Theories for Toll Traffic Engineering in the U. S. A.*

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Present toll trunk traffic engineering practices in the United States are reviewed, and various congestion formulas compared with data obtained on long distance traffic. Customer habits upon meeting busy channels are noted and a theory developed describing the probable result of permitting subscribers to have direct dialing access to high delay toll trunk groups.

Continent-wide automatic alternate routing plans are described briefly, in which near no-delay service will permit direct customer dialing. The presence of non-random overflow traffic from high usage groups complicates the estimation of correct quantities of alternate paths. Present methods of solving graded multiple problems are reviewed and found unadaptable to the variety of trunking arrangements occurring in the toll plan.

Evidence is given that the principal fluctuation characteristics of overflow-type of non-random traffic are described by their mean and variance. An approximate probability distribution of simultaneous calls for this kind of non-random traffic is developed, and found to agree satisfactorily with theoretical overflow distributions and those seen in traffic simulations.

A method is devised using "equivalent random" traffic, which has good loss predictive ability under the "lost calls cleared" assumption, for a diverse field of alternate route trunking arrangements. Loss comparisons are made with traffic simulation results and with observations in exchanges.

Working curves are presented by which multi-alternate route trunking systems can be laid out to meet economic and grade of service criteria. Examples of their application are given.

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1. INTRODUCTION

It has long been the stated aim of the Bell System to make it easily and economically possible for any telephone customer in the United States to reach any other telephone in the world. The principal effort in this direction by the American Telephone and Telegraph Company and its associated operating companies is, of course, confined to interconnecting the telephones in the United States, and to providing communication channels between North America and the other countries of the world. Since the United States is some 1500 miles from north to south and 3000 miles from east to west, to realize even the aim of fast and economical service between customers is a problem of great magnitude; it has engaged our planning engineers for many years.

There are now 52 million telephones in the United States, over 80 per cent of which are equipped with dials. Until quite recently most telephone users were limited in their direct dialing to the local or immediately surrounding areas and long distance operators were obliged to build up a circuit with the aid of a "through" operator at each switching point.

Both speed and economy dictated the automatic build-up of long toll circuits without the intervention of more than the originating toll operator. The development of the No. 4-type toll crossbar switching system with its ability to accept, translate, and pass on the necessary digits (or equivalent information) to the distant office made this method of operation possible and feasible. It was introduced during World War II, and now by means of it and allied equipment, 55 per cent of all long distance calls (over 25 miles) are completed by the originating operator.

As more elaborate switching and charge-recording arrangements were developed, particularly in metropolitan areas, the distances which customers themselves might dial measurably increased. This expansion of the local dialing area was found to be both economical and pleasing to the users. It was then not too great an effort to visualize customers dialing to all other telephones in the United States and neighboring countries, and perhaps ultimately across the sea.

The physical accomplishment of nationwide direct distance dialing which is now gradually being introduced has involved, as may well be imagined, an immense amount of advance study and fundamental planning. Adequate transmission and signalling with up to eight intertoll trunks in tandem, a nationwide uniform numbering plan simple enough to be used accurately and easily by the ordinary telephone caller, provision for automatic recording of who called whom and how long he talked, with subsequent automatic message accounting, are a few of many problems which have required solution. How they are being met is a romantic story beyond the scope of the present paper. The references given in the bibliography at the end contain much of the history as well as the plans for the future.

2. PRESENT TOLL TRAFFIC ENGINEERING PRACTICE

There are today approximately 116,000 intertoll trunks (over 25 miles in length) in the Bell System, apportioned among some 13,000 trunk groups. A small segment of the 2,600 toll centers which they interconnect is shown in Fig. 1. Most of these intertoll groups are presently traffic engineered to operate according to one of several so-called T-schedules: T-8, T-15, T-30, T-60, or T-120. The number following T (T for Toll) is

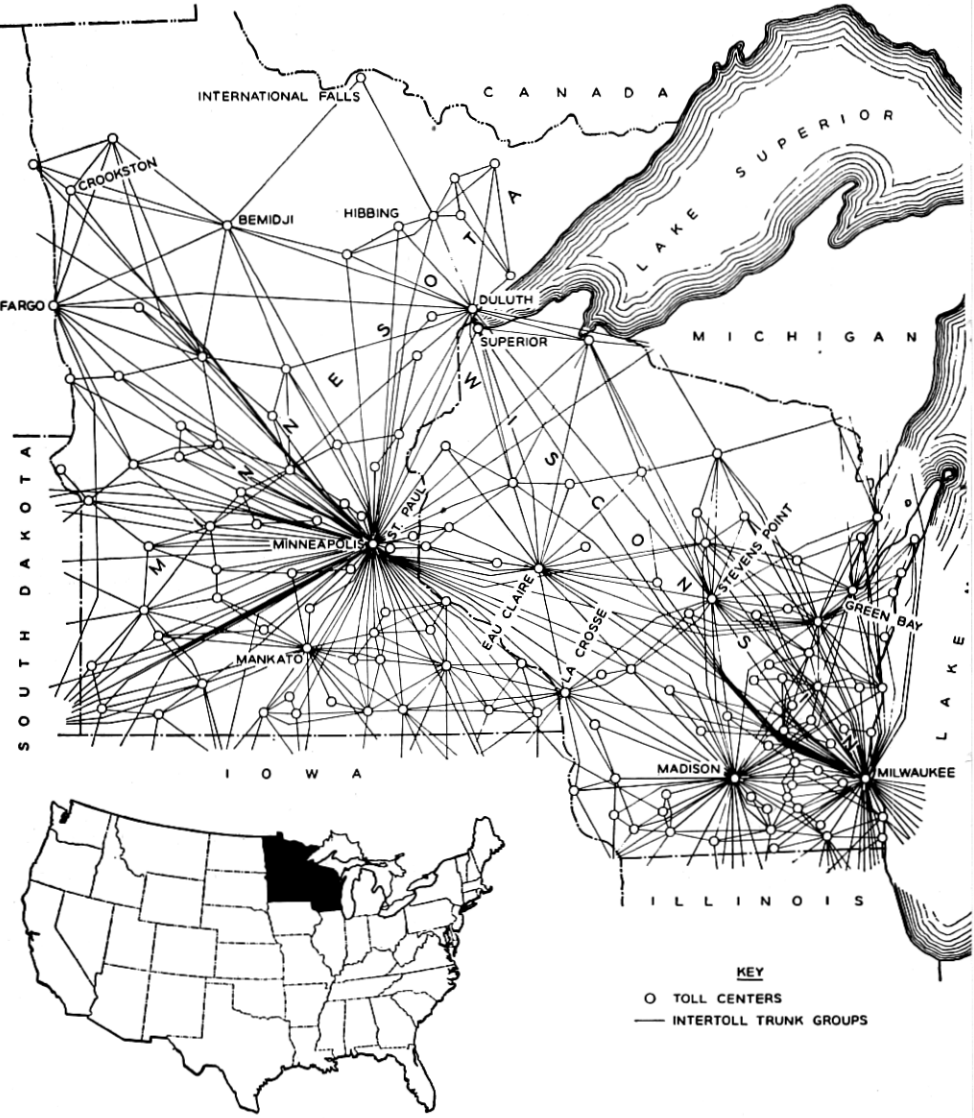

Fig. 1 — Principal intertoll trunk groups in Minnesota and Wisconsin.

