	0.899^-	$1-q^7-q^8$	0.912		1.000

---

* The interval for $x = 5$ vanishes for $n = 12$.

TABLE III B — TO BE USED WITH PROCEDURE R_1 IN ANY $G(m, n)$ SITUATION WITH $2 \leqq m \leqq n \leqq 16$ OR m VERY LARGE

Each polynomial shown with its unique real root in the unit interval $0 < q < 1$; these roots determining the region where the next test group (taken from the defective set) is of size x.

The exponential symbols $+$, $-$ indicate only the relative magnitude of two or three different roots that are equal to three decimal places (i.e., $a^- < a^+$ or $a^- < a < a^+$).

m	The unique real root in the unit interval of the polynomial shown is the dividing point between:													
	$x = 1$ and $x = 2$		$x = 2$ and $x = 3$		$x = 3$ and $x = 4$		$x = 4$ and $x = 5$		$x = 5$ and $x = 6$		$x = 6$ and $x = 7$		$x = 7$ and $x = 8$	
2														
3														
4	$1-q^2-q^3$	0.755												
5	$1-q^2-q^3-q^4$	0.682												
6	$1-q^2-q^3-q^4-q^5$	0.652												
7	$1-q^2-q^3-q^4-q^5-q^6$	0.637	$1-q^4-q^5$	0.857										
8	$1-q^2-q^3- \cdots -q^7$	0.629	$1-q^3-q^4$	0.819	$1-q^6-q^7$	0.899^-								
9	$1-q^2-q^3- \cdots -q^8$	0.624	$1-q^3-q^6-q^7$	0.803	$1-q^5-q^6$	0.881								
10	$1-q^2-q^3- \cdots -q^9$	0.622	$1-q^3-q^5-q^6$	0.782	$1-q^4-q^5$	0.857								
11	$1-q^2-q^3- \cdots -q^{10}$	0.620	$1-q^3-q^5-q^8-q^9$	0.779	$1-q^4-q^9-q^{10}$	0.857								
12	$1-q^2-q^3- \cdots -q^{11}$	0.619^+	$1-q^3-q^5-q^7-q^8$	0.768	$1-q^4-q^8-q^9$	0.846								
13	$1-q^2-q^3- \cdots -q^{12}$	0.619	$1-q^3-q^5-q^7-q^{10}-q^{11}$	0.767	$1-q^4-q^8-q^{11}-q^{12}$	0.838	$1-q^8-q^9$	0.922						
14	$1-q^2-q^3- \cdots -q^{13}$	0.619^-	$1-q^3-q^5-q^7-q^9-q^{10}$	0.762^+	$1-q^4-q^7-q^8$	0.834	$1-q^7-q^8$	0.912^-	$1-q^{10}-q^{11}$	0.936				
15	$1-q^2-q^3- \cdots -q^{14}$	0.618^+	$1-q^3-q^5-q^7-q^9-q^{12}-q^{13}$	0.762^-	$1-q^4-q^7-q^{11}-q^{12}$	0.832	$1-q^6-q^7$	0.899^-	$1-q^9-q^{10}$	0.930	$1-q^{12}-q^{13}$	0.946		
16	$1-q^2-q^3- \cdots -q^{15}$	0.618	$1-q^3-q^5-q^7-q^9-q^{11}-q^{12}$	0.759	$1-q^4-q^7-q^{11}-q^{14}-q^{15}$	0.829	$1-q^6-q^{11}-q^{12}$	0.888^+	$1-q^8-q^9$	0.922	$1-q^{11}-q^{12}$	0.941	$1-q^{14}-q^{15}$	0.953
∞	$1-q-q^2$	0.618^-	$1-q^2-q^3$	0.755	$1-q^3-q^4$	0.819	$1-q^4-q^5$	0.857	$1-q^5-q^6$	0.881	$1-q^6-q^7$	0.899	$1-q^7-q^8$	0.912^-

TABLE IV A — FORMULAE FOR $F_1^*(m)$ FOR PROCEDURE R_1 AND VALUES OF THE NEXT TEST-GROUP SIZE x FOR $m = 2(1)16$

	q-interval	x	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}	q^{13}	q^{14}	q^{15}	
$F_1^*(2)$	0.000 to 1.000	1	1	1															
$F_1^*(3)$	0.000 to 1.000	1	1	2	2														
$F_1^*(4)$	0.000 to 0.755	1	1	2	3	3													†
	0.755 to 1.000	2	2	2	2	2													
$F_1^*(5)$	0.000 to 0.682	1	1	2	3	4	4												†
	0.682 to 1.000	2	2	2	2	3	3												
$F_1^*(6)$	0.000 to 0.652	1	1	2	3	4	5	5											†
	0.652 to 0.755	2	2	2	2	3	4	4											†
	0.755 to 1.000	2	2	2	3	3	3	3											
$F_1^*(7)$	0.000 to 0.637	1	1	2	3	4	5	6	6										†
	0.637 to 0.682	2	2	2	2	3	4	5	5										†
	0.682 to 0.857	2	2	2	3	3	3	4	4										†
	0.857 to 1.000	3	2	3	3	3	3	3	3										
$F_1^*(8)$	0.000 to 0.629	1	1	2	3	4	5	6	7	7									†
	0.629 to 0.652	2	2	2	2	3	4	5	6	6									†
	0.652 to 0.755	2	2	2	3	3	3	4	5	5									†
	0.755 to 0.819	2	2	2	3	3	4	4	4	4									†
	0.819 to 0.899^-	3	2	3	3	3	3	3	4	4									
	0.899^- to 1.000	4	3	3	3	3	3	3	3	3									
$F_1^*(9)$	0.000 to 0.624	1	1	2	3	4	5	6	7	8	8								†
	0.624 to 0.637	2	2	2	2	3	4	5	6	7	7								†
	0.637 to 0.682	2	2	2	3	3	3	4	5	6	6								†
	0.682 to 0.803	2	2	2	3	3	4	4	4	5	5								†
	0.803 to 0.881	3	2	3	3	3	3	4	4	4	4								†
	0.881 to 1.000	4	3	3	3	3	3	3	3	4	4								
$F_1^*(10)$	0.000 to 0.622	1	1	2	3	4	5	6	7	8	9	9							†
	0.622 to 0.629	2	2	2	2	3	4	5	6	7	8	8							†
	0.629 to 0.652	2	2	2	3	3	3	4	5	6	7	7							†
	0.652 to 0.755	2	2	2	3	3	4	4	4	5	6	6							†
	0.755 to 0.782	2	2	2	3	3	4	4	5	5	5	5							†
	0.782 to 0.857	3	2	3	3	3	3	4	4	4	5	5							†
	0.857 to 1.000	4	3	3	3	3	3	3	4	4	4	4							
$F_1^*(11)$	0.000 to 0.620	1	1	2	3	4	5	6	7	8	9	10	10						†
	0.620 to 0.624	2	2	2	2	3	4	5	6	7	8	9	9						†
	0.624 to 0.637	2	2	2	3	3	3	4	5	6	7	8	8						†
	0.637 to 0.682	2	2	2	3	3	4	4	4	5	6	7	7						†
	0.682 to 0.779	2	2	2	3	3	4	4	5	5	5	6	6						†
	0.779 to 0.819	3	2	3	3	3	3	4	4	5	5	5	5						†
	0.819 to 0.857	3	2	3	3	3	4	4	4	4	4	5	5						†
	0.857 to 1.000	4	3	3	3	3	3	4	4	4	4	4	4						
$F_1^*(12)$	0.000 to 0.619	1	1	2	3	4	5	6	7	8	9	10	11	11					†
	0.619 to 0.622	2	2	2	2	3	4	5	6	7	8	9	10	10					†
	0.622 to 0.629	2	2	2	3	3	3	4	5	6	7	8	9	9					†
	0.629 to 0.652	2	2	2	3	3	4	4	4	5	6	7	8	8					†
	0.652 to 0.755	2	2	2	3	3	4	4	5	5	5	6	7	7					†
	0.755 to 0.768	2	2	2	3	3	4	4	5	5	6	6	6	6					†
	0.768 to 0.803	3	2	3	3	3	3	4	4	5	5	5	6	6					†
	0.803 to 0.846^+	3	2	3	3	3	4	4	4	4	5	5	5	5					†
	0.846^+ to 0.899^-	4	3	3	3	3	3	4	4	4	4	4	5	5					†
	0.899^- to 1.000	4	3	3	3	3	4	4	4	4	4	4	4	4					

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† These equations are not needed to compute $H_1(n)$-formulae if the experiment *starts* with a "pure binomial" set of any size N (i.e., if $m = 0$ at the outset).

TABLE IV A — *Continued*

	q-interval	x	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}	q^{13}	q^{14}	q^{15}	
$F_1^*(13)$	0.000 to 0.619	1	1	2	3	4	5	6	7	8	9	10	11	12	12				†
	0.619 to 0.620	2	2	2	2	3	4	5	6	7	8	9	10	11	11				†
	0.620 to 0.624	2	2	2	3	3	3	4	5	6	7	8	9	10	10				†
	0.624 to 0.637	2	2	2	3	3	4	4	4	5	6	7	8	9	9				†
	0.637 to 0.682	2	2	2	3	3	4	4	5	5	5	6	7	8	8				†
	0.682 to 0.767	2	2	2	3	3	4	4	5	5	6	6	6	7	7				†
	0.767 to 0.782	3	2	3	3	3	3	4	4	5	5	6	6	6	6				†
	0.782 to 0.834	3	2	3	3	3	4	4	4	4	5	5	5	6	6				†
	0.834 to 0.881	4	3	3	3	3	3	4	4	4	4	5	5	5	5				†
	0.881 to 0.922	4	3	3	3	3	4	4	4	4	4	4	4	5	5				†
	0.922 to 1.000	5	3	3	3	4	4	4	4	4	4	4	4	4	4				
$F_1^*(14)$	0.000 to 0.619^-	1	1	2	3	4	5	6	7	8	9	10	11	12	13	13			†
	0.619^- to 0.619^+	2	2	2	2	3	4	5	6	7	8	9	10	11	12	12			†
	0.619^+ to 0.622	2	2	2	3	3	3	4	5	6	7	8	9	10	11	11			†
	0.622 to 0.629	2	2	2	3	3	4	4	4	5	6	7	8	9	10	10			†
	0.629 to 0.652	2	2	2	3	3	4	4	5	5	5	6	7	8	9	9			†
	0.652 to 0.755	2	2	2	3	3	4	4	5	5	6	6	6	7	8	8			†
	0.755 to 0.762^+	2	2	2	3	3	4	4	5	5	6	6	7	7	7	7			†
	0.762^+ to 0.779	3	2	3	3	3	3	4	4	5	5	6	6	6	7	7			†
	0.779 to 0.819	3	2	3	3	3	4	4	4	4	5	5	6	6	6	6			†
	0.819 to 0.834	3	2	3	3	3	4	4	4	5	5	5	5	5	6	6			†
	0.834 to 0.857	4	3	3	3	3	3	4	4	4	4	5	5	5	6	6			†
	0.857 to 0.912^-	4	3	3	3	3	4	4	4	4	4	4	5	5	5	5			†
	0.912^- to 0.936	5	3	3	3	4	4	4	4	4	4	4	4	4	5	5			
	0.936 to 1.000	6	3	3	4	4	4	4	4	4	4	4	4	4	4	4			
$F_1^*(15)$	0.000 to 0.618^+	1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	14		†
	0.618^+ to 0.619	2	2	2	2	3	4	5	6	7	8	9	10	11	12	13	13		†
	0.619 to 0.620	2	2	2	3	3	3	4	5	6	7	8	9	10	11	12	12		†
	0.620 to 0.624	2	2	2	3	3	4	4	4	5	6	7	8	9	10	11	11		†
	0.624 to 0.637	2	2	2	3	3	4	4	5	5	5	6	7	8	9	10	10		†
	0.637 to 0.682	2	2	2	3	3	4	4	5	5	6	6	6	7	8	9	9		†
	0.682 to 0.762	2	2	2	3	3	4	4	5	5	6	6	7	7	7	8	8		†
	0.762 to 0.768	3	2	3	3	3	3	4	4	5	5	6	6	7	7	7	7		†
	0.768 to 0.803	3	2	3	3	3	4	4	4	4	5	5	6	6	6	7	7		†
	0.803 to 0.832	3	2	3	3	3	4	4	4	5	5	5	5	6	6	6	6		†
	0.832 to 0.857	4	3	3	3	3	3	4	4	4	5	5	5	5	5	6	6		†
	0.857 to 0.899^-	4	3	3	3	3	4	4	4	4	4	5	5	5	5	5	5		†
	0.899^- to 0.930	5	3	3	3	4	4	4	4	4	4	4	4	5	5	5	5		†
	0.930 to 0.946	6	3	3	4	4	4	4	4	4	4	4	4	4	4	5	5		
	0.946 to 1.000	7	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4		
$F_1^*(16)$	0.000 to 0.618^-	1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	15	†
	0.618^- to 0.619^-	2	2	2	2	3	4	5	6	7	8	9	10	11	12	13	14	14	†
	0.619^- to 0.619^+	2	2	2	3	3	3	4	5	6	7	8	9	10	11	12	13	13	†
	0.619^+ to 0.622	2	2	2	3	3	4	4	4	5	6	7	8	9	10	11	12	12	†
	0.622 to 0.629	2	2	2	3	3	4	4	5	5	5	6	7	8	9	10	11	11	†
	0.629 to 0.652	2	2	2	3	3	4	4	5	5	6	6	6	7	8	9	10	10	†
	0.652 to 0.755	2	2	2	3	3	4	4	5	5	6	6	7	7	7	8	9	9	†
	0.755 to 0.759	2	2	2	3	3	4	4	5	5	6	6	7	7	8	8	8	8	†
	0.759 to 0.767	3	2	3	3	3	3	4	4	5	5	6	6	7	7	7	8	8	†
	0.767 to 0.782	3	2	3	3	3	4	4	4	4	5	5	6	6	7	7	7	7	†
	0.782 to 0.829	3	2	3	3	3	4	4	4	5	5	5	5	6	6	6	7	7	†
	0.829 to 0.846^+	4	3	3	3	3	3	4	4	4	5	5	5	5	6	6	6	6	†
	0.846^+ to 0.888^+	4	3	3	3	3	4	4	4	4	4	5	5	5	5	5	6	6	†
	0.888^+ to 0.922	5	3	3	3	4	4	4	4	4	4	4	5	5	5	5	5	5	†
	0.922 to 0.941	6	3	3	4	4	4	4	4	4	4	4	4	4	5	5	5	5	†
	0.941 to 0.953	7	3	4	4	4	4	4	4	4	4	4	4	4	4	4	5	5	
	0.953 to 1.000	8	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	

The exponential symbols +, − indicate only the relative magnitude of two different roots that are equal to three decimal places (i.e., $a^- < a^+$).

TABLE IV B — FORMULAE FOR $H_1(n)$ AND VALUES OF THE NEXT SAMPLE SIZE x FOR ANY H-SITUATION EXPRESSED AS FUNCTIONS OF q FOR $n = 2(1)12$ WHEN THE PROCEDURE R_1 IS USED.

The integer shown below q^y opposite $H_1(n)$ is the coefficient of q^y in the polynomial formula for $H_1(n)$.

	q-interval	x	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}
$H_1(2)$	0.000 to 0.618	1	2												
	0.618 to 1.000	2	3	−1	−1										
$H_1(3)$	0.000 to 0.618	1	3												
	0.618 to 0.707	2	5	−3	−1	1									
	0.707 to 1.000	3	5	−2	−1	−1									
$H_1(4)$	0.000 to 0.618	1	4												
	0.618 to 0.707	2	7	−5	0	1	−1								
	0.707 to 0.786	2	7	−4	−1	−1	1								
	0.786 to 1.000	4	8	−4	−2	−1									
$H_1(5)$	0.000 to 0.618	1	5												
	0.618 to 0.707	2	9	−7	1	0	−1	1							
	0.707 to 0.755	2	9	−6	0	−1	1	−1							
	0.755 to 0.786	3	9	−5	−1	−2	1								
	0.786 to 0.817	3	10	−6	−2	−1	0	1							
	0.817 to 1.000	5	11	−7	−2	0	0	−1							
$H_1(6)$	0.000 to 0.618	1	6												
	0.618 to 0.707	2	11	−9	2	−1	0	1	−1						
	0.707 to 0.755	2	11	−8	1	−2	1	−1	1						
	0.755 to 0.786	3	11	−6	−2	−2	2	0	−1						
	0.786 to 0.817	3	12	−7	−3	−1	1	1	−1						
	0.817 to 0.844	3	13	−9	−2	0	0	−1	1						
	0.844 to 1.000	6	14	−10	−1	0	−1	−1							
$H_1(7)$	0.000 to 0.618	1	7												
	0.618 to 0.707	2	13	−11	3	−2	1	0	−1	1					
	0.707 to 0.755	2	13	−10	2	−3	2	−1	1	−1					
	0.755 to 0.786	3	13	−7	−3	−2	2	0	−1	1					
	0.786 to 0.817	3	14	−8	−4	0	1	0	−1						
	0.817 to 0.819	3	15	−10	−3	1	0	−2	1						
	0.819 to 0.844	4	16	−11	−3	0	0	−1	1						
	0.844 to 0.869	4	17	−13	−1	−1	−1	0	0	1					
	0.869 to 1.000	7	17	−12	−1	−1	−1	−1							
$H_1(8)$	0.000 to 0.618	1	8												
	0.618 to 0.707	2	15	−13	4	−3	2	−1	0	1	−1				
	0.707 to 0.755	2	15	−12	3	−4	3	−2	1	−1	1				
	0.755 to 0.786	3	15	−8	−4	−2	3	0	−2	1					
	0.786 to 0.812	3	16	−9	−5	0	1	0	−1	0	1				
	0.812 to 0.817	4	17	−10	−5	0	1	0	−1						
	0.817 to 0.819	4	18	−12	−4	1	0	−2	1						
	0.819 to 0.844	4	19	−14	−3	0	1	−1							
	0.844 to 0.869	4	20	−16	−1	−1	0	0	−1	1					
	0.869 to 0.885	4	20	−15	−2	−1	0	−1	0	0	1				
	0.885 to 0.899^-	8	20	−14	−2	−1	−1	−1	1	0	−1				
	0.899^- to 1.000	8	21	−15	−2	−1	−1	−1							

TABLE IV B — *Continued*

	q-interval	x	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}
$H_1(9)$	0.000 to 0.618	1	9												
	0.618 to 0.707	2	17	−15	5	−4	3	−2	1	0	−1	1			
	0.707 to 0.755	2	17	−14	4	−5	4	−3	2	−1	1	−1			
	0.755 to 0.786	3	17	−9	−5	−2	4	−1	−2	2	0	−1			
	0.786 to 0.812	3	18	−10	−6	0	2	−1	−1	1	1	−1			
	0.812 to 0.817	3	19	−12	−5	0	2	−1	−1	1					
	0.817 to 0.819	3	20	−14	−4	1	1	−3	1	1					
	0.819 to 0.824	3	21	−16	−3	0	2	−2	0	1					
	0.824 to 0.844	4	22	−17	−3	0	2	−2	0	1	0	−1			
	0.844 to 0.857	4	23	−19	−1	−1	1	−1	−1	2	0	−1			
	0.857 to 0.869	5	23	−19	−1	0	0	−1	−1	1					
	0.869 to 0.885	5	23	−18	−2	0	0	−2	0	0	1				
	0.885 to 0.896	5	23	−17	−3	0	−1	−1	1	−1	−1	2			
	0.896 to 0.899^-	9	24	−17	−3	−1	−1	−1	1	0	−1				
	0.899^- to 1.000	9	25	−19	−2	−1	−1	−1	0	1	0	−1			
$H_1(10)$	0.000 to 0.618	1	10												
	0.618 to 0.707	2	19	−17	6	−5	4	−3	2	−1	0	1	−1		
	0.707 to 0.755	2	19	−16	5	−6	5	−4	3	−2	1	−1	1		
	0.755 to 0.786	3	19	−10	−6	−2	5	−2	−2	2	0	−1	1		
	0.786 to 0.812	3	20	−11	−7	0	3	−2	0	1	0	−1			
	0.812 to 0.817	3	21	−13	−6	0	3	−2	0	1	−1				
	0.817 to 0.819	3	22	−15	−5	1	2	−4	2	1	−1				
	0.819 to 0.824	4	24	−18	−4	0	2	−2	0	1					
	0.824 to 0.840	4	25	−20	−3	0	2	−2	0	1	0	−1	1		
	0.840 to 0.844	5	25	−20	−3	1	2	−3	0	1	0	−1			
	0.844 to 0.857	5	26	−22	−1	0	1	−2	−1	2	0	−1			
	0.857 to 0.869	5	26	−22	−1	1	−1	−1	−1	1	1	0	−1		
	0.869 to 0.885	5	26	−21	−2	1	−1	−2	0	0	2	0	−1		
	0.885 to 0.896	5	26	−20	−3	1	−2	−1	1	−1	0	2	−1		
	0.896 to 0.899^-	5	27	−21	−3	0	−1	−1	1	0	−1	0	1		
	0.899^- to 0.904	6	28	−23	−1	−1	−1	−1	0	1	−1	−1	2		
	0.904 to 1.000	10	29	−23	−2	−1	−1	−1	1	1	−1	−1			
$H_1(11)$	0.000 to 0.618	1	11												
	0.618 to 0.707	2	21	−19	7	−6	5	−4	3	−2	1	0	−1	1	
	0.707 to 0.755	2	21	−18	6	−7	6	−5	4	−3	2	−1	1	−1	
	0.755 to 0.786	3	21	−11	−7	−2	6	−3	−2	3	0	−2	1		
	0.786 to 0.812	3	22	−12	−8	0	4	−3	0	1	0	−1	0	1	
	0.812 to 0.817	3	23	−14	−7	1	3	−3	0	1	−1				
	0.817 to 0.819	3	24	−16	−6	2	2	−5	2	1	−1				
	0.819 to 0.824	4	27	−21	−4	0	3	−2	−1	1					
	0.824 to 0.840	4	28	−23	−3	0	3	−2	−1	1	0	−1	1		
	0.840 to 0.844	4	28	−23	−3	1	2	−3	0	1	0	−1	0	1	
	0.844 to 0.856	4	29	−25	−1	0	1	−2	−1	2	0	−1	0	1	
	0.856 to 0.857	5	29	−25	−1	1	0	−2	−1	3	0	−2			
	0.857 to 0.869	5	29	−25	−1	2	−2	−1	−1	2	1	−1	−1		
	0.869 to 0.881	5	29	−24	−2	2	−2	−2	0	1	2	−1	−1		
	0.881 to 0.885	6	29	−24	−1	1	−2	−2	0	0	2	0	−1		
	0.885 to 0.896	6	29	−23	−2	1	−3	−1	1	−1	0	2	−1		
	0.896 to 0.899^-	6	30	−24	−2	0	−2	−1	1	0	−1	0	1		
	0.899^- to 0.904	6	31	−26	0	−2	−1	−1	0	1	−1	0	2	−1	
	0.904 to 0.912	6	32	−27	−1	−1	−1	−1	1	0	−1	0	0	1	
	0.912 to 1.000	11	33	−27	−2	−1	−1	0	1	0	−1	−1			

TABLE IV B — *Continued*

	q-interval	x	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}
$H_1(12)$	0.000 to 0.618	1	12												
	0.618 to 0.707	2	23	−21	8	−7	6	−5	4	−3	2	−1	0	1	−1
	0.707 to 0.755	2	23	−20	7	−8	7	−6	5	−4	3	−2	1	−1	1
	0.755 to 0.786	3	23	−12	−8	−2	7	−4	−2	4	−1	−2	2	0	−1
	0.786 to 0.812	3	24	−13	−9	0	5	−4	0	2	−1	−1	1	1	−1
	0.812 to 0.817	3	25	−15	−8	1	3	−3	0	2	−2	0	1		
	0.817 to 0.819	3	26	−17	−7	2	2	−5	2	2	−2	0	1		
	0.819 to 0.824	4	30	−24	−4	0	4	−3	−1	1	1	0	−1		
	0.824 to 0.840	4	31	−26	−3	0	4	−3	−1	1	1	−1			
	0.840 to 0.844	4	31	−26	−3	1	3	−4	0	1	1	−1	−1	1	
	0.844 to 0.856	4	32	−28	−1	0	2	−3	−1	2	1	−1	−1	1	
	0.856 to 0.857	4	32	−28	−1	1	0	−2	−1	3	0	−2	0	0	1
	0.857 to 0.869	4	32	−28	−1	2	−2	−1	−1	2	1	−1	−1	0	1
	0.869 to 0.881	6	32	−27	−1	2	−3	−2	0	1	2	−1	−1		
	0.881 to 0.885	6	32	−27	0	0	−2	−2	0	0	3	0	−2		
	0.885 to 0.896	6	32	−26	−1	0	−3	−1	1	−1	1	2	−2		
	0.896 to 0.899⁻	6	33	−27	−1	−1	−2	−1	1						
	0.899⁻ to 0.904	6	34	−29	1	−3	−1	−1	0	1	0	0	1	−1	
	0.904 to 0.912⁻	6	35	−30	0	−2	−1	−1	1	0	0	0	−1	1	
	0.912⁻ to 0.912⁺	7	35	−29	−1	−2	−1	−1	1	0	−1	0	0	1	
	0.912⁺ to 0.919	7	36	−30	−2	−1	−1	0	0	0	−1	−1	1	0	1
	0.919 to 1.000	12	37	−31	−2	−1	0	0	0	0	−1	−1			

The exponential symbols +, − indicate only the relative magnitude of two different roots that are equal to three decimal places (i.e., $a^- < a^+$).