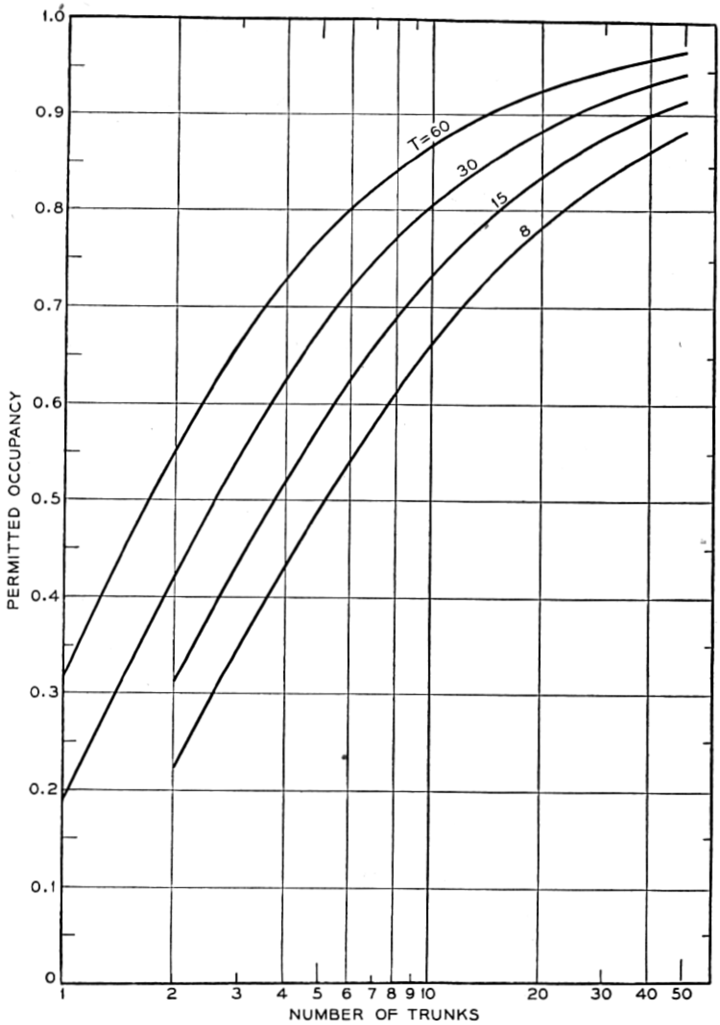

Fig. 2 — Permitted intertoll trunk occupancy for a 6.5-minute usage time per message.

the expected, or average, delay in seconds for calls to obtain an idle trunk in that group during the average Busy Season Busy Hour. In 1954 the system "average trunk speed" was approximately 30 seconds, resulting from operating the majority of the groups at a busy-hour trunking efficiency of 75 to 85 per cent in the busy season.

The T-engineering tables show permissible call minutes of use for a wide range of group sizes, and several selections of message holding times. They were constructed following summarization of many observations of load and resultant average delays on ringdown (non-dial) intertoll trunks.[1] Fig. 2 shows the permissible occupancy (efficiency) of various trunk group sizes for 6.5 minutes of use per message, for a variety of T-schedules. It is perhaps of some interest that the best fitting curves relating average delay and load were found to be the well-known Pollaczek-Crommelin delay curves for constant holding time — this in spite of the fact that the circuit holding times were far indeed from having a constant value.

A second, and probably not uncorrelated, observation was that the per cent "No-Circuit" (NC) reported on the operators' tickets showed consistently lower values than were measured on group-busy timing devices. Although not thoroughly documented, this disparity has generally been attributed to the reluctance of an operator to admit immediately the presence of an NC condition. She exhibits a certain tolerance (very difficult to measure) before actually recording a delay which would require her to adopt a prescribed procedure for the subsequent handling of the call.* There are then two measures of the No-Circuit condition which are of some interest, the "NC encountered" by operators, and the "NC existing" as measured by timing devices.

It has long been observed that the distribution of numbers n of simultaneous calls found on T-engineered ringdown intertoll groups is in remarkable agreement with the individual probability terms of the Erlang "lost calls" formula,

$$f(n) = \frac{\dfrac{a'^n e^{-a'}}{n!}}{\displaystyle\sum_{n=0}^{c} \frac{a'^n e^{-a'}}{n!}} \tag{1}$$

where c = number of paths in the group,

a' = an enhanced average load submitted such that $a'[1 - E_{1,c}(a')] = L$, the actual load carried, and

$E_{1,c}(a') = f(c)$ = Erlang loss probability (commonly called Erlang B in America).

An example of the agreement of observations with (1) is shown in Fig. 3, where the results of switch counts made some years ago on many ringdown circuit groups of size 3 are summarized. A wide range of "sub-

* Upon finding No-Circuit, an operator is instructed to try again in 30 seconds and 60 seconds (before giving an NC report to the customer), followed by additional attempts 5 minutes and 10 minutes later if necessary.

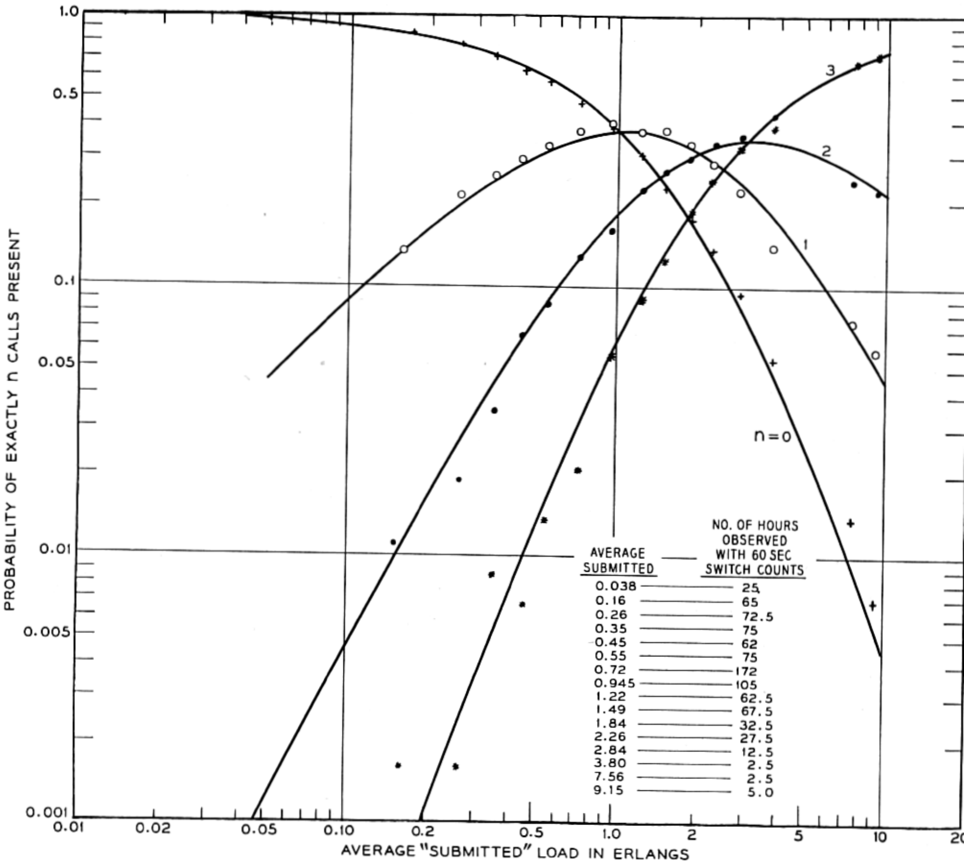

Fig. 3 — Distributions of simultaneous calls on three-trunk toll groups at Albany and Buffalo.

mitted" loads a' to produce the observed carried loads is required. On Fig. 4 are shown the corresponding comparisons of theory and observations for the proportions of time all paths are busy ("NC Existing") for 2-, 4-, 5-, 7-, and 9-circuit groups. Good agreement has also been observed for circuit groups up to 20 trunks. This has been found to be a stable relationship, in spite of the considerable variation in the actual practices in ringdown operation on the resubmission of delayed calls. Since the estimation of traffic loads and the subsequent administration of ringdown toll trunks has been performed principally by means of Group Busy Timers (which cumulate the duration of NC time), the Erlang relationship just described has been of great importance.

With the recent rapid increase in operator dialed intertoll groups, it might be expected that the above discrepancy between "% NC encountered" and "% NC existing" would disappear — for an operator now initiates each call unaware of the momentary state of the load on any particular intertoll group. By the use of peg count meters (which count calls offered) and overflow call counters, this change has in fact been observed to occur. Moreover, since the initial re-trial intervals are commonly fairly short (30 seconds) subsequent attempts tend to find some of the previous congestion still existing, so that the ratio of overflow to peg count readings now *exceeds* slightly the "% NC existing." This situation is illustrated in Fig. 5, which shows data taken on an operator-

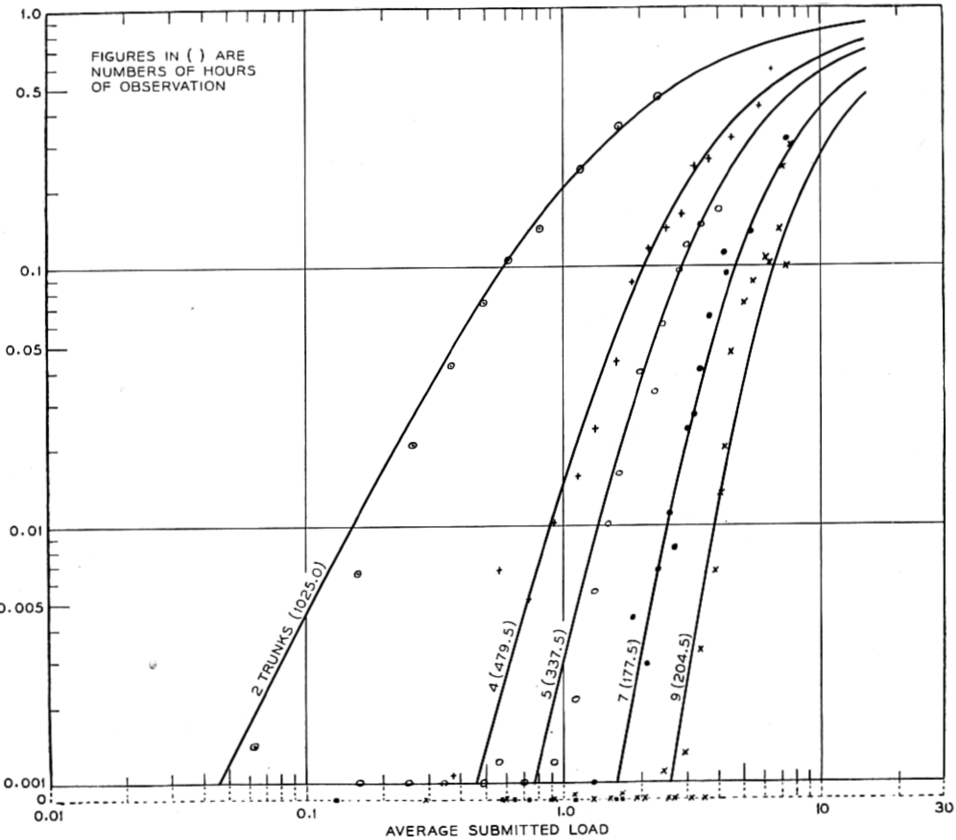

Fig. 4 — Observed proportions of time all trunks were busy on Albany and Buffalo groups of 2, 4, 5, 7, and 9 trunks.

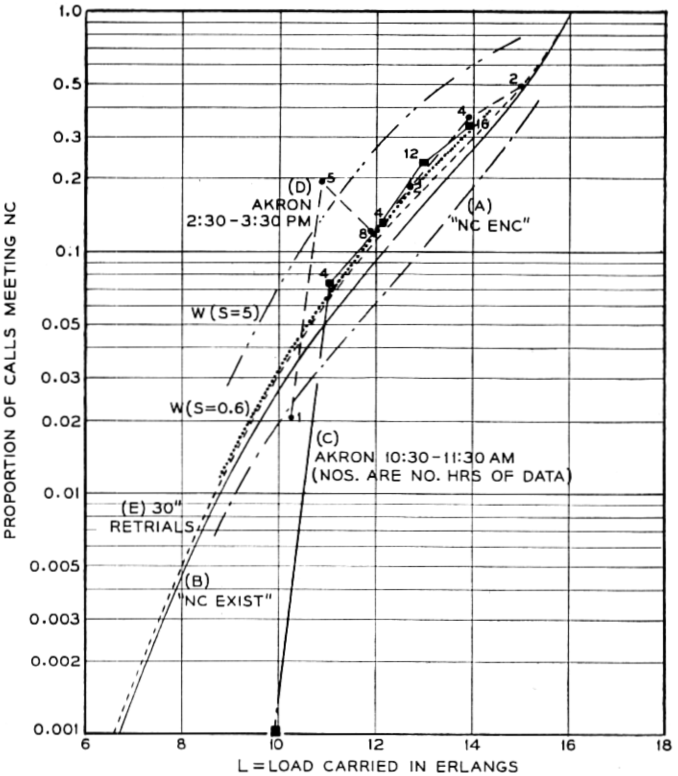

Fig. 5 — Comparison of NC data on a 16-trunk T-engineered toll group with various load versus NC theories.

dialed T-engineered group of 16 trunks between Newark, N. J., and Akron, Ohio. Curve A shows the empirically determined "NC encountered" relationship described above for ringdown operation; Curve B gives the corresponding theoretical "NC existing" values. Lines C and D give the operator-dialing results, for morning and afternoon busy hours. The observed points are now seen generally to be significantly above Curve B.*

At the same time as this change in the "NC encountered" was occurring, due to the introduction of operator toll dialing, there seems to have been little disturbance to the traditional relationship between load

* The observed point at 11 erlangs which is clearly far out of agreement with the remainder of the data was produced by a combination of high-trend hours and an hour in which an operator apparently made many re-trials in rapid succession.