TABLE IV C — FORMULAE FOR $G_1^*(m, n)$ AND THE VALUES OF THE NEXT SAMPLE SIZE x FOR CERTAIN G-SITUATIONS EXPRESSED AS FUNCTIONS OF q FOR $n = 2(1)12$ AND $q \geqq 0.850$ WHEN PROCEDURE R_1 IS USED

The integer shown below q^y opposite $G_1^*(m,n)$ is the coefficient of q^y in the polynomial formula for $G_1^*(m,n)$.

n	m	q-interval	x	$G_1^*(m,n)$											
				1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}
2	2	0.850 to 1.000	1	2	1										
3	2	0.850 to 1.000	1	4	1	−1									
	3	0.850 to 1.000	1	4	2	1									
4	2	0.850 to 1.000	1	6	2	−2	−2								
	4	0.850 to 1.000	2	7	3	1									
5	2	0.850 to 1.000	1	9	2	−4	−2	−1							
	3	0.850 to 1.000	1	9	3	1	−3	−2							
	5	0.850 to 1.000	2	10	3	1	1	1							
6	2	0.850 to 1.000	1	12	2	−6	−2	−1	−1						
	3	0.850 to 1.000	1	12	3	1	−4	−2	−2						
	6	0.850 to 1.000	2	13	3	2	2	1							
7	2	0.850 to 1.000	1	15	2	−8	−2	−1	−1	−1					
	3	0.850 to 1.000	1	15	3	2	−6	−3	−2	−1					
	4	0.850 to 1.000	2	16	3	2	1	−5	−3	−2					
	7	0.850 to 0.857	2	16	3	3	2	1	1	1					
		0.857 to 1.000	3	16	4	3	2	1							
8	2	0.850 to 0.869	1	18	2	−11	−2	−1	−1	−1	1				
		0.869 to 1.000	1	18	3	−11	−2	−1	−2	−1					
	3	0.850 to 0.869	1	18	3	2	−9	−3	−1	−1					
		0.869 to 1.000	1	18	4	2	−9	−3	−2	−1	−1				
	4	0.850 to 0.869	2	19	3	2	1	−7	−3	−2					
		0.869 to 1.000	2	19	4	2	1	−7	−4	−2	−1				
	8	0.850 to 0.869	3	19	4	3	2	1	1	1	2				
		0.869 to 0.899^-	3	19	5	3	2	1	0	1	1				
		0.899^- to 1.000	4	20	5	3	2	1							
9	2	0.850 to 0.869	1	21	2	−14	−2	−1	−1	−1	1	1			
		0.869 to 0.885	1	21	3	−14	−2	−1	−2	−1	0	1			
		0.885 to 0.899^-	1	21	4	−14	−2	−2	−2	0	0	−1			
		0.899^- to 1.000	1	22	3	−14	−2	−2	−2	−1					
	3	0.850 to 0.869	1	21	3	2	−12	−2	−1	−2	0	1			
		0.869 to 0.885	1	21	4	2	−12	−2	−2	−2	−1	1			
		0.885 to 0.899^-	1	21	5	2	−12	−3	−2	−1	−1	−1			
		0.899^- to 1.000	1	22	4	2	−12	−3	−2	−2	−1				
	4	0.850 to 0.869	2	22	3	2	1	−9	−3	−2					
		0.869 to 0.885	2	22	4	2	1	−9	−4	−2	−1				
		0.885 to 0.899^-	2	22	5	2	1	−10	−4	−1	−1	−2			
		0.899^- to 1.000	2	23	4	2	1	−10	−4	−2	−1	−1			

TABLE IV C — *Continued*

n	m	q-*interval*	x	$G_1^*(m,n)$											
				1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}
9	5	0.850 to 0.869	2	22	3	2	2	2	−7	−4	−1				
		0.869 to 0.885	2	22	4	2	2	2	−8	−4	−2				
		0.885 to 0.899^-	2	22	5	2	2	1	−8	−3	−2	−2			
		0.899^- to 1.000	2	23	4	2	2	1	−8	−4	−2	−1			
	9	0.850 to 0.869	3	22	4	3	2	2	2	1	2	2			
		0.869 to 0.881	3	22	5	3	2	2	1	1	1	2			
		0.881 to 0.885	4	23	5	3	2	2	0	0	1	2			
		0.885 to 0.899^-	4	23	6	3	2	1	0	1	1				
		0.899^- to 1.000	4	24	5	3	2	1	0	0	1	1			
10	2	0.850 to 0.857	1	24	2	−17	−2	0	−1	−1	1	1	−1		
		0.857 to 0.869	1	24	2	−17	−1	−1	−1	−1	0	1			
		0.869 to 0.885	1	24	3	−17	−2	−1	−2	−1	0	1	1		
		0.885 to 0.896	1	24	4	−17	−2	−2	−2	0	0	−1	1		
		0.896 to 0.899^-	1	25	4	−17	−3	−2	−2	0	1	−1	−1		
		0.899^- to 1.000	1	26	3	−17	−3	−2	−2	−1	1	0	−1		
	3	0.850 to 0.857	1	24	3	2	−15	−1	−2	−2	1	1			
		0.857 to 0.869	1	24	3	2	−14	−2	−2	−2	0	1	1		
		0.869 to 0.885	1	24	4	2	−14	−2	−3	−2	−1	1	1		
		0.885 to 0.896	1	24	5	2	−14	−3	−3	−1	−1	−1	1		
		0.896 to 0.899^-	1	25	5	2	−15	−3	−3	−1	0	−1	−1		
		0.899^- to 1.000	1	26	4	2	−15	−3	−3	−2	0	0	−1		
	4	0.850 to 0.857	2	25	3	2	1	−11	−3	−2					
		0.857 to 0.869	2	25	3	2	2	−12	−3	−2	−1	0	1		
		0.869 to 0.885	2	25	4	2	2	−12	−4	−2	−2	0	1		
		0.885 to 0.896	2	25	5	2	2	−13	−4	−1	−2	−2	1		
		0.896 to 0.899^-	2	26	5	2	1	−13	−4	−1	−1	−2	−1		
		0.899^- to 1.000	2	27	4	2	1	−13	−4	−2	−1	−1	−1		
	5	0.850 to 0.857	2	25	3	2	2	3	−10	−4	0	0	−1		
		0.857 to 0.869	2	25	3	2	3	2	−10	−4	−1				
		0.869 to 0.885	2	25	4	2	3	2	−11	−4	−2				
		0.885 to 0.896	2	25	5	2	3	1	−11	−3	−2	−2			
		0.896 to 0.899^-	2	26	5	2	2	1	−11	−3	−1	−2	−2		
		0.899^- to 1.000	2	27	4	2	2	1	−11	−4	−1	−1	−2		
	6	0.850 to 0.857	2	25	3	3	2	3	1	−8	−2	−1	−1		
		0.857 to 0.869	2	25	3	3	3	2	1	−8	−3	−1			
		0.869 to 0.885	2	25	4	3	3	2	0	−8	−4	−1			
		0.885 to 0.896	2	25	5	3	3	1	0	−7	−4	−3			
		0.896 to 0.899^-	2	26	5	3	2	1	0	−7	−3	−3	−2		
		0.899^- to 1.000	2	27	4	3	2	1	0	−8	−3	−2	−2		
	10	0.850 to 0.857	3	25	4	3	2	3	2	1	3	3	2		
		0.857 to 0.869	4	26	4	3	3	2	1	1	2	2	2		
		0.869 to 0.885	4	26	5	3	3	2	0	1	1	2	2		
		0.885 to 0.896	4	26	6	3	3	1	0	2	1	0	2		
		0.896 to 0.899^-	4	27	6	3	2	1	0	2	2				
		0.899^- to 1.000	4	28	5	3	2	1	0	1	2	1			
11	2	0.850 to 0.857	1	27	2	−20	−1	0	−1	−2	1	2	−1	−1	
		0.857 to 0.869	1	27	2	−20	0	−1	−1	−2	0	2	0	−1	
		0.869 to 0.885	1	27	3	−20	−1	−1	−2	−2	0	2	1	−1	
		0.885 to 0.896	1	27	4	−20	−2	−2	−2	0	0	−1	1	1	
		0.896 to 0.899^-	1	28	4	−20	−3	−2	−2	0	1	−1	−1	1	
		0.899^- to 0.904	1	29	3	−20	−3	−2	−2	−1	1	0	−1	1	
		0.904 to 1.000	1	30	3	−21	−3	−2	−2	0	1	0	−1	−1	

TABLE IV C — *Continued*

n	m	q-interval	x	$G_1^*(m,n)$											
				1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}
11	3	0.850 to 0.857	1	27	3	2	−17	−1	−2	−2	1	1	0	−1	
		0.857 to 0.869	1	27	3	2	−16	−2	−2	−2	0	1	1	−1	
		0.869 to 0.885	1	27	4	2	−16	−3	−3	−2	−1	2	1		
		0.885 to 0.896	1	27	5	2	−16	−4	−3	−1	−1	0	1		
		0.896 to 0.899^-	1	28	5	2	−17	−4	−3	−1	0	0	−1		
		0.899^- to 0.904	1	29	4	3	−18	−4	−3	−2	0	0	−1	1	
		0.904 to 1.000	1	30	4	2	−18	−4	−3	−1	0	0	−1	−1	
	4	0.850 to 0.857	2	28	3	2	2	−14	−3	−3	0	1			
		0.857 to 0.869	2	28	3	2	3	−15	−3	−3	−1	1	1		
		0.869 to 0.885	2	28	4	2	3	−15	−4	−3	−2	1	1		
		0.885 to 0.896	2	28	5	2	3	−16	−4	−2	−2	−1	1		
		0.896 to 0.899^-	2	29	5	2	2	−16	−4	−2	−1	−1	−1		
		0.899^- to 0.904	2	30	4	3	1	−16	−4	−3	−1	−1	−1	1	
		0.904 to 1.000	2	31	4	2	1	−16	−4	−2	−1	−1	−1	−1	
	5	0.850 to 0.857	2	28	3	2	3	3	−13	−4	0	0	−1		
		0.857 to 0.869	2	28	3	2	4	2	−13	−4	−1				
		0.869 to 0.885	2	28	4	2	4	2	−14	−4	−2				
		0.885 to 0.896	2	28	5	2	4	1	−14	−3	−2	−2			
		0.896 to 0.899^-	2	29	5	2	3	1	−14	−3	−1	−1	−2	−2	
		0.899^- to 0.904	2	30	4	2	3	1	−14	−4	−1	−1	−2		
		0.904 to 1.000	2	31	4	1	3	1	−14	−3	−1	−1	−2	−2	
	6	0.850 to 0.857	2	28	3	3	3	3	1	−11	−2	0	−1	−1	
		0.857 to 0.869	2	28	3	3	4	2	1	−11	−3	0	0	−1	
		0.869 to 0.885	2	28	4	3	4	2	0	−11	−4	0	0	−1	
		0.885 to 0.896	2	28	5	3	4	1	0	−10	−4	−2	0	−1	
		0.896 to 0.899^-	2	29	5	3	3	1	0	−10	−3	−2	−2	−1	
		0.899^- to 0.904	2	30	4	4	2	1	0	−11	−3	−2	2		
		0.904 to 1.000	2	31	4	3	2	1	0	−10	−3	−2	−2	−2	
	11	0.850 to 0.857	3	28	4	3	3	4	2	1	3	3	2	2	
		0.857 to 0.869	4	29	4	3	4	2	2	1	2	3	2	1	
		0.869 to 0.885	4	29	5	3	4	2	1	1	1	3	2	1	
		0.885 to 0.896	4	29	6	3	4	1	1	2	1	1	2	1	
		0.896 to 0.899^-	4	30	6	3	3	1	1	2	2	1	0	1	
		0.899^- to 0.904	4	31	5	4	2	1	1	2	2	1	0	2	
		0.904 to 1.000	4	32	5	3	2	1	1	2	2	1			
12	2	0.850 to 0.856	1	30	2	−23	−1	1	−1	−3	1	2	−1	−1	1
		0.856 to 0.857	1	30	2	−23	0	0	−1	−3	2	2	2	−2	−1
		0.857 to 0.869	1	30	2	−23	1	−1	−2	−2	1	2	0	−1	−1
		0.869 to 0.881	1	30	3	−23	0	−1	−3	−2	1	2	1	−1	−1
		0.881 to 0.885	1	30	3	−22	−1	−1	−3	−2	0	2	2	−1	−1
		0.885 to 0.896	1	30	4	−22	−2	−2	−3	0	0	−1	2	1	−1
		0.896 to 0.899^-	1	31	4	−23	−3	−2	−2	0	1	−1	−1	1	1
		0.899^- to 0.904	1	32	3	−23	−3	−2	−2	−1	1	0	−1	1	1
		0.904 to 0.912^-	1	33	3	−24	−3	−2	−2	0	1	0	−1	−1	1
		0.912^- to 1.000	1	34	3	−25	−3	−2	−1	0	1	0	−2	−1	
	3	0.850 to 0.856	1	30	3	2	−20	0	−2	−2	0	1	1	−1	
		0.856 to 0.857	1	30	3	2	−19	−1	−2	−2	1	1	0	−1	−1
		0.857 to 0.869	1	30	3	2	−18	−2	−2	−2	0	1	1	−1	−1
		0.869 to 0.881	1	30	4	2	−18	−3	−3	−2	−1	2	1	0	−1
		0.881 to 0.885	1	30	4	3	−19	−3	−3	−2	−2	2	2	0	−1
		0.885 to 0.896	1	30	5	3	−19	−5	−3	−1	−1	0	1	0	1
		0.896 to 0.899^-	1	31	5	3	−20	−5	−3	−1	0	0	−1	0	1
		0.899^- to 0.904	1	32	4	4	−22	−4	−3	−2	0	0	0	1	
		0.904 to 0.912^-	1	33	4	3	−22	−4	−3	−1	0	0	0	−1	
		0.912^- to 1.000	1	34	4	2	−22	−4	−2	−1	0	0	−1	−1	−1