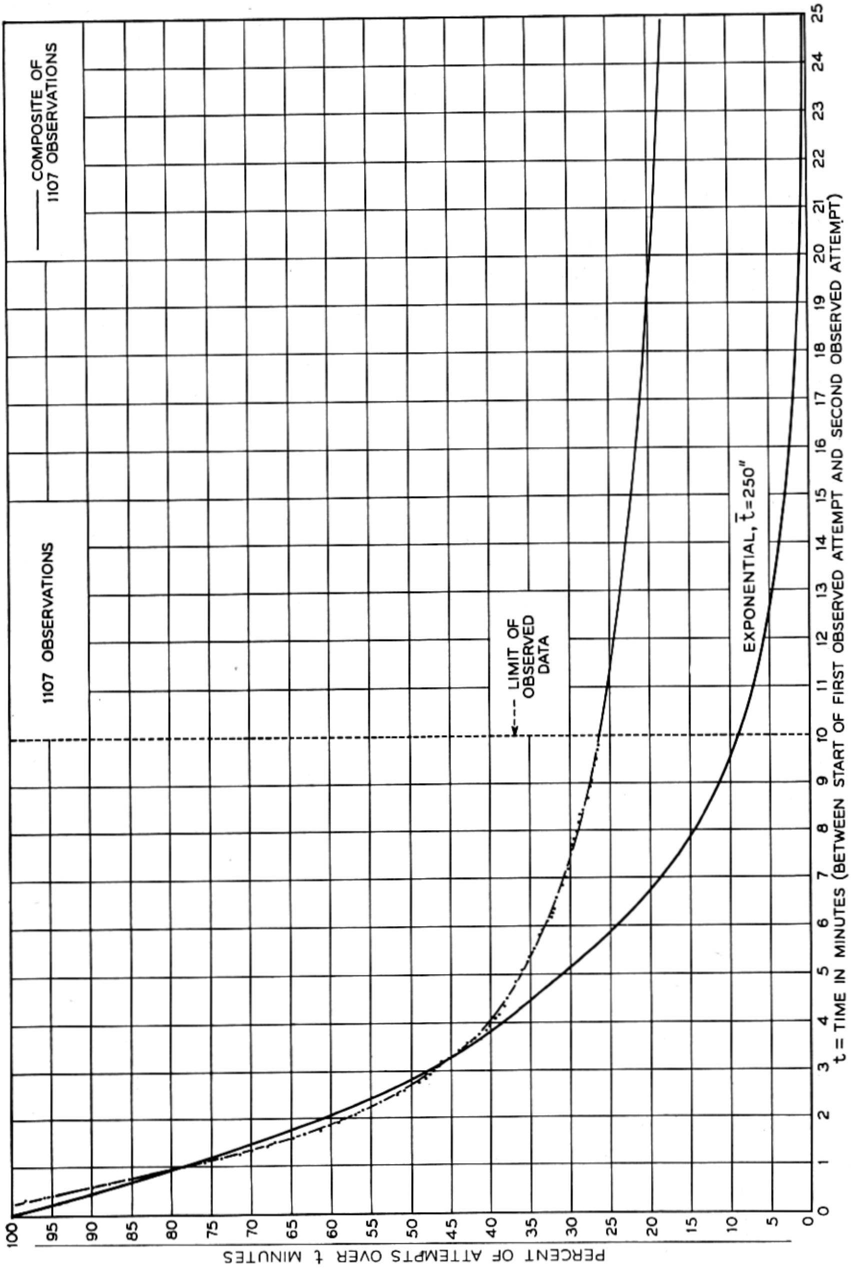

Fig. 6 — Time to second attempt after a subscriber receives a busy signal.

carried and "% NC existing." C. J. Truitt of the A.T. & T. Co. studied a number of operator-dialed T-engineered groups at Newark, New Jersey, in 1954 with a traffic usage recorder (TUR) and group-busy timers, and found the relationship of equation (1) still good. (This analysis has not been published.)

A study by Dr. L. Kosten has provided an estimate of the probability that when an NC condition has been found, it will also appear at a time τ later.[2] When this modification is made, the expected load-versus-NC relationship is shown by Curve E on Fig. 5. (The re-trial time here was taken as the operators' nominal 30 seconds; with 150-second circuit-use time the return is 0.2 holding time.) The observed NC's are seen to lie slightly above the E-curve. This could be explained either on the basis that Kosten's analysis is a lower limit, or that the operators did not strictly observe the 30-second return schedule, or, more probably, a combination of both.

3. CUSTOMERS DIALING ON GROUPS WITH CONSIDERABLE DELAY

It is not to be expected that customers could generally be persuaded to wait a designated constant or minimum re-trial time on their calls which meet the NC condition. Little actual experience has been accumulated on customers dialing long distance calls on high-delay circuits. However, it is plausible that they would follow the re-trial time distributions of customers making local calls, who encounter paths-busy or line-busy signals (between which they apparently do not usually distinguish). Some information on re-trial times was assembled in 1944 by C. Clos[3] by observing the action of customers who received the busy signal on 1,100 local calls in the City of New York. As seen in Fig. 6, the return times, after meeting "busy," exhibit a marked tendency toward the exponential distribution, after allowance for a minimum interval required for re-dialing.

An exponential distribution with average of 250 seconds has been fitted by eye on Fig. 6, to the earlier — and more critical — customer return times. This may seem an unexpectedly long wait in the light of individual experience; however it is probably a fair estimate, especially since, following the collection of the above data, it has become common practice for American operating companies in their instructional literature to advise customers receiving the busy signal to "hang up, wait a few minutes, and try again."

The mathematical representation of the situation assuming exponential return times is easily formulated. Let there be x actual trunks, and imagine y waiting positions, where y is so large that few calls are rejected.* Assume that the offered load is a erlangs, and that the calls have exponential conversation holding times of unit average duration. Finally let the average return time for calls which have advanced to the waiting positions, be $1/s$ times that of the unit conversation time. The statistical equilibrium equation can then be written for the probability $f(m, n)$ that m calls are in progress on the x trunks and n calls are waiting on the y storage positions:

$$\begin{aligned} f(m, n) = {} & af(m-1, n)\,dt + s(n+1)\,f(m-1, n+1)\,dt \\ & + (m+1)f(m+1, n)\,dt + af(x, n-1)\,dt\star \\ & + [1 - (a\star\star\star + sn\star\star)\,dt - m\,dt]f(m, n) \end{aligned} \tag{2}$$

where $0 \leqq m \leqq x$, $0 \leqq n \leqq y$, and the special limiting situations are recognized by:

$\star$ Include term only when $m = x$

$\star\star$ Omit sn when $m = x$

$\star\star\star$ Omit a when $m = x$ and $n = y$

Equation (2) reduces to

$$\begin{aligned} (a\star\star\star + sn\star\star + m)f(m, n) = {} & af(m-1, n) \\ & + s(n+1)f(m-1, n+1) \\ & + (m+1)f(m+1, n) + af(x, n-1)\star \end{aligned} \tag{3}$$

Solution of (3) is most easily effected for moderate values of x and y by first setting $f(x, y) = 1.000000$ and solving for all other $f(m, n)$ in terms of $f(x, y)$. Normalizing through $\sum_{m=0}^{x} \sum_{n=0}^{y} f(m, n) = 1.0$, then gives the entire $f(m, n)$ array.

The proportion of time "NC exists," will, of course be

$$\sum_{n=0}^{y} f(x, n) \tag{4}$$

and the load carried is

$$L = \sum_{m=0}^{x} \sum_{n=0}^{y} mf(m, n) \tag{5}$$

The proportion of call attempts meeting NC, including all re-trials

* The quantity y can also be chosen so that some calls are rejected, thus roughly describing those calls abandoned after the first attempt.

will be

$$W(x, a, s) = \frac{\text{Expected overflow calls per unit time}}{\text{Expected calls offered per unit time}}$$

$$= \frac{\sum_{n=0}^{y} (a + sn)f(x, n)}{\sum_{m=0}^{x} \sum_{n=0}^{y} (a + sn)f(m, n)} = \frac{s\bar{n} + af(x, y)}{a + s\bar{n}} \tag{6}$$

in which $\bar{n} = \sum_{m=0}^{x} \sum_{n=0}^{y} nf(m, n)$. And when y is chosen so large that $f(x, y)$ is negligible, as we shall use it here,

$$L = a \tag{5'}$$

$$W(x, a, s) = \frac{s\bar{n}}{a + s\bar{n}} \tag{6'}$$

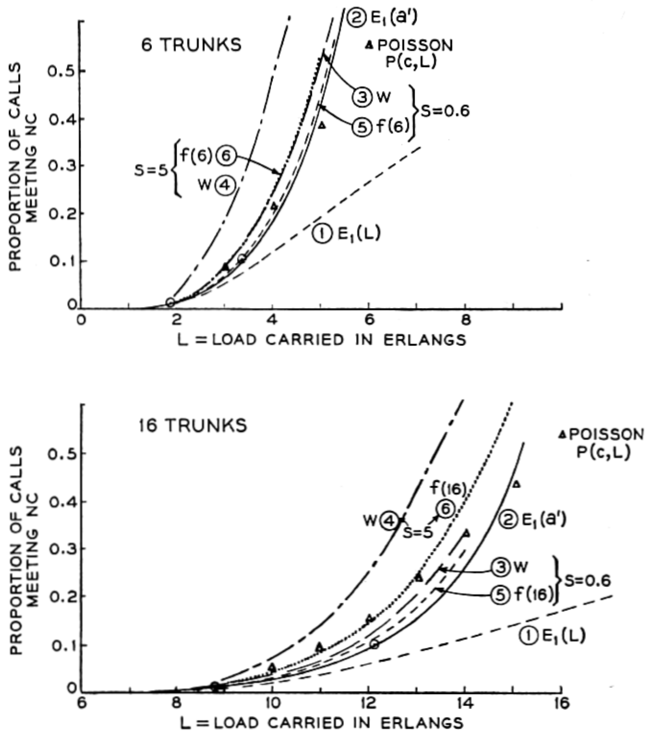

Fig. 7 — Comparison of trunking formulas.

This formula provides a means for estimating the grade of service which customers might be expected to receive if asked to dial their calls over moderate-delay or high-delay trunk groups. For a circuit use length of 150 seconds, and an average return time of 250 seconds (as on Fig. 6), both exponential, the load-versus-proportion-NC curves for 6 and 16 trunks are given as curves (3) on Fig. 7. For example with an offered (= carried) load of a = 4.15 erlangs on 6 trunks we should expect to find 27.5 per cent of the total attempts resulting in failure.

For comparison with a fixed return time of NC-calls, the W-formula curves for exponential returns of 30 seconds ($s = 5$) and 250 seconds ($s = 0.6$) averages are shown on Fig. 5. The first is far too severe an assumption for operator performance, giving NC's nearly double those actually observed (and those given by theory for a 30-second constant return time). The 250-second average return, however, lies only slightly above the 30-second constant return curve and is in good agreement with the data. Although not logically an adequate formula for interpreting Peg Count and Overflow registrations on T-engineered groups under operator dialing conditions, the W-formula apparently could be used for this purpose with suitable s-values determined empirically.

3.1. *Comparison of Some Formulas for Estimating Customers' NC Service on Congested Groups*

As has been previously observed, a large proportion of customers who receive a busy signal, return within a few minutes (on Fig. 6, 75 per cent of the customers returned within 10 minutes). It is well known too, that under adverse service conditions subscriber attempts (to reach a particular distant office for example) tend to produce an inflated estimate of the true offered load. A count of calls carried (or a direct measurement of load carried) will commonly be a closer estimate of the offered load than a count of attempts. An exception may occur when a large proportion of attempts is lost, indicating an offered load possibly in excess even of the number of paths provided. Under the latter condition it is difficult to estimate the true offered load by any method, since not all the attempts can be expected to return repeatedly until served; instead, a significant number will be abandoned somewhere through the trials. In most other circumstances, however, the carried load will prove a reasonably good estimate of the true offered load in systems not provided with alternate paths.

This is a matter of especial interest for both toll and local operation in America since principal future reliance for load measurement is expected to be placed on automatically processed TUR data, and as the TUR is a switch counting device the results will be in terms of load carried. Moreover, the quantity now obtained in many local exchanges is load carried.* Visual switch counting of line finders and selectors off-normal is widely practiced in step-by-step and panel offices; a variety of electromechanical switch counting devices is also to be found in crossbar offices. It is common to take load-carried figures as equal to load-offered when using conventional trunking tables to ascertain the proper provision of trunks or switches. Fig. 7 compares the NC predictions made by a number of the available load-loss formulas when load carried is used as the entry variable.

The lowest curves (1) on Fig. 7 are from the Erlang lost calls formula E_1 (or B) with load carried L used as the offered load a. At low losses, say 0.01 or less, either L or $a = L/[1 - E_1(a)]$ can be used indiscriminately as the entry in the E_1 formula. If however considerably larger losses are encountered and calls are not in reality "cleared" upon meeting NC, it will no longer be satisfactory to substitute L for a. In this circumstance it is common to calculate a fictitious load a' to submit to the c paths such that the load carried, $a'[1 - E_{1,c}(a')]$, equals the desired L. (This was the process used in Section 2 to obtain "% NC existing.") The curves (2) on Fig. 7 show this relation; physically it corresponds to an initially offered load of L erlangs (or L call arrivals per average holding time), whose overflow calls return again and again until successful but without disturbing the randomness of the input. Thus if the loss from this enhanced random traffic is E, then the total trials seen per holding time will be $L(1 + E + E^2 + \cdots) = L/(1 - E) = a'$, the apparent arrival rate of new calls, but actually of new calls plus return attempts.

The random resubmission of calls may provide a reasonable description of operation under certain circumstances, presumably when re-trials are not excessive. Kosten[2] has discussed the dangers here and provided upper and lower limit formulas and curves for estimating the proportions of NC's to be expected when re-trials are made at any specified fixed return time. His lower bounds (lower bound because the change in congestion character caused by the returning calls is ignored) are shown by open dots on Fig. 7 for return times of 1.67 holding times. They lie above curves (2) (although only very slightly because of the relatively long return time) since they allow for the fact that a call shortly returning

* In fact, it is difficult to see how any estimate of offered load, other than carried load, can be obtained with useful reliability.

after meeting a busy signal will have a higher probability of again finding all paths busy, than would a randomly originated call.

The curves (3) show the W-formula previously developed in this section, which contemplates exponential return times on all NC attempts. The average return time here is also taken as 1.67 holding times. These curves lie higher than Kosten's values for two reasons. First, the altered congestion due to return calls is allowed for; and second, with exponential returns nearly two-thirds of the return times are shorter than the average, and of these, the shortest ones will have a relatively high probability of failure upon re-trying. If the customers were to return with exponential times after waiting an average of only 0.2 holding time (e.g., 30 seconds wait for 150-second calls) the W-curves would rise markedly to the positions shown by (4).

Curves (5) and (6) give the proportions of time that all paths are busy (equation 4) under the W-formula assumptions corresponding to NC curves (3) and (4) respectively; their upward displacement from the random return curves (2) reflects the disturbance to the group congestion produced by the non-random return of the delayed calls. (The limiting position for these curves is, of course, given by Erlang's E_2 (or C) delay formula.) As would be expected, curve (6) is above (5) since the former contemplates exponential returns with average of 0.2 holding time, as against 1.67 for curve (5). Neither the (5)-curves nor the open dots of constant 30-second return times show a marked increase over curves (2). This appears to explain why the relationship of load carried versus "NC existing" (as charted in Figs. 3 and 4) was found so insensitive to variable operating procedures in handling subsequent attempts in toll ringdown operation, and again, why it did not appreciably change under operator dialing.