TABLE IV C — *Continued*

n	m	q-interval	x	$G_1^*(m,n)$											
				1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}
12	4	0.850 to 0.856	2	31	3	2	2	−16	−3	−3	0	1			
		0.856 to 0.857	2	31	3	2	3	−17	−3	−3	1	1	−1	0	−1
		0.857 to 0.869	2	31	3	2	4	−18	−3	−3	0	1	0	0	−1
		0.869 to 0.881	2	31	4	2	4	−18	−5	−3	−1	1	1		
		0.881 to 0.885	2	31	4	3	3	−18	−5	−3	−2	1	2		
		0.885 to 0.896	2	31	5	3	3	−19	−5	−2	−2	−1	2		
		0.896 to 0.899^-	2	32	5	3	2	−19	−5	−2	−1	−1			
		0.899^- to 0.904	2	33	4	4	1	−19	−5	−3	−1	−1	0	1	
		0.904 to 0.912^-	2	34	4	3	1	−19	−5	−2	−1	−1	0	−1	
		0.912^- to 1.000	2	35	4	2	1	−19	−4	−2	−1	−1	−1	−1	−1
	6	0.850 to 0.856	2	31	3	3	3	4	1	−14	−2	0	−1	−1	1
		0.856 to 0.857	2	31	3	3	4	3	1	−14	−1	0	−2	−1	
		0.857 to 0.869	2	31	3	3	5	2	1	−14	−2	0	−1	−1	
		0.869 to 0.881	2	31	4	3	5	2	0	−14	−3	0	−1	−1	
		0.881 to 0.885	2	31	4	4	4	2	0	−14	−4	0	0	−1	
		0.885 to 0.896	2	31	5	4	4	1	0	−13	−4	−2	0	−1	
		0.896 to 0.899^-	2	32	5	4	3	1	0	−13	−3	−2	−2	−1	
		0.899^- to 0.904	2	33	4	5	2	1	0	−14	−3	−2	−2		
		0.904 to 0.912^-	2	34	4	4	2	1	0	−13	−3	−2	−2	−2	
		0.912^- to 1.000	2	35	4	3	2	1	1	−13	−3	−2	−3	−2	−1
	7	0.850 to 0.856	2	31	3	3	3	4	2	1	−9	−2	−1	−1	
		0.856 to 0.857	2	31	3	3	4	3	2	1	−8	−2	−2	−1	−1
		0.857 to 0.869	3	31	4	3	5	2	1	0	−9	−2	−1	−1	−1
		0.869 to 0.881	3	31	5	3	5	2	0	0	−10	−2	−1	−1	−1
		0.881 to 0.885	3	31	5	4	4	2	0	0	−11	−2	0	−1	−1
		0.885 to 0.896	3	31	6	4	4	1	0	1	−11	−4	0	−1	−1
		0.896 to 0.899^-	3	32	6	4	3	1	0	1	−10	−4	−2	−1	−1
		0.899^- to 0.904	3	33	5	5	2	1	0	0	−10	−4	−2	0	−1
		0.904 to 0.912^-	3	34	5	4	2	1	0	1	−10	−4	−2	−2	−1
		0.912^- to 1.000	3	35	5	3	2	1	1	1	−10	−4	−3	−2	−2
	12	0.850 to 0.856	4	32	4	3	3	4	2	1	3	3	2	2	3
		0.856 to 0.857	4	32	4	3	4	3	2	1	4	3	1	2	2
		0.857 to 0.869	4	32	4	3	5	2	2	1	3	3	2	2	2
		0.869 to 0.881	4	32	5	3	5	2	1	1	2	3	2	2	2
		0.881 to 0.885	4	32	5	4	4	2	1	1	1	3	3	2	2
		0.885 to 0.896	4	32	6	4	4	1	1	2	1	1	3	2	2
		0.896 to 0.899^-	4	33	6	4	3	1	1	2	2	1	1	2	2
		0.899^- to 0.904	4	34	5	5	2	2	1	1	2	1	1	2	1
		0.904 to 0.912^-	4	35	5	4	2	2	1	2	2	1	1	0	1
		0.912^- to 1.000	4	36	5	3	2	2	2	2	2	1			

The exponential symbols +, − indicate only the relative magnitude of two different roots that are equal to three decimal places (i.e., $a^- < a^+$).

TABLE V A — EXPECTED NUMBER OF TESTS REQUIRED AND SIZE x OF THE NEXT SAMPLE TO BE TAKEN FOR ANY $G_1(m,n)$ AND ANY $H_1(n)$ SITUATION WHEN PROCEDURE R_1 IS USED AND $q = 0.90$

n	$H_1(n)$	x	$G_1(m,n)$							
			$m = 2$ $x = 1$	$m = 3$ $x = 1$	$m = 4$ $x = 2$	$m = 5$ $x = 2$	$m = 6$ $x = 2$	$m = 7$ $x = 3$	$m = 8$ $x = 4$	$m = 9$ $x = 4$
1	1.000	1								
2	1.290	2	1.526							
3	1.661	3	2.153	2.439						
4	2.051	4	2.485	2.971	3.056					
5	2.490	5	2.866	3.325	3.547	3.637				
6	2.943	6	3.282	3.727	3.925	4.115	4.148			
7	3.414	7	3.728	4.157	4.343	4.512	4.621	4.628		
8	3.904	8	4.191	4.613	4.784	4.943	5.033	5.100	5.100	
9	4.395	9	4.672	5.085	5.250	5.396	5.476	5.528	5.575	5.543
10	4.872	6	5.163	5.570	5.728	5.868	5.938	5.980	6.014	6.021
11	5.327	6	5.646	6.056	6.210	6.346	6.411	6.445	6.470	6.466
12	5.790	6	6.111	6.528	6.687	6.821	6.883	6.914	6.933	6.922
13	6.261	7	6.570	6.993	7.157	7.294	7.356	7.385	7.401	7.385
14	6.732	7	7.037	7.456	7.623	7.765	7.829	7.857	7.872	7.853
15	7.213	7	7.509	7.925	8.089	8.232	8.300	8.330	8.345	8.324
16	7.695	7	7.985	8.400	8.561	8.702	8.770	8.802	8.819	8.798
17	8.161	6	8.467	8.878	9.038	9.176	9.242	9.275	9.293	9.273
18	8.629	6	8.940	9.354	9.513	9.651	9.715	9.746	9.764	9.746
19	9.100	7	9.407	9.825	9.986	10.124	10.188	10.218	10.235	10.217
20	9.572	7	9.877	10.294	10.458	10.596	10.660	10.690	10.706	10.687
21	10.044	7	10.348	10.764	10.927	11.068	11.133	11.163	11.179	11.159
22	10.520	7	10.820	11.236	11.398	11.538	11.605	11.635	11.651	11.632
23	10.996	6	11.295	11.709	11.871	12.010	12.076	12.108	12.124	12.105
24	11.466	6	11.770	12.184	12.345	12.484	12.549	12.580	12.597	12.578
25	11.937	7	12.243	12.658	12.819	12.957	13.022	13.052	13.069	13.051
26	12.408	7	12.714	13.130	13.292	13.430	13.495	13.525	13.541	13.522
27	12.881	7	13.185	13.601	13.763	13.902	13.967	13.997	14.014	13.994
28	13.353	7	13.657	14.073	14.235	14.374	14.440	14.470	14.486	14.467
29	13.827	7	14.129	14.545	14.707	14.846	14.912	14.943	14.959	14.939
30	14.301	7	14.603	15.018	15.179	15.319	15.384	15.415	15.431	15.412
31	14.773	7	15.077	15.491	15.653	15.792	15.857	15.888	15.904	15.885
32	15.244	7	15.549	15.965	16.126	16.264	16.329	16.360	16.377	16.358
33	15.717	7	16.021	16.437	16.598	16.737	16.802	16.833	16.849	16.830
34	16.189	7	16.493	16.909	17.071	17.210	17.275	17.305	17.321	17.302
35	16.661	7	16.965	17.381	17.543	17.682	17.747	17.778	17.794	17.775
36	17.135	7	17.438	17.853	18.015	18.154	18.219	18.250	18.266	18.247
37	17.608	7	17.910	18.326	18.487	18.626	18.692	18.723	18.739	18.720
38	18.080	7	18.384	18.799	18.960	19.099	19.164	19.195	19.212	19.192
39	18.552	7	18.856	19.272	19.433	19.572	19.637	19.668	19.684	19.665
40	19.024	7	19.328	19.744	19.906	20.044	20.110	20.140	20.157	20.137
41	19.497	7	19.801	20.216	20.378	20.517	20.582	20.613	20.629	20.610
42	19.969	7	20.273	20.688	20.850	20.989	21.055	21.085	21.102	21.082
43	20.442	7	20.745	21.161	21.323	21.462	21.527	21.558	21.574	21.555
44	20.915	7	21.218	21.634	21.795	21.934	21.999	22.030	22.047	22.027
45	21.387	7	21.691	22.106	22.268	22.407	22.472	22.503	22.519	22.500
46	21.860	7	22.164	22.579	22.740	22.879	22.945	22.975	22.992	22.972
47	22.332	7	22.636	23.051	23.213	23.352	23.417	23.448	23.464	23.445
48	22.805	7	23.108	23.524	23.685	23.825	23.890	23.920	23.937	23.917
49	23.277	7	23.581	23.996	24.158	24.297	24.362	24.393	24.409	24.390
50	23.750	7	24.053	24.469	24.630	24.769	24.835	24.865	24.882	24.862

TABLE V A — *Continued*

n	$H_1(n)$	x	$G_1(m,n)$							
			$m = 2$ $x = 1$	$m = 3$ $x = 1$	$m = 4$ $x = 2$	$m = 5$ $x = 2$	$m = 6$ $x = 2$	$m = 7$ $x = 3$	$m = 8$ $x = 4$	$m = 9$ $x = 4$
51	24.222	7	24.526	24.941	25.103	25.242	25.307	25.338	25.354	25.335
52	24.695	7	24.998	25.414	25.575	25.714	25.780	25.810	25.827	25.807
53	25.167	7	25.471	25.886	26.048	26.187	26.252	26.283	26.299	26.280
54	25.640	7	25.943	26.359	26.521	26.660	26.725	26.755	26.772	26.752
55	26.112	7	26.416	26.831	26.993	27.132	27.197	27.228	27.244	27.225
56	26.585	7	26.888	27.304	27.465	27.605	27.670	27.700	27.717	27.697
57	27.057	7	27.361	27.776	27.938	28.077	28.142	28.173	28.189	28.170
58	27.530	7	27.833	28.249	28.410	28.550	28.615	28.645	28.662	28.643
59	28.002	7	28.306	28.721	28.883	29.022	29.087	29.118	29.134	29.115
60	28.475	7	28.779	29.194	29.356	29.495	29.560	29.590	29.607	29.588
61	28.947	7	29.251	29.666	29.828	29.967	30.032	30.063	30.079	30.060
62	29.420	7	29.723	30.139	30.301	30.440	30.505	30.535	30.552	30.533
63	29.892	7	30.196	30.611	30.773	30.912	30.977	31.008	31.024	31.005
64	30.365	7	30.668	31.084	31.246	31.385	31.450	31.480	31.497	31.478
65	30.837	7	31.141	31.556	31.718	31.857	31.922	31.953	31.969	31.950
66	31.310	7	31.614	32.029	32.191	32.330	32.395	32.425	32.442	32.423
67	31.782	7	32.086	32.502	32.663	32.802	32.867	32.898	32.914	32.895
68	32.255	7	32.559	32.974	33.136	33.275	33.340	33.370	33.387	33.368
69	32.727	7	33.031	33.446	33.608	33.747	33.812	33.843	33.859	33.840
70	33.200	7	33.504	33.919	34.081	34.220	34.285	34.316	34.332	34.313
71	33.672	7	33.976	34.392	34.553	34.692	34.757	34.788	34.804	34.785
72	34.145	7	34.449	34.864	35.026	35.165	35.230	35.261	35.277	35.258
73	34.617	7	34.921	35.337	35.498	35.637	35.702	35.733	35.749	35.730
74	35.090	7	35.394	35.809	35.971	36.110	36.175	36.206	36.222	36.203
75	35.562	7	35.866	36.282	36.443	36.582	36.647	36.678	36.694	36.675
76	36.035	7	36.339	36.754	36.916	37.055	37.120	37.151	37.167	37.148
77	36.507	7	36.811	37.227	37.388	37.527	37.592	37.623	37.639	37.620
78	36.980	7	37.284	37.699	37.861	38.000	38.065	38.096	38.112	38.093
79	37.453	7	37.756	38.172	38.333	38.472	38.537	38.568	38.584	38.565
80	37.925	7	38.229	38.644	38.806	38.945	39.010	39.041	39.057	39.038
81	38.398	7	38.701	39.117	39.278	39.417	39.482	39.513	39.530	39.510
82	38.870	7	39.174	39.589	39.751	39.890	39.955	39.986	40.002	39.983
83	39.343	7	39.646	40.062	40.223	40.362	40.427	40.458	40.475	40.455
84	39.815	7	40.119	40.534	40.696	40.835	40.900	40.931	40.947	40.928
85	40.288	7	40.591	41.007	41.168	41.307	41.373	41.403	41.420	41.400
86	40.760	7	41.064	41.479	41.641	41.780	41.845	41.876	41.892	41.873
87	41.233	7	41.536	41.952	42.113	42.252	42.318	42.348	42.365	42.345
88	41.705	7	42.009	42.424	42.586	42.725	42.790	42.821	42.837	42.818
89	42.178	7	42.481	42.897	43.058	43.197	43.263	43.293	43.310	43.290
90	42.650	7	42.954	43.369	43.531	43.670	43.735	43.766	43.782	43.763
91	43.123	7	43.426	43.842	44.003	44.142	44.208	44.238	44.255	44.235
92	43.595	7	43.899	44.314	44.476	44.615	44.680	44.711	44.727	44.708
93	44.068	7	44.371	44.787	44.948	45.087	45.153	45.183	45.200	45.180
94	44.540	7	44.844	45.259	45.421	45.560	45.625	45.656	45.672	45.653
95	45.013	7	45.316	45.732	45.893	46.032	46.098	46.128	46.145	46.125
96	45.485	7	45.789	46.204	46.366	46.505	46.570	46.601	46.617	46.598
97	45.958	7	46.261	46.677	46.838	46.978	47.043	47.073	47.090	47.070
98	46.430	7	46.734	47.149	47.311	47.450	47.515	47.546	47.562	47.543
99	46.903	7	47.206	47.622	47.784	47.923	47.988	48.018	48.035	48.016
100	47.375	7	47.679	48.094	48.256	48.395	48.460	48.491	48.507	48.488

TABLE V B—EXPECTED NUMBER OF TESTS REQUIRED AND SIZE x OF THE NEXT SAMPLE TO BE TAKEN FOR ANY $G(m,n)$ AND ANY $H_1(n)$ SITUATION WHEN PROCEDURE R_1 IS USED AND $q = 0.95$

n	$H_1(n)$	x	$G_1(m,n)$																	
			$m = 2$ $x = 1$	$m = 3$ $x = 1$	$m = 4$ $x = 2$	$m = 5$ $x = 2$	$m = 6$ $x = 2$	$m = 7$ $x = 3$	$m = 8$ $x = 4$	$m = 9$ $x = 4$	$m = 10$ $x = 4$	$m = 11$ $x = 4$	$m = 12$ $x = 4$	$m = 13$ $x = 5$	$m = 14$ $x = 6$	$m = 15$ $x = 7$	$m = 16$ $x = 7$	$m = 17$ $x = 8$	$m = 18$ $x = 8$	$m = 19$ $x = 8$
1	1.000	1																		
2	1.148	2	1.513																	
3	1.340	3	2.076	2.385																
4	1.538	4	2.246	2.818	2.898															
5	1.771	5	2.441	2.998	3.268	3.409														
6	2.009	6	2.657	3.206	3.462	3.749	3.810													
7	2.252	7	2.893	3.430	3.679	3.953	4.130	4.150												
8	2.499	8	3.134	3.668	3.908	4.175	4.341	4.457	4.455											
9	2.767	9	3.379	3.911	4.148	4.408	4.568	4.675	4.753	4.780										
10	3.039	10	3.637	4.164	4.398	4.655	4.808	4.908	4.978	5.074	5.082									
11	3.315	11	3.907	4.427	4.656	4.909	5.059	5.153	5.218	5.306	5.373	5.368								
12	3.594	12	4.180	4.699	4.923	5.171	5.318	5.409	5.467	5.550	5.610	5.657	5.643							
13	3.878	13	4.458	4.975	5.196	5.440	5.583	5.671	5.726	5.804	5.859	5.899	5.932	5.913						
14	4.166	14	4.739	5.254	5.474	5.716	5.855	5.939	5.991	6.066	6.116	6.152	6.178	6.201	6.179					
15	4.458	15	5.025	5.538	5.756	5.997	6.133	6.214	6.262	6.334	6.382	6.413	6.435	6.451	6.467	6.443				
16	4.753	16	5.316	5.826	6.042	6.281	6.416	6.494	6.540	6.608	6.653	6.681	6.699	6.712	6.721	6.732	6.704			
17	5.051	17	5.609	6.118	6.332	6.569	6.702	6.779	6.822	6.888	6.930	6.955	6.971	6.979	6.985	6.990	6.993	6.961		
18	5.348	18	5.906	6.414	6.626	6.861	6.992	7.067	7.109	7.173	7.211	7.234	7.247	7.253	7.256	7.257	7.255	7.251	7.214	
19	5.648	19	6.203	6.710	6.922	7.155	7.285	7.358	7.398	7.461	7.498	7.518	7.528	7.532	7.532	7.530	7.524	7.516	7.504	7.465
20	5.940	11	6.502	7.009	7.219	7.452	7.580	7.652	7.691	7.752	7.787	7.806	7.814	7.815	7.813	7.809	7.800	7.788	7.772	7.757
21	6.220	12	6.798	7.305	7.516	7.748	7.876	7.947	7.985	8.044	8.078	8.096	8.102	8.101	8.097	8.090	8.079	8.065	8.046	8.026
22	6.499	12	7.083	7.595	7.808	8.041	8.169	8.240	8.277	8.336	8.369	8.386	8.391	8.389	8.382	8.374	8.361	8.345	8.323	8.301
23	6.780	12	7.363	7.879	8.095	8.330	8.460	8.531	8.568	8.627	8.659	8.675	8.679	8.677	8.669	8.659	8.644	8.626	8.603	8.578
24	7.064	13	7.643	8.159	8.378	8.616	8.747	8.820	8.858	8.916	8.949	8.964	8.968	8.964	8.956	8.945	8.929	8.909	8.884	8.858
25	7.348	13	7.926	8.440	8.659	8.899	9.033	9.107	9.146	9.205	9.238	9.253	9.257	9.252	9.243	9.232	9.215	9.194	9.167	9.139
26	7.633	13	8.210	8.723	8.941	9.181	9.316	9.392	9.432	9.492	9.526	9.542	9.545	9.541	9.531	9.519	9.501	9.480	9.451	9.423
27	7.921	14	8.494	9.008	9.225	9.464	9.598	9.676	9.718	9.779	9.813	9.829	9.833	9.829	9.819	9.806	9.788	9.766	9.737	9.707
28	8.210	14	8.781	9.293	9.510	9.748	9.882	9.959	10.002	10.064	10.100	10.117	10.121	10.117	10.107	10.094	10.076	10.053	10.024	9.993
29	8.500	14	9.069	9.581	9.796	10.034	10.167	10.243	10.286	10.349	10.386	10.403	10.408	10.405	10.395	10.382	10.364	10.341	10.311	10.280
30	8.791	14	9.358	9.870	10.085	10.321	10.454	10.530	10.571	10.634	10.671	10.690	10.695	10.692	10.683	10.671	10.652	10.629	10.599	10.567

TABLE V B — *Continued*

n	$H_1(n)$	x	$G_1(m,n)$																	
			$m = 2$ $x = 1$	$m = 3$ $x = 1$	$m = 4$ $x = 2$	$m = 5$ $x = 2$	$m = 6$ $x = 2$	$m = 7$ $x = 3$	$m = 8$ $x = 4$	$m = 9$ $x = 4$	$m = 10$ $x = 4$	$m = 11$ $x = 4$	$m = 12$ $x = 4$	$m = 13$ $x = 5$	$m = 14$ $x = 6$	$m = 15$ $x = 7$	$m = 16$ $x = 7$	$m = 17$ $x = 8$	$m = 18$ $x = 8$	$m = 19$ $x = 8$
31	9.084	15	9.649	10.160	10.374	10.610	10.742	10.817	10.858	10.920	10.957	10.976	10.982	10.980	10.971	10.959	10.940	10.917	10.887	10.855
32	9.377	15	9.941	10.451	10.665	10.901	11.032	11.106	11.146	11.208	11.244	11.263	11.269	11.267	11.259	11.248	11.229	11.206	11.176	11.144
33	9.669	15	10.234	10.743	10.957	11.192	11.323	11.396	11.436	11.497	11.532	11.550	11.557	11.555	11.547	11.536	11.518	11.495	11.465	11.433
34	9.956	13	10.527	11.036	11.249	11.484	11.614	11.687	11.726	11.787	11.822	11.839	11.845	11.843	11.835	11.825	11.807	11.785	11.755	11.723
35	10.240	13	10.816	11.327	11.540	11.775	11.905	11.978	12.017	12.077	12.111	12.128	12.133	12.131	12.123	12.112	12.095	12.073	12.044	12.012
36	10.524	13	11.102	11.614	11.829	12.064	12.195	12.268	12.306	12.366	12.400	12.417	12.422	12.419	12.411	12.400	12.383	12.361	12.332	12.300
37	10.810	13	11.386	11.899	12.116	12.352	12.483	12.556	12.595	12.655	12.689	12.706	12.710	12.707	12.698	12.687	12.670	12.648	12.620	12.588
38	11.096	13	11.671	12.184	12.401	12.639	12.771	12.845	12.884	12.944	12.978	12.994	12.999	12.995	12.986	12.975	12.957	12.935	12.907	12.876
39	11.383	13	11.957	12.469	12.686	12.924	13.057	13.132	13.172	13.232	13.266	13.282	13.287	13.283	13.274	13.262	13.245	13.223	13.194	13.163
40	11.671	14	12.243	12.756	12.972	13.209	13.343	13.418	13.459	13.519	13.554	13.570	13.575	13.571	13.562	13.550	13.532	13.510	13.481	13.450
41	11.959	14	12.531	13.043	13.258	13.496	13.629	13.704	13.745	13.807	13.842	13.858	13.863	13.859	13.850	13.838	13.820	13.798	13.768	13.737
42	12.247	14	12.819	13.330	13.546	13.783	13.915	13.990	14.032	14.094	14.129	14.146	14.151	14.147	14.138	14.126	14.108	14.085	14.056	14.024
43	12.536	14	13.107	13.618	13.833	14.070	14.203	14.277	14.318	14.380	14.416	14.434	14.439	14.435	14.426	14.414	14.396	14.373	14.344	14.312
44	12.826	14	13.395	13.907	14.122	14.358	14.490	14.565	14.605	14.667	14.703	14.721	14.726	14.723	14.714	14.702	14.684	14.661	14.632	14.600
45	13.116	14	13.685	14.196	14.411	14.647	14.779	14.853	14.893	14.955	14.990	15.008	15.014	15.011	15.003	14.991	14.972	14.950	14.920	14.888
46	13.406	14	13.975	14.485	14.700	14.936	15.068	15.142	15.182	15.243	15.278	15.296	15.301	15.299	15.291	15.279	15.261	15.238	15.208	15.177
47	13.696	13	14.265	14.776	14.990	15.226	15.357	15.431	15.471	15.532	15.566	15.584	15.589	15.587	15.579	15.567	15.549	15.527	15.497	15.465
48	13.982	13	14.555	15.066	15.280	15.515	15.647	15.720	15.760	15.821	15.855	15.872	15.878	15.875	15.867	15.855	15.838	15.815	15.786	15.754
49	14.268	13	14.843	15.354	15.569	15.805	15.936	16.009	16.049	16.109	16.144	16.161	16.166	16.163	16.155	16.143	16.126	16.104	16.074	16.042
50	14.555	13	15.129	15.641	15.857	16.093	16.224	16.298	16.337	16.398	16.432	16.449	16.454	16.451	16.443	16.431	16.413	16.392	16.362	16.331
51	14.842	13	15.416	15.928	16.144	16.381	16.512	16.586	16.626	16.686	16.721	16.738	16.742	16.739	16.730	16.719	16.701	16.679	16.650	16.619
52	15.130	13	15.702	16.215	16.430	16.668	16.800	16.874	16.914	16.974	17.009	17.026	17.031	17.027	17.018	17.007	16.989	16.967	16.937	16.906
53	15.418	14	15.990	16.502	16.717	16.954	17.087	17.162	17.202	17.262	17.297	17.314	17.319	17.315	17.306	17.295	17.277	17.254	17.225	17.194
54	15.706	14	16.277	16.789	17.005	17.242	17.374	17.449	17.489	17.550	17.585	17.602	17.607	17.603	17.595	17.583	17.565	17.542	17.513	17.481
55	15.994	14	16.565	17.077	17.292	17.529	17.661	17.736	17.777	17.838	17.873	17.890	17.895	17.891	17.883	17.871	17.853	17.830	17.801	17.769
56	16.282	14	16.853	17.365	17.580	17.817	17.949	18.024	18.064	18.125	18.161	18.178	18.183	18.179	18.171	18.159	18.141	18.118	18.089	18.057
57	16.570	14	17.142	17.653	17.868	18.105	18.237	18.311	18.352	18.413	18.448	18.466	18.471	18.467	18.459	18.447	18.429	18.406	18.377	18.345
58	16.859	14	17.430	17.941	18.156	18.393	18.525	18.599	18.639	18.701	18.736	18.753	18.759	18.755	18.747	18.735	18.717	18.694	18.665	18.633
59	17.149	14	17.719	18.230	18.445	18.681	18.813	18.887	18.927	18.989	19.024	19.041	19.046	19.043	19.035	19.023	19.005	18.983	18.953	18.921
60	17.438	14	18.008	18.519	18.734	18.970	19.102	19.176	19.216	19.277	19.312	19.329	19.334	19.331	19.323	19.311	19.293	19.271	19.241	19.210