Finally, through the two fields of curves on Fig. 7 is indicated the Poisson summation $P(c, L)$ with load carried L used as the entering variable. The fact that these values approach closely the (2) and (3) sets of curves over a considerable range of NC's should reassure those who have been concerned that the Poisson engineering tables were not useful for losses larger than a few per cent.*

4. SERVICE REQUIREMENTS FOR DIRECT DISTANCE DIALING BY CUSTOMERS

As shown by the W-curves (3) on Fig. 7, the attempt failures by customers resulting from their tendency to re-try shortly following an NC

* Reference may be made also to a throwdown by C. Clos (Ref. 3) using the return times of Fig. 6; his "% NC" results agreed closely with the Poisson predictions.

would be expected to exceed slightly the values for completely random re-trials. These particular curves are based on a re-trial interval of 1.67 times the average circuit-use time. Such moderation on the part of the customer is probably attainable through instructional literature and other means if the customer believes the "NC" or "busy" to be caused by the called party's actually using his telephone (the usual case in local practice). It would be considerably more difficult, however, to dissuade the customer from re-trying at a more rapid rate if the circuit NC's should generally approach or exceed actual called-party busies, a condition of which he would sooner or later become aware. His attempts might then be more nearly described by the (4) curves on Fig. 7 corresponding to an average exponential return of only 0.2 holding time—or even higher. Such a result would not only displease the user, but also result in the requirement of increased switching control equipment to handle many more wasted attempts.

If subscribers are to be given satisfactory direct dialing access to the intertoll trunk network, it appears then that the probability of finding NC even in the busy hours must be kept to a low figure. The following engineering objective has tentatively been selected: *The calls offered to the "final" group of trunks in an alternate route system should receive no more than 3 per cent* NC(P.03) *during the network busy season busy hour.* (If there are no alternate routes, the direct group is the "final" route.)

Since in the nationwide plan there will be a final route between each of some 2,600 toll centers and its next higher center, and the majority of calls offered to high usage trunks will be carried without trying their final route (or routes), the over-all point-to-point service, while not easy to estimate, will apparently be quite satisfactory for customer dialing.

5. ECONOMICS OF TOLL ALTERNATE ROUTING

In a general study of the economics of a nationwide toll switching plan, made some years ago by engineers of the American Telephone and Telegraph Company, it was concluded that a toll line plant sufficient to give the then average level of service (about T-40) with ordinary single-route procedures could, if operated on a multi-alternate route basis, give the desired P.03 service on final routes with little, if any, increase in toll line investment.* On the other hand to attain a similar P.03 grade of service by liberalizing a typical intertoll group of 10 trunks working presently

* This, of course, does not reflect the added costs of the No. 4 switching equipment.

at a T-40 grade of service and an occupancy of 0.81 would require an increase of 43 per cent (to 14.3 trunks), with a corresponding decrease in occupancy to 0.57. The possible savings in toll lines with alternate routing are therefore considerable in a system which must provide a service level satisfactory for customer dialing.

In order to take fullest advantage of the economies of alternate routing, present plans call for five classes of toll offices. There will be a large number of so-called End Offices, a smaller number of Toll Centers, and progressively fewer Primary Centers (about 150), Sectional Centers (about 40) and Regional Centers (9), one of which will be the National Center, to be used as the "home" switching point of the other eight Regional Centers.* Primary and higher centers will be arranged to perform automatic alternate routing and are called Control Switching Points (CSP's). Each class of office will "home" on a higher class of office (not necessarily the next higher one); the toll paths between them are called "final routes." As described in Section 4, these final routes will be provided to give low delays, so that between each principal toll point and every other one there will be available a succession of approximately P.03 engineered trunk groups. Thus if the more direct and heavily loaded interconnecting paths commonly provided are busy there will still be a good chance of making immediate connection over final routes.

Fig. 8 illustrates the manner in which automatic alternate routing will operate in comparison with present-day operator routing. On a call from Syracuse, N. Y., to Miami, Florida, (a distance of some 1,250 miles), under present-day operation, the Syracuse operator signals Albany, and requests a trunk to Miami. With T-schedule operation the Syracuse-Miami traffic might be expected to encounter as much as 25 per cent NC during the busy hour, and approximately 4 per cent NC for the whole day, producing perhaps a two-minute over-all speed of service in the busy season.

With the proposed automatic alternate routing plan, all points on the chart will have automatic switching systems.† The customer (or the operator until customer dialing arrangements are completed) will dial a ten-digit code (three-digit area code 305 for Florida plus the listed Miami seven-digit telephone number) into the machine at Syracuse. The various routes which then might conceivably be tried automatically

* See the bibliography (particularly Pilliod and Truitt) for details of the general trunking plan.

† The notation used on the diagram of Fig. 8 is: Open circle — Primary Center (Syracuse, Miami); Triangle — Sectional Center (Albany, Jacksonville); Square — Regional Center (White Plains, Atlanta, St. Louis; St. Louis is also the National Center).

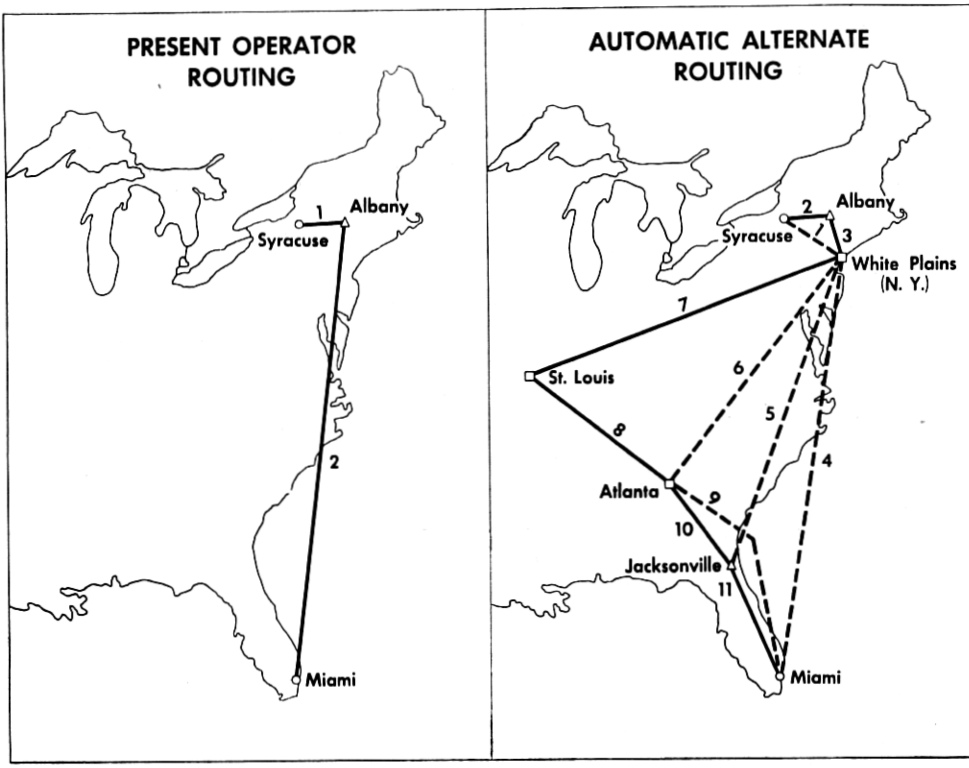

Fig. 8 — Present and proposed methods of handling a call from Syracuse, N. Y., to Miami, Florida.

are shown on the diagram numbered in the order of trial; in this particular layout shown, a maximum of eleven circuit groups could be tested for an idle path if each high usage group should be found NC. Dotted lines show the high usage routes, which if found busy will overflow to the final groups represented by solid lines. The switching equipment at each point upon finding an idle circuit passes on the required digits to the next machine.