61	17.726	13	18.297	18.808	19.023	19.259	19.390	19.464	19.504	19.565	19.600	19.617	19.622	19.619	19.611	19.599	19.582	19.559	19.530	19.498
62	18.013	13	18.585	19.097	19.311	19.547	19.679	19.753	19.793	19.854	19.888	19.905	19.911	19.907	19.899	19.888	19.870	19.847	19.818	19.786
63	18.300	13	18.873	19.385	19.600	19.836	19.967	20.041	20.081	20.142	20.177	20.194	20.199	20.196	20.187	20.175	20.158	20.136	20.106	20.074
64	18.588	13	19.160	19.672	19.887	20.124	20.256	20.330	20.369	20.430	20.465	20.482	20.487	20.484	20.475	20.463	20.446	20.423	20.394	20.363
65	18.875	13	19.448	19.959	20.175	20.412	20.544	20.618	20.658	20.718	20.753	20.770	20.775	20.772	20.763	20.751	20.734	20.711	20.682	20.651
66	19.163	14	19.735	20.247	20.462	20.699	20.831	20.906	20.946	21.006	21.041	21.058	21.063	21.060	21.051	21.039	21.021	20.999	20.970	20.938
67	19.451	14	20.023	20.535	20.750	20.987	21.119	21.194	21.234	21.295	21.329	21.346	21.351	21.348	21.339	21.327	21.310	21.287	21.258	21.226
68	19.740	14	20.311	20.823	21.038	21.275	21.407	21.481	21.522	21.582	21.617	21.634	21.639	21.636	21.627	21.615	21.598	21.575	21.546	21.514
69	20.028	14	20.599	21.111	21.326	21.563	21.695	21.769	21.809	21.870	21.905	21.922	21.927	21.924	21.915	21.904	21.886	21.863	21.834	21.802
70	20.316	14	20.887	21.399	21.614	21.851	21.982	22.057	22.097	22.158	22.193	22.210	22.215	22.212	22.204	22.192	22.174	22.151	22.122	22.090
71	20.604	14	21.175	21.687	21.902	22.139	22.270	22.345	22.385	22.446	22.481	22.498	22.503	22.500	22.492	22.480	22.462	22.439	22.410	22.378
72	20.892	14	21.464	21.975	22.190	22.427	22.559	22.633	22.673	22.734	22.769	22.786	22.791	22.788	22.780	22.768	22.750	22.727	22.698	22.666
73	21.181	14	21.752	22.263	22.478	22.715	22.847	22.921	22.961	23.022	23.057	23.074	23.079	23.076	23.068	23.056	23.038	23.016	22.986	22.954
74	21.470	13	22.040	22.552	22.767	23.003	23.135	23.209	23.249	23.310	23.345	23.362	23.367	23.364	23.356	23.344	23.326	23.304	23.274	23.243
75	21.757	13	22.329	22.840	23.055	23.292	23.423	23.498	23.538	23.598	23.633	23.650	23.655	23.652	23.644	23.632	23.614	23.592	23.562	23.531
76	22.045	13	22.617	23.129	23.344	23.580	23.712	23.786	23.826	23.887	23.921	23.939	23.944	23.940	23.932	23.920	23.902	23.880	23.850	23.819
77	22.333	13	22.905	23.417	23.632	23.868	24.000	24.074	24.114	24.175	24.210	24.227	24.232	24.229	24.220	24.208	24.190	24.168	24.139	24.107
78	22.621	13	23.193	23.704	23.920	24.156	24.288	24.362	24.402	24.463	24.498	24.515	24.520	24.517	24.508	24.496	24.478	24.456	24.427	24.395
79	22.909	14	23.480	23.992	24.207	24.444	24.576	24.650	24.690	24.751	24.786	24.803	24.808	24.805	24.796	24.784	24.766	24.744	24.715	24.683
80	23.197	14	23.768	24.280	24.495	24.732	24.864	24.938	24.978	25.039	25.074	25.091	25.096	25.093	25.084	25.072	25.054	25.032	25.003	24.971
81	23.485	14	24.056	24.568	24.783	25.020	25.152	25.226	25.266	25.327	25.362	25.379	25.384	25.381	25.372	25.360	25.343	25.320	25.291	25.259
82	23.773	14	24.345	24.856	25.071	25.308	25.440	25.514	25.554	25.615	25.650	25.667	25.672	25.669	25.660	25.649	25.631	25.608	25.579	25.547
83	24.061	14	24.633	25.144	25.359	25.596	25.728	25.802	25.842	25.903	25.938	25.955	25.960	25.957	25.948	25.937	25.919	25.896	25.867	25.835
84	24.349	14	24.921	25.432	25.647	25.884	26.016	26.090	26.130	26.191	26.226	26.243	26.248	26.245	26.237	26.225	26.207	26.184	26.155	26.123
85	24.637	14	25.209	25.720	25.935	26.172	26.304	26.378	26.418	26.479	26.514	26.531	26.536	26.533	26.525	26.513	26.495	26.473	26.443	26.411
86	24.926	14	25.497	26.008	26.224	26.460	26.592	26.666	26.706	26.767	26.802	26.819	26.824	26.821	26.813	26.801	26.783	26.761	26.731	26.699
87	25.214	14	25.785	26.297	26.512	26.748	26.880	26.954	26.994	27.055	27.090	27.107	27.112	27.109	27.101	27.089	27.071	27.049	27.019	26.988
88	25.502	13	26.073	26.585	26.800	27.036	27.168	27.242	27.283	27.343	27.378	27.395	27.400	27.397	27.389	27.377	27.359	27.337	27.307	27.276
89	25.790	13	26.362	26.873	27.088	27.325	27.456	27.531	27.571	27.632	27.666	27.684	27.689	27.685	27.677	27.665	27.647	27.625	27.595	27.564
90	26.078	13	26.650	27.161	27.376	27.613	27.745	27.819	27.859	27.920	27.955	27.972	27.977	27.974	27.965	27.953	27.935	27.913	27.883	27.852
91	26.366	14	26.938	27.449	27.664	27.901	28.033	28.107	28.147	28.208	28.243	28.260	28.265	28.262	28.253	28.241	28.223	28.201	28.171	28.140
92	26.654	14	27.226	27.737	27.952	28.189	28.321	28.395	28.435	28.496	28.531	28.548	28.553	28.550	28.541	28.529	28.511	28.489	28.460	28.428
93	26.942	14	27.514	28.025	28.240	28.477	28.609	28.683	28.723	28.784	28.819	28.836	28.841	28.838	28.829	28.817	28.800	28.777	28.748	28.716
94	27.230	14	27.802	28.313	28.528	28.765	28.897	28.971	29.011	29.072	29.107	29.124	29.129	29.126	29.117	29.105	29.088	29.065	29.036	29.004
95	27.518	14	28.090	28.601	28.816	29.053	29.185	29.259	29.299	29.360	29.395	29.412	29.417	29.414	29.405	29.394	29.376	29.353	29.324	29.292
96	27.806	14	28.378	28.889	29.104	29.341	29.473	29.547	29.587	29.648	29.683	29.700	29.705	29.702	29.693	29.682	29.664	29.641	29.612	29.580
97	28.094	14	28.666	29.177	29.393	29.629	29.761	29.835	29.875	29.936	29.971	29.988	29.993	29.990	29.982	29.970	29.952	29.929	29.900	29.868
98	28.382	14	28.954	29.465	29.681	29.917	30.049	30.123	30.163	30.224	30.259	30.276	30.281	30.278	30.270	30.258	30.240	30.218	30.188	30.156
99	28.670	14	29.242	29.754	29.969	30.205	30.337	30.411	30.451	30.512	30.547	30.564	30.569	30.566	30.558	30.546	30.528	50.506	30.476	30.445
100	28.959	14	29.530	30.042	30.257	30.493	30.625	30.699	30.739	30.800	30.835	30.852	30.857	30.854	30.846	30.834	30.816	30.794	30.764	30.733

TABLE V C — EXPECTED NUMBER OF TESTS REQUIRED AND SIZE x OF THE NEXT SAMPLE TO BE TAKEN FOR ANY $G_1(m,n)$ AND ANY $H_1(n)$-SITUATION WHEN PROCEDURE R_1 IS USED AND $q=0.99$

n	$H_1(n)$	x	$G_1(m,n)$																
			$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$	$m = 20$	$m = 30$	$m = 40$	$m = 50$	$m = 60$	$m = 70$	$m = 80$	$m = 90$	$m = 100$
			$x = 1$	$x = 1$	$x = 2$	$x = 2$	$x = 2$	$x = 3$	$x = 4$	$x = 4$	$x = 4$	$x = 8$	$x = 14$	$x = 16$	$x = 18$	$x = 28$	$x = 32$	$x = 32$	$x = 32$	$x = 35$
1	1.000	1	—																	
2	1.030	2	1.503																	
3	1.070	3	2.015	2.343																
4	1.110	4	2.050	2.697	2.779															
5	1.159	5	2.090	2.733	3.053	3.240														
6	1.208	6	2.134	2.776	3.092	3.468	3.560													
7	1.258	7	2.184	2.823	3.137	3.510	3.757	3.798												
8	1.308	8	2.233	2.872	3.184	3.556	3.800	3.974	3.986											
9	1.366	9	2.283	2.922	3.234	3.603	3.847	4.018	4.145	4.237										
10	1.425	10	2.337	2.974	3.286	3.655	3.896	4.066	4.191	4.385	4.447									
11	1.484	11	2.396	3.030	3.340	3.708	3.949	4.117	4.241	4.433	4.585									
12	1.543	12	2.455	3.089	3.397	3.763	4.003	4.171	4.293	4.483	4.634									
13	1.603	13	2.514	3.148	3.455	3.821	4.059	4.226	4.347	4.536	4.685									
14	1.662	14	2.573	3.207	3.515	3.879	4.117	4.282	4.402	4.591	4.739									
15	1.722	15	2.632	3.266	3.574	3.939	4.176	4.340	4.459	4.647	4.794									
16	1.782	16	2.692	3.326	3.633	3.998	4.235	4.399	4.517	4.704	4.851									
17	1.849	17	2.752	3.386	3.693	4.057	4.294	4.458	4.577	4.762	4.908									
18	1.916	18	2.815	3.448	3.754	4.119	4.355	4.519	4.637	4.822	4.967									
19	1.984	19	2.883	3.513	3.818	4.181	4.417	4.580	4.698	4.884	5.028									
20	2.051	20	2.950	3.580	3.883	4.246	4.481	4.643	4.761	4.946	5.090	5.773								
21	2.119	21	3.018	3.647	3.951	4.312	4.546	4.708	4.824	5.009	5.153	5.871								
22	2.187	22	3.085	3.715	4.018	4.379	4.612	4.773	4.889	5.073	5.216	5.928								
23	2.255	23	3.153	3.783	4.086	4.447	4.680	4.839	4.955	5.138	5.281	5.986								
24	2.324	24	3.221	3.851	4.154	4.515	4.747	4.907	5.021	5.204	5.346	6.046								
25	2.392	25	3.290	3.919	4.222	4.583	4.815	4.975	5.089	5.271	5.412	6.107								
26	2.461	26	3.358	3.987	4.290	4.651	4.883	5.043	5.157	5.339	5.479	6.169								
27	2.530	27	3.427	4.056	4.359	4.719	4.952	5.111	5.225	5.407	5.547	6.232								
28	2.599	28	3.495	4.125	4.427	4.788	5.020	5.179	5.293	5.475	5.615	6.296								
29	2.668	29	3.564	4.194	4.496	4.856	5.089	5.248	5.362	5.543	5.684	6.360								
30	2.738	30	3.634	4.263	4.565	4.925	5.157	5.317	5.431	5.612	5.752	6.426	6.676							

31	2.807	31	3.703	4.332	4.634	4.994	5.227	5.386	5.499	5.681	5.821	6.492	6.763							
32	2.877	32	3.773	4.401	4.704	5.064	5.296	5.455	5.568	5.749	5.890	6.558	6.824							
33	2.952	33	3.842	4.471	4.773	5.133	5.365	5.524	5.638	5.819	5.959	6.625	6.887							
34	3.027	34	3.915	4.543	4.844	5.204	5.436	5.594	5.708	5.888	6.028	6.692	6.950							
35	3.101	35	3.989	4.616	4.917	5.276	5.507	5.665	5.778	5.959	6.099	6.761	7.014							
36	3.176	36	4.064	4.690	4.990	5.348	5.579	5.737	5.850	6.030	6.170	6.830	7.080							
37	3.252	37	4.139	4.765	5.065	5.422	5.652	5.810	5.922	6.102	6.241	6.900	7.146							
38	3.327	38	4.214	4.840	5.140	5.497	5.726	5.883	5.995	6.175	6.314	6.970	7.213							
39	3.402	39	4.289	4.915	5.215	5.572	5.801	5.958	6.069	6.248	6.387	7.041	7.281							
40	3.478	40	4.365	4.991	5.290	5.647	5.876	6.033	6.143	6.322	6.460	7.112	7.349	7.486						
41	3.554	41	4.440	5.066	5.366	5.723	5.952	6.108	6.218	6.396	6.534	7.183	7.418	7.570						
42	3.630	42	4.516	5.142	5.441	5.798	6.027	6.183	6.294	6.472	6.609	7.255	7.487	7.635						
43	3.706	43	4.592	5.218	5.517	5.874	6.103	6.259	6.369	6.547	6.684	7.328	7.557	7.701						
44	3.782	44	4.668	5.294	5.593	5.950	6.178	6.334	6.445	6.622	6.760	7.400	7.628	7.767						
45	3.859	45	4.744	5.370	5.669	6.025	6.254	6.410	6.520	6.698	6.835	7.474	7.699	7.835						
46	3.935	46	4.821	5.446	5.745	6.102	6.330	6.486	6.596	6.774	6.911	7.547	7.770	7.903						
47	4.012	47	4.897	5.522	5.821	6.178	6.406	6.562	6.672	6.850	6.987	7.621	7.842	7.972						
48	4.089	48	4.974	5.599	5.898	6.254	6.483	6.638	6.748	6.926	7.063	7.695	7.915	8.042						
49	4.166	49	5.051	5.676	5.975	6.331	6.559	6.715	6.825	7.002	7.139	7.770	7.987	8.112						
50	4.243	50	5.128	5.753	6.052	6.408	6.636	6.791	6.901	7.079	7.215	7.845	8.060	8.183	8.211					
51	4.321	51	5.205	5.830	6.129	6.485	6.713	6.868	6.978	7.155	7.292	7.920	8.134	8.254	8.293					
52	4.398	52	5.282	5.907	6.206	6.562	6.790	6.945	7.055	7.232	7.369	7.996	8.207	8.325	8.361					
53	4.476	53	5.360	5.985	6.283	6.639	6.867	7.022	7.132	7.309	7.446	8.072	8.281	8.397	8.430					
54	4.554	54	5.438	6.062	6.361	6.717	6.945	7.100	7.209	7.386	7.523	8.148	8.356	8.470	8.499					
55	4.632	55	5.515	6.140	6.438	6.794	7.022	7.177	7.287	7.464	7.600	8.225	8.430	8.543	8.569					
56	4.711	56	5.594	6.218	6.516	6.872	7.100	7.255	7.364	7.541	7.677	8.301	8.505	8.616	8.639					
57	4.789	57	5.672	6.296	6.594	6.950	7.178	7.333	7.442	7.619	7.755	8.378	8.581	8.690	8.710					
58	4.868	58	5.750	6.375	6.763	7.028	7.256	7.411	7.520	7.697	7.833	8.455	8.656	8.763	8.782					
59	4.947	59	5.829	6.453	6.751	7.107	7.334	7.489	7.598	7.775	7.910	8.532	8.732	8.838	8.854					
60	5.026	60	5.908	6.532	6.830	7.185	7.413	7.567	7.676	7.853	7.989	8.609	8.808	8.912	8.927	8.891				

TABLE V C — *Continued*

n	$H_1(n)$	x	$G_1(m,n)$																	
			$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$	$m = 9$	$m = 10$	$m = 20$	$m = 30$	$m = 40$	$m = 50$	$m = 60$	$m = 70$	$m = 80$	$m = 90$	$m = 100$
			$x = 1$	$x = 1$	$x = 2$	$x = 2$	$x = 2$	$x = 3$	$x = 4$	$x = 4$	$x = 4$	$x = 8$	$x = 14$	$x = 16$	$x = 18$	$x = 28$	$x = 32$	$x = 32$	$x = 32$	$x = 35$
61	5.105	61	5.987	6.611	6.909	7.264	7.491	7.646	7.755	7.931	8.067	8.687	8.884	8.987	8.999	8.972				
62	5.185	62	6.066	6.690	6.988	7.343	7.570	7.724	7.833	8.010	8.145	8.765	8.961	9.062	9.073	9.042				
63	5.265	63	6.145	6.769	7.067	7.422	7.649	7.803	7.912	8.088	8.224	8.842	9.038	9.137	9.146	9.113				
64	5.345	64	6.225	6.849	7.146	7.501	7.728	7.883	7.991	8.167	8.303	8.920	9.116	9.213	9.220	9.184				
65	5.425	65	6.305	6.929	7.226	7.581	7.808	7.962	8.070	8.247	8.382	8.999	9.193	9.289	9.294	9.256				
66	5.506	66	6.385	7.009	7.306	7.661	7.887	8.041	8.150	8.326	8.461	9.077	9.270	9.365	9.369	9.328				
67	5.587	67	6.466	7.089	7.386	7.741	7.967	8.121	8.230	8.406	8.541	9.156	9.348	9.442	9.444	9.401				
68	5.667	68	6.546	7.170	7.467	7.821	8.048	8.201	8.310	8.485	8.620	9.234	9.426	9.518	9.520	9.475				
69	5.748	69	6.627	7.250	7.547	7.901	8.128	8.282	8.390	8.565	8.700	9.313	9.504	9.595	9.595	9.548				
70	5.830	70	6.708	7.331	7.628	7.982	8.209	8.362	8.470	8.646	8.781	9.393	9.583	9.673	9.671	9.622	9.560			
71	5.911	71	6.789	7.412	7.709	8.063	8.289	8.443	8.551	8.726	8.861	9.472	9.661	9.750	9.747	9.697	9.641			
72	5.992	72	6.870	7.493	7.790	8.144	8.370	8.524	8.632	8.807	8.941	9.551	9.740	9.828	9.824	9.772	9.714			
73	6.074	73	6.952	7.575	7.871	8.225	8.451	8.605	8.713	8.888	9.022	9.631	9.819	9.906	9.900	9.847	9.786			
74	6.155	74	7.033	7.656	7.953	8.307	8.533	8.686	8.794	8.969	9.103	9.711	9.898	9.984	9.977	9.922	9.859			
75	6.237	75	7.115	7.737	8.034	8.388	8.614	8.767	8.875	9.050	9.184	9.791	9.977	10.063	10.054	9.998	9.933			
76	6.319	76	7.196	7.819	8.115	8.469	8.695	8.848	8.956	9.131	9.265	9.872	10.057	10.142	10.132	10.074	10.007			
77	6.401	77	7.278	7.901	8.197	8.551	8.777	8.930	9.037	9.212	9.347	9.952	10.136	10.220	10.209	10.150	10.081			
78	6.483	78	7.360	7.983	8.279	8.633	8.858	9.011	9.119	9.294	9.428	10.033	10.216	10.299	10.287	10.227	10.156			
78	6.565	79	7.442	8.064	8.361	8.714	8.940	9.093	9.201	9.375	9.510	10.113	10.296	10.378	10.365	10.304	10.231			
80	6.647	80	7.524	8.146	8.443	8.796	9.022	9.175	9.282	9.457	9.591	10.194	10.376	10.458	10.444	10.381	10.307	10.221		
81	6.729	81	7.606	8.229	8.525	8.878	9.104	9.257	9.364	9.539	9.673	10.275	10.456	10.537	10.522	10.458	10.383	10.302		
82	6.812	82	7.688	8.311	8.607	8.960	9.186	9.339	9.446	9.621	9.755	10.357	10.537	10.617	10.601	10.536	10.459	10.376		
83	6.895	83	7.771	8.393	8.689	9.043	9.268	9.421	9.528	9.703	9.837	10.438	10.617	10.697	10.680	10.613	10.535	10.450		
84	6.977	84	7.853	8.476	8.772	9.125	9.351	9.503	9.611	9.785	9.919	10.520	10.698	10.777	10.759	10.691	10.612	10.525		
85	7.060	85	7.936	8.559	8.854	9.208	9.433	9.586	9.693	9.867	10.001	10.601	10.779	10.857	10.839	10.770	10.689	10.600		
86	7.143	86	8.019	8.641	8.937	9.290	9.516	9.668	9.775	9.950	10.084	10.683	10.860	10.937	10.918	10.848	10.766	10.676		
87	7.227	87	8.102	8.724	9.020	9.373	9.599	9.751	9.858	10.032	10.166	10.765	10.941	11.017	10.998	10.927	10.844	10.752		
88	7.310	88	8.185	8.807	9.103	9.456	9.681	9.834	9.941	10.115	10.249	10.847	11.022	11.098	11.078	11.006	10.922	10.828		
89	7.393	89	8.268	8.890	9.186	9.539	9.764	9.917	10.024	10.198	10.331	10.929	11.104	11.179	11.158	11.085	10.999	10.904		
90	7.477	90	8.352	8.974	9.269	9.622	9.848	10.000	10.107	10.281	10.414	11.011	11.186	11.260	11.238	11.164	11.078	10.981	10.881	

91	7.561	91	8.435	9.057	9.353	9.706	9.931	10.083	10.190	10.364	10.497	11.094	11.267	11.341	11.318	11.244	11.156	11.058	10.962	
92	7.644	92	8.519	9.141	9.436	9.789	10.014	10.166	10.273	10.447	10.580	11.176	11.349	11.422	11.399	11.323	11.235	11.135	11.037	
93	7.728	93	8.603	9.225	9.520	9.873	10.098	10.250	10.356	10.530	10.664	11.259	11.431	11.503	11.479	11.403	11.313	11.213	11.113	
94	7.813	94	8.687	9.308	9.604	9.956	10.181	10.333	10.440	10.614	10.747	11.342	11.514	11.585	11.560	11.483	11.393	11.291	11.189	
95	7.897	95	8.771	9.392	9.688	10.040	10.265	10.417	10.524	10.697	10.831	11.425	11.596	11.666	11.641	11.564	11.472	11.369	11.266	
96	7.981	96	8.855	9.477	9.772	10.124	10.349	10.501	10.607	01.781	10.914	11.508	11.678	11.748	11.722	11.644	11.551	11.447	11.342	
97	8.066	97	8.939	9.561	9.856	10.208	10.433	10.585	10.691	10.865	10.998	11.591	11.761	11.830	11.803	11.725	11.631	11.526	11.420	
98	8.151	98	9.024	9.645	9.940	10.293	10.517	10.669	10.775	10.949	11.082	11.674	11.844	11.912	11.885	11.805	11.711	11.605	11.497	
99	8.235	99	9.108	9.730	10.025	10.377	10.602	10.753	10.860	11.033	11.166	11.758	11.927	11.994	11.966	11.886	11.791	11.684	11.575	
100	8.320	100	9.193	9.815	10.109	10.462	10.686	10.838	10.944	11.118	11.250	11.841	12.010	12.077	12.048	11.967	11.871	11.763	11.653	11.547

VALUES OF THE NEXT TEST-GROUP SIZE x UNDER PROCEDURE R_1 FOR $q = 0.99$ AND $m = 2(1)100$

	1	2	3	4	5	6	7	8	9	10
0+		1	1	2	2	2	3	4	4	4
10+	4	4	5	6	7	8	8	8	8	8
20+	8	8	8	8	9	10	11	12	13	14
30+	15	16	16	16	16	16	16	16	16	16
40+	16	16	16	16	16	16	16	16	17	18
50+	19	20	21	22	23	24	25	26	27	28
60+	29	30	31	32	32	32	32	32	32	32
70+	32	32	32	32	32	32	32	32	32	32
80+	32	32	32	32	32	32	32	32	32	32
90+	32	32	32	32	32	32	33	34	35	35

TABLE VI— EXPECTED NUMBER OF TESTS REQUIRED FOR PROCEDURE R_2

The integer below q^ν opposite $H_2(n)$ is the coefficient of q^ν in the polynomial formula for $H_2(n)$.

	q-interval	x	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}
$H_2(2)$	0.000 to 0.618	1	2												
	0.618 to 1.000	2	3	−1	−1										
$H_2(3)$	0.000 to 0.618	1	3												
	0.618 to 0.755	2	5	−3	−1	1									
	0.755 to 1.000	3	5	−2	−1	−1									
$H_2(4)$	0.000 to 0.618	1	4												
	0.618 to 0.755	2	7	−5	0	1	−1								
	0.755 to 0.819	3	7	−3	−2	−2	2								
	0.819 to 1.000	4	8	−4	−2	−1									
$H_2(5)$	0.000 to 0.618	1	5												
	0.618 to 0.755	2	9	−7	1	0	−1	1							
	0.755 to 0.819	3	9	−4	−3	−2	3	−1							
	0.819 to 0.857	4	11	−7	−2	−1	−1	2							
	0.857 to 1.000	5	11	−7	−2	0	0	−1							
$H_2(6)$	0.000 to 0.618	1	6												
	0.618 to 0.755	2	11	−9	2	−1	0	1	−1						
	0.755 to 0.819	3	11	−5	−4	−2	4	−1	−1						
	0.819 to 0.857	4	14	−10	−2	−1	−1	3	−1						
	0.857 to 0.881	5	14	−10	−2	1	−1	−3	3						
	0.881 to 0.885	6	14	−10	−1	0	−1	−1							
	0.885 to 1.000	6	14	−9	−2	−1	0	−1							
$H_2(7)$	0.000 to 0.618	1	7												
	0.618 to 0.755	2	13	−11	3	−2	1	0	−1	1					
	0.755 to 0.819	3	13	−6	−5	−2	5	−2	−2	2					
	0.819 to 0.857	4	17	−13	−2	−1	−1	4	−1	−1					
	0.857 to 0.881	5	17	−13	−2	2	−2	−4	5	−1					
	0.881 to 0.885	6	17	−13	0	−1	−1	−1	−2	3					
	0.885 to 0.899⁻	6	17	−11	−3	−2	2	−2	−2	3					
	0.899⁻ to 1.000	7	17	−11	−3	−1	1	−2							
$H_2(8)$	0.000 to 0.618	1	8												
	0.618 to 0.755	2	15	−13	4	−3	2	−1	0	1	−1				
	0.755 to 0.819	3	15	−7	−6	−2	6	−3	−2	3	−1				
	0.819 to 9.857	4	20	−16	−2	−1	0	4	−2	−1					
	0.857 to 0.881	5	20	−16	−2	3	−3	−5	7	−1	−1				
	0.881 to 0.885	6	20	−16	1	−2	−1	−1	−3	5	−1				
	0.885 to 0.899⁻	6	20	−13	−4	−3	4	−3	−3	5	−1				
	0.899⁻ to 0.912	7	20	−13	−4	−1	1	−2	0	−2	3				
	0.912 to 0.932	8	20	−13	−4	−1	1	−2	1	0	−1				
	0.932 to 1.000	8	20	−14	−4	−1	1	−2							
$H_2(9)$	0.000 to 0.618	1	9												
	0.618 to 0.755	2	17	−15	5	−4	3	−2	1	0	−1	1			
	0.755 to 0.819	3	17	−8	−7	−2	7	−4	−2	4	−1	−1			
	0.819 to 0.857	4	23	−19	−2	−1	1	3	−2	−1	−1	2			
	0.857 to 0.881	5	23	−19	−2	4	−4	−5	8	−2	−1				
	0.881 to 0.890	6	23	−19	2	−3	−1	−1	−4	7	−1	−1			
	0.890 to 0.899⁻	6	23	−15	−5	−4	6	−4	−4	7	−1	−1			
	0.899⁻ to 0.912	7	23	−15	−5	−1	1	−2	0	−3	5	−1			
	0.912 to 0.922	8	23	−15	−5	−1	1	−2	2	−1	−4	4			
	0.922 to 0.938	9	24	−16	−5	−1	1	−2	1	0	−1				
	0.938 to 1.000	9	25	−18	−4	−1	1	−2	0	1	0	−1			