While the routing possibilities shown are factual, only in rare instances would a call be completed over the final route via St. Louis. Even in the busy season busy hour just a small portion of the calls would be expected to be switched as many as three times. And only a fraction of one per cent of all calls in the busy hour should encounter NC. As a result the service will be fast. When calls are handled by a toll operator, the customer will not ordinarily need to hang up when NC is obtained. When he himself dials, a second trial after a short wait following NC should have a high probability of success.

Not many situations will be as complex as shown in Fig. 8; commonly several of the links between centers will be missing, the particular ones retained having been chosen from suitable economic studies. A large number of switching arrangements will be no more involved than the illustrative one shown in Fig. 9(a), centering on the Toll Center of Bloomsburg, Pennsylvania. The dashed lines indicate high usage groups from Bloomsburg to surrounding toll centers; since Bloomsburg "homes" on Scranton this is a final route as denoted by the solid line. As an example of the operation, consider a call at Bloomsburg destined for Williamsport. Upon finding all direct trunks busy, a second trial is made via Harrisburg; and should no paths in the Harrisburg group be available, a third and final trial is made through the Scranton group.

In considering the traffic flow of a network such as illustrated at Bloomsburg it is convenient to employ the conventional form of a two-stage graded multiple having "legs" of varying sizes and traffic loads individual to each, as shown in Fig. 9(b). Here only the circuits immediately outgoing from the toll center are shown; the parcels of traffic

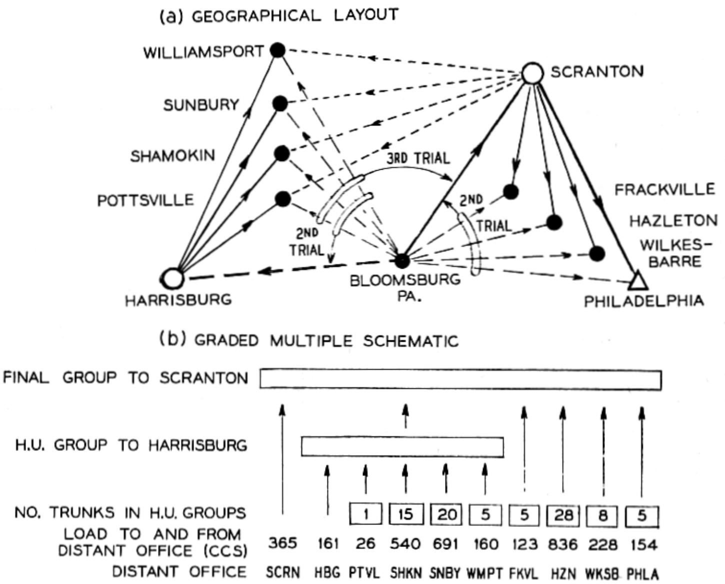

Fig. 9 — Automatic alternate routing for direct distance dialing at Bloomsburg, Pa.

calculated for each further connecting route will be recorded as part of the offered load for consideration when the next higher switching center is engineered. It is implicitly assumed that a call which has selected one of the alternate route paths will be successful in finding the necessary paths available from the distant switching point onward. This is not quite true but is believed generally to be close enough for engineering purposes, and permits ignoring the return attempt problem.

6. NEW PROBLEMS IN THE ENGINEERING AND ADMINISTRATION OF INTERTOLL GROUPS RESULTING FROM ALTERNATE ROUTING

With the greatly increased teamwork among groups of intertoll trunks which supply overflow calls to an alternate route, an unexpected increase or flurry in the offered load to any one can adversely affect the service to all. The high efficiency of the alternate route networks also reduces their overload carrying ability. Conversely, the influence of an underprovision of paths in the final alternate route may be felt by many groups which overflow to it. With non-alternate route arrangements only the single groups having these flurries would be affected.

Administratively, an alternate route trunk layout may well prove easier to monitor day by day than a large number of separate and independent intertoll groups, since a close check on the service given on the final routes only may be sufficient to insure that all customers are being served satisfactorily. When rearrangements are indicated, how-

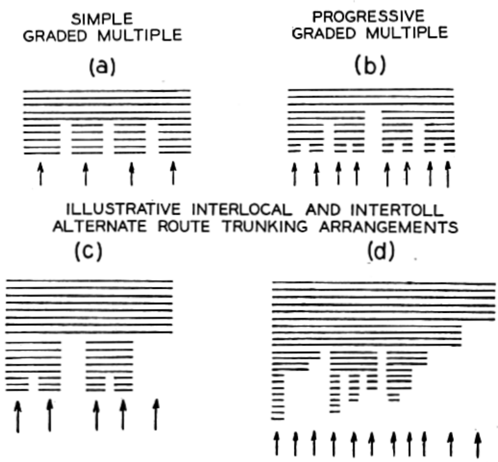

Fig. 10 — Graded multiples and alternate route trunking arrangements.

ever, the determination of the proper place to take action, and the desirable extent, may sometimes be difficult to determine. Suitable traffic measuring devices must be provided with these latter problems in mind.

For engineering purposes, it will be highly desirable:

(1) To be able to estimate the load-service relationships with any specified loads offered to a particular intertoll alternate routing network; and

(2) To know the day-to-day busy hour variations in the various groups' offered loads during the busy season, so that the general grade of service given to customers can be estimated.

The balance of this paper will review the studies which have been made in the Bell System toward a practicable method for predicting the grade of service given in an alternate route network under any given loads. Analyses of the day-to-day load variations and their effects on customer dialing service are currently being made, and will be reported upon later.

7. LOAD-SERVICE RELATIONSHIPS IN ALTERNATE ROUTE SYSTEMS

In their simplest form, alternate route systems appear as symmetrical graded multiples, as shown in Fig. 10(a) and 10(b). Patterns such as these have long been used in local automatic systems to partially overcome the trunking efficiency limitations imposed by limited access switches. The traffic capacity of these arrangements has been the subject of much study by theory and "throwdowns" (simulated traffic studies) both in the United States and abroad. Field trials have substantiated the essential accuracy of the trunking tables which have resulted.

In toll alternate route systems as contemplated in America, however, there will seldom be the symmetry of pattern found in local graded multiples, nor does maximum switch size generally produce serious limitation on the access. The "legs" or first-choice trunk groups will vary widely in size; likewise the number of such groups overflowing calls jointly to an alternate route may cover a considerable range. In all cases a given group, whether or not a link of an alternate route, will have one or more parcels of traffic for which it is the first-choice route. [See the right-hand parcel of offered traffic on Fig. 10(c).] Often this first routed traffic will be the bulk of the load offered to the group, which also serves as an alternate route for other traffic.

The simplest of the approximate formulas developed for solving the local graded multiple problems are hopelessly unwieldy when applied to such arrangements as shown in Fig. 10(d). Likewise it is impracticable to solve more than a few of the infinite variety of arrangements by means of "throwdowns."

However, for both engineering (planning for future trunk provisions) and administration (current operating) of trunks in these multi-alternate routing systems, a rapid, simple, but reasonably accurate method is required. The basis for the method which has been evolved for Bell System use will be described in the following pages.