TABLE VI — *Continued*

	q-interval	x	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}	q^{12}
$H_2(10)$	0.000 to 0.618	1	10												
	0.618 to 0.755	2	19	−17	6	−5	4	−3	2	−1	0	1	−1		
	0.755 to 0.819	3	19	−9	−8	−2	8	−5	−2	5	−2	−2	2		
	0.819 to 0.857	4	26	−22	−2	−1	2	2	−2	−1	−1	3	−1		
	0.857 to 0.881	5	26	−22	−2	5	−5	−5	8	−2	0	0	−1		
	0.881 to 0.890	6	26	−22	3	−4	−1	−1	−4	8	−2	−1			
	0.890 to 0.899^-	6	26	−17	−6	−5	8	−5	−4	8	−2	−1			
	0.899^- to 0.912	7	26	−17	−6	−1	1	−2	0	−4	7	−1	−1		
	0.912 to 0.922	8	26	−17	−6	−1	1	−2	3	−2	−6	7	−1		
	0.922 to 0.930	9	28	−20	−5	−1	1	−2	1	1	−2	−3	4		
	0.930 to 0.938	10	28	−20	−5	−1	1	−1	1	−1	−1				
	0.938 to 0.960	10	29	−22	−4	−1	1	−1	0	0	0	−1			
	0.960 to 1.000	10	29	−22	−4	0	1	−3	0	1	0	−1			
$H_2(11)$	0.000 to 0.618	1	11												
	0.618 to 0.755	2	21	−19	7	−6	5	−4	3	−2	1	0	−1	1	
	0.755 to 0.819	3	21	−10	−9	−2	9	−6	−2	6	−3	−2	3	−1	
	0.819 to 0.857	4	29	−25	−2	−1	3	1	−2	−1	−1	4	−1	−1	
	0.857 to 0.881	5	29	−25	−2	6	−6	−5	8	−2	1	−1	−3	3	
	0.881 to 0.890	6	29	−25	4	−5	−1	−1	−4	8	−2	0	0	−1	
	0.890 to 0.899^-	6	29	−19	−7	−6	10	−6	−4	8	−2	0	0	−1	
	0.899^- to 0.912	7	29	−19	−7	−1	1	−2	0	−4	8	−2	−1		
	0.912 to 0.922	8	29	−19	−7	−1	1	−2	4	−3	−8	10	−1	−1	
	0.922 to 0.930	9	32	−24	−5	−1	1	−2	1	2	−3	−5	7	−1	
	0.930 to 0.936	10	32	−24	−5	−1	1	0	0	−2	1	−1	3	4	
	0.936 to 0.938	11	32	−24	−5	0	1	−2	1	−1	−1				
	0.938 to 0.960	11	33	−26	−4	0	1	−2	0	0	0	−1			
	0.960 to 1.000	11	33	−26	−4	1	0	−4	2	1	−1	−1			
$H_2(12)$	0.000 to 0.618	1	12												
	0.618 to 0.755	2	23	−21	8	−7	6	−5	4	−3	2	−1	0	1	−1
	0.755 to 0.819	3	23	−11	−10	−2	10	−7	−2	7	−4	−2	4	−1	−1
	0.819 to 0.857	4	32	−28	−2	−1	4	0	−2	−1	0	4	−2	−1	
	0.857 to 0.881	5	32	−28	−2	7	−7	−5	8	−2	2	−2	−4	5	−1
	0.881 to 0.890	6	32	−28	5	−6	−1	−1	−4	8	−1	0	−1	−1	
	0.890 to 0.899^-	6	32	−21	−8	−7	12	−7	−4	9	−2	−1	0	−1	
	0.899^- to 0.912	7	32	−21	−8	−1	1	−2	0	−4	8	−2	0	0	−1
	0.912 to 0.922	8	32	−21	−8	−1	1	−2	5	−4	−9	12	−2	−1	
	0.922 to 0.930	9	36	−28	−5	−1	1	−2	1	3	−4	−7	10	−1	−1
	0.930 to 0.936	10	36	−28	−5	−1	1	1	−1	−3	3	−2	−5	7	−1
	0.936 to 0.938	11	36	−28	−5	1	0	−3	3	−2	−2	2	−1	−3	4
	0.938 to 0.941	11	37	−30	−4	1	0	−3	2	−1	−1	1	−1	−3	4
	0.941 to 0.960	12	37	−30	−4	1	0	−3	2	−1	0	0	−1		
	0.960 to 0.972	12	37	−30	−4	2	−1	−5	4	0	−1	0	−1		
	0.972 to 1.000	12	37	−29	−4	0	0	−4	2	1	−1	−1			

The exponential symbols +, − indicate only the relative magnitude of two different roots that are equal to three decimal places (i.e., $a^- < a^+$).

TABLE VII — THE DIVIDING POINTS BETWEEN x AND $x + 1$ FOR THE INFORMATION PROCEDURE R_2

G-Situation

x	m									
	4	6	7	8	9	11	12	13	14	16
1	0.7549	0.6518	0.6369	0.6289	0.6245^{-}	0.6204	0.6195^{-}	0.6189	0.6186	0.6182
2		0.8899	0.8376	0.8087	0.7913	0.7728	0.7677	0.7642	0.7617	0.7586
3				0.9378	0.9016	0.8631	0.8524	0.8446	0.8388	0.8313
4						0.9340	0.9160	0.9031	0.8935^{-}	0.8806
5							0.9723	0.9528	0.9384	0.9193
6									0.9796	0.9529
7										0.9844

G-Situation

m										x
10	20	30	40	50	60	70	80	90	100	
0.6219	0.6181	0.6180	0.6180	0.6180	0.6180	0.6180	0.6180	0.6180	0.6180	1
0.7802	0.7560	0.7549	0.7549	0.7549	0.7549	0.7549	0.7549	0.7549	0.7549	2
0.8786	0.8241	0.8198	0.8193	0.8192	0.8192	0.8192	0.8192	0.8192	0.8192	3
0.9601	0.8677	0.8587	0.8571	0.8568	0.8567	0.8567	0.8567	0.8567	0.8567	4
	0.8999	0.8854	0.8823	0.8816	0.8814	0.8813	0.8813	0.8813	0.8813	5
	0.9262	0.9056	0.9008	0.8993	0.8989	0.8987	0.8987	0.8987	0.8987	6
	0.9490	0.9218	0.9150	0.9129	0.9121	0.9118	0.9117	0.9116	0.9116	7
	0.9699	0.9355^{-}	0.9267	0.9236	0.9225^{-}	0.9220	0.9218	0.9217	0.9216	8
	0.9900	0.9474	0.9364	0.9325^{+}	0.9309	0.9302	0.9299	0.9297	0.9296	9
		0.9582	0.9449	0.9400	0.9380	0.9370	0.9365^{+}	0.9363	0.9362	10
		0.9682	0.9524	0.9466	0.9440	0.9428	0.9422	0.9418	0.9417	11
		0.9776	0.9593	0.9524	0.9493	0.9478	0.9470	0.9466	0.9463	12
		0.9867	0.9655^{+}	0.9576	0.9540	0.9522	0.9512	0.9507	0.9504	13
		0.9956	0.9714	0.9624	0.9582	0.9561	0.9549	0.9543	0.9539	14
46	0.9991		0.9770	0.9668	0.9620	0.9596	0.9583	0.9575^{-}	0.9570	15
45	0.9982		0.9823	0.9709	0.9656	0.9628	0.9613	0.9604	0.9598	16
44	0.9973		0.9874	0.9747	0.9689	0.9658	0.9640	0.9630	0.9623	17
43	0.9964		0.9925^{-}	0.9784	0.9719	0.9685^{-}	0.9665^{+}	0.9654	0.9646	18
42	0.9955^{+}		0.9975^{-}	0.9820	0.9748	0.9710	0.9689	0.9676	0.9667	19
41	0.9946	0.9989		0.9854	0.9776	0.9734	0.9711	0.9696	0.9687	20
40	0.9937	0.9978		0.9887	0.9802	0.9757	0.9731	0.9715^{-}	0.9705^{-}	21
39	0.9928	0.9967		0.9920	0.9827	0.9779	0.9750^{+}	0.9733	0.9721	22
38	0.9919	0.9956		0.9952	0.9852	0.9799	0.9769	0.9750^{-}	0.9737	23
37	0.9909	0.9944		0.9984	0.9876	0.9819	0.9786	0.9765^{+}	0.9752	24
36	0.9899	0.9933	0.9986		0.9899	0.9838	0.9803	0.9780	0.9766	25
35	0.9889	0.9921	0.9972		0.9922	0.9856	0.9819	0.9795^{-}	0.9779	26
34	0.9879	0.9910	0.9957		0.9944	0.9874	0.9834	0.9809	0.9792	27
33	0.9869	0.9898	0.9943		0.9967	0.9892	0.9849	0.9822	0.9804	28
32	0.9858	0.9886	0.9928		0.9989	0.9909	0.9863	0.9835^{-}	0.9816	29
31	0.9847	0.9873	0.9914	0.9981		0.9926	0.9877	0.9847	0.9827	30
30	0.9836	0.9860	0.9899	0.9962		0.9943	0.9891	0.9859	0.9838	31
29	0.9824	0.9847	0.9883	0.9943		0.9959	0.9904	0.9871	0.9848	32
28	0.9812	0.9834	0.9868	0.9924		0.9976	0.9918	0.9882	0.9858	33
27	0.9800	0.9820	0.9852	0.9904		0.9992	0.9931	0.9893	0.9868	34
26	0.9786	0.9805^{+}	0.9835^{+}	0.9884	0.9974		0.9943	0.9904	0.9878	35
25	0.9773	0.9790	0.9818	0.9864	0.9947		0.9956	0.9914	0.9887	36
24	0.9758	0.9775^{-}	0.9800	0.9843	0.9920		0.9969	0.9925^{-}	0.9896	37
23	0.9743	0.9758	0.9782	0.9822	0.9893		0.9981	0.9935^{+}	0.9905^{+}	38
22	0.9727	0.9741	0.9763	0.9800	0.9866		0.9994	0.9945^{+}	0.9914	39
21	0.9709	0.9722	0.9742	0.9776	0.9837	0.9961		0.9955^{+}	0.9922	40
20	0.9691	0.9702	0.9721	0.9752	0.9808	0.9921		0.9965^{+}	0.9931	41
19	0.9671	0.9681	0.9698	0.9727	0.9778	0.9881		0.9975^{+}	0.9939	42
18	0.9649	0.9659	0.9674	0.9700	0.9746	0.9840		0.9985^{+}	0.9948	43
17	0.9626	0.9634	0.9648	0.9671	0.9713	0.9798		0.9995^{+}	0.9956	44
16	0.9600	0.9608	0.9619	0.9640	0.9678	0.9754	0.9935^{-}		0.9964	45
15	0.9572	0.9578	0.9588	0.9606	0.9640	0.9708	0.9869		0.9972	46
14	0.9541	0.9546	0.9554	0.9570	0.9599	0.9659	0.9802		0.9980	47
13	0.9505^{-}	0.9509	0.9516	0.9529	0.9555^{-}	0.9607	0.9733		0.9988	48
12	0.9464	0.9467	0.9473	0.9484	0.9506	0.9551	0.9660		0.9996	49
11	0.9417	0.9420	0.9424	0.9433	0.9451	0.9489	0.9582	0.9872		
10	0.9362	0.9364	0.9367	0.9374	0.9388	0.9419	0.9498	0.9742		
9	0.9297	0.9298	0.9300	0.9305^{-}	0.9315^{+}	0.9340	0.9404	0.9608		
8	0.9216	0.9217	0.9218	0.9222	0.9229	0.9248	0.9298	0.9463		
7	0.9116	0.9116	0.9117	0.9119	0.9124	0.9137	0.9174	0.9303		
6	0.8987	0.8987	0.8987	0.8988	0.8991	0.8999	0.9024	0.9120	0.9645^{-}	
5	0.8813	0.8813	0.8813	0.8813	0.8814	0.8818	0.8834	0.8898	0.9276	
4	0.8567	0.8567	0.8567	0.8567	0.8567	0.8569	0.8576	0.8612	0.8862	
3	0.8192	0.8192	0.8192	0.8192	0.8192	0.8192	0.8194	0.8209	0.8346	
2	0.7549	0.7549	0.7549	0.7549	0.7549	0.7549	0.7549	0.7552	0.7599	
1	0.6180	0.6180	0.6180	0.6180	0.6180	0.6180	0.6180	0.6180	0.6184	0.6823
x	95	85	75	65	55	45	35	25	15	5
	m									

TABLE VII — *Continued*

H-Situation

x	q	x	q
1	0.6180	51	0.9866
2	0.7549	52	0.9869
3	0.8192	53	0.9871
4	0.8567	54	0.9874
5	0.8813	55	0.9876
6	0.8987	56	0.9878
7	0.9116	57	0.9880
8	0.9216	58	0.9882
9	0.9296	59	0.9884
10	0.9361	60	0.9886
11	0.9415^-	61	0.9888
12	0.9460	62	0.9890
13	0.9499	63	0.9891
14	0.9533	64	0.9893
15	0.9563	65	0.9895^-
16	0.9588	66	0.9896
17	0.9612	67	0.9898
18	0.9632	68	0.9899
19	0.9651	69	0.9901
20	0.9667	70	0.9902
21	0.9683	71	0.9904
22	0.9697	72	0.9905^-
23	0.9709	73	0.9906
24	0.9721	74	0.9907
25	0.9732	75	0.9909
26	0.9742	76	0.9910
27	0.9751	77	0.9911
28	0.9760	78	0.9912
29	0.9768	79	0.9913
30	0.9775^+	80	0.9914
31	0.9782	81	0.9915^+
32	0.9789	82	0.9916
33	0.9795^+	83	0.9917
34	0.9801	84	0.9918
35	0.9807	85	0.9919
36	0.9812	86	0.9920
37	0.9817	87	0.9921
38	0.9822	88	0.9922
39	0.9826	89	0.9923
40	0.9830	90	0.9924
41	0.9834	91	0.9925^-
42	0.9838	92	0.9925^+
43	0.9842	93	0.9926
44	0.9845^+	94	0.9927
45	0.9849	95	0.9928
46	0.9852	96	0.9928
47	0.9855^+	97	0.9929
48	0.9858	98	0.9930
49	0.9861	99	0.9931
50	0.9864	100	0.9931

The exponents $^+$ and $^-$ are to be used for rounding in the usual manner.

TABLE VIII — POLYNOMIAL EXPRESSIONS FOR $F_2^*(m)$ FOR PROCEDURE R_2

m	Coefficient of												q-interval	Next Test Group Size
	1	q	q^2	q^3	q^4	q^5	q^6	q^7	q^8	q^9	q^{10}	q^{11}		
2	1	1											$0.000 \leqq q \leqq 1.000$	1
3	1	2	2										$0.000 \leqq q \leqq 1.000$	1
4	2	2	2	2									$0.755 \leqq q \leqq 1.000$	2
5	2	2	2	3	3								$0.682 < q \leqq 1.000$	2
6	2	2	3	3	3	3							$0.652 < q \leqq 0.890$	2
	2	3	3	2	3	3							$0.890 < q \leqq 1.000$	3
7	2	3	3	3	3	3	3						$0.838 \leqq q \leqq 1.000$	3
8	2	3	3	3	3	3	4	4					$0.809 < q < 0.938$	3
	3	3	3	3	3	3	3	3					$0.938 \leqq q \leqq 1.000$	4
9	3	3	3	3	3	3	3	4	4				$0.902 \leqq q \leqq 1.000$	4
10	3	3	3	3	3	4	4	3	4	4			$0.879 < q \leqq 0.960$	4
	3	3	3	4	4	3	3	3	4	4			$0.960 < q \leqq 1.000$	5
11	3	3	3	4	4	3	4	4	3	4	4		$0.934 \leqq q \leqq 1.000$	5
12	3	3	3	4	4	3	4	4	4	4	4	4	$0.916 < q \leqq 0.972$	5
	3	4	4	3	4	4	3	4	4	3	4	4	$0.972 < q \leqq 1.000$	6