7.1. *The "Peaked" Character of Overflow Traffic*

The difficulty in predicting the load-service relationship in alternate route systems has lain in the non-random character of the traffic overflowing a first set of paths to which calls may have been randomly offered. This non-randomness is a well appreciated phenomenon among traffic engineers. If adequate trunks are provided for accommodating the momentary traffic peaks, the time-call level diagram may appear as in Fig. 11(a), (average level of 9.5 erlangs). If however a more limited number of trunks, say $x = 12$, is provided, the peaks of Fig. 11(a) will be clipped, and the overflow calls will either be "lost" or they may be handled on a subsequent set of paths y. The momentary loads seen on y then appear as in Fig. 11(b). It will readily be seen that a given average load on the y trunks will have quite different fluctuation characteristics than if it had been found on the x trunks. There will be more occurrences of large numbers of calls, and also longer intervals when few or no calls are present. This gives rise to the expression that overflow traffic is "peaked."

Peaked traffic requires more paths than does random traffic to operate at a specified grade of delayed or lost calls service. And the increase in paths required will depend upon the degree of peakedness of the traffic involved. A measure of peakedness of overflow traffic is then required which can be easily determined from a knowledge of the load offered and the number of trunks in the group immediately available.

In 1923, G. W. Kendrick, then with the American Telephone and Telegraph Company, undertook to solve the graded multiple problem through an application of Erlang's statistical equilibrium method. His principal contribution (in an unpublished memorandum) was to set up the equations for describing the existence of calls on a full access group of $x + y$ paths, arranged so that arriving calls always seek service first in the x-group, and then in the y-group when the x are all busy.

Let $f(m, n)$ be the probability that at a random instant m calls exist on the x paths and n calls on the y paths, when an average Poisson load of a erlangs is submitted to the $x + y$ paths. The general state equation for all possible call arrangements, is

$$(a^* + m + n)f(m, n) = (m + 1)f(m + 1, n) + (n + 1)f(m, n + 1) + af(m - 1, n) + af(x, n - 1)\ddagger \quad (7)$$

in which the term marked (‡) is to be included only when $m = x$, and * indicates that the a in this term is to be omitted when $m + n = x + y$. m and n may take values only in the intervals, $0 \leqq m \leqq x$; $0 \leqq n \leqq y$. As written, the equation represents the "lost calls cleared" situation.

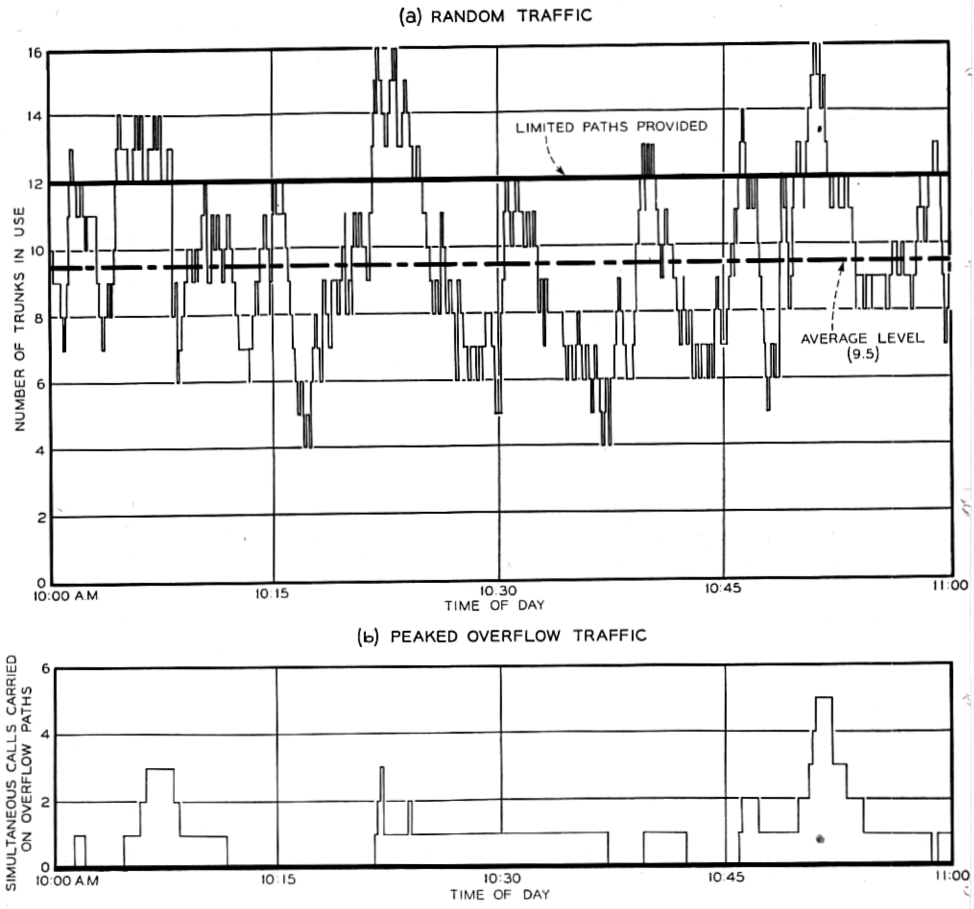

Fig. 11 — Production of peakedness in overflow traffic.

By choosing $x + y$ large compared with the submitted load a a "lost calls held" situation or infinite-overflow-trunks result can be approached as closely as desired.

Kendrick suggested solving the series of simultaneous equations (7) by determinants, and also by a method of continued fractions. However little of this numerical work was actually undertaken until several years later.

Early in 1935 Miss E. V. Wyckoff of Bell Telephone Laboratories became interested in the solution of the $(x + 1)(y + 1)$ lost calls cleared simultaneous equations leading to all terms in the $f(m, n)$ distribution. She devised an order of substituting one equation in the next which provided an entirely practical and relatively rapid means for the numerical solution of almost any set of these equations. By this method a considerable number of $f(m, n)$ distributions on x, y type multiples with varying load levels were calculated.

From the complete m, n matrix of probabilities, one easily obtains the distribution $\theta_m(n)$ of overflow calls when exactly m are present on the lower group of x trunks; or by summing on m, the $\theta(n)$ distribution without regard to m, is realized. A number of other procedures for obtaining the $f(m, n)$ values have been proposed. All involve lengthy computations, very tedious for solution by desk calculating machines, and most do not have the ready checks of the Wyckoff-method available at regular points through the calculations.

In 1937 Kosten[4] gave the following expression for $f(m, n)$:

$$f(m, n) = (-1)^n \varphi_0(x) \sum_{i=0}^{\infty} \binom{i}{n} \frac{(-a)^i}{i!} \cdot \frac{\varphi_i(m)}{\varphi_{i+1}(x)\varphi_i(x)} \qquad (8)$$

where

$$\varphi_0(x) = \frac{a^x e^{-a}}{x!}$$

and for $i > 0$,

$$\varphi_i(x) = e^{-a} \sum_{j=0}^{x} \binom{i + j - 1}{j} \frac{a^{x-j}}{(x - j)!}$$

These equations, too, are laborious to calculate if the load and numbers of trunks are not small. It would, of course, be possible to program a modern automatic computer to do this work with considerable rapidity.

The corresponding application of the statistical equilibrium equations to the graded multiple problem was visualized by Kendrick who, however, went only so far as to write out the equation for the three-trunk case consisting of two subgroups of one trunk each and one common overflow trunk.

Instead of solving the enormously elaborate system of equations describing all the calls which could simultaneously be present in a large multiple, several ingenious methods of convoluting the

$$\theta(n) = \sum_{m=0}^{x} f(m, n)$$

overflow distributions from the individual legs of a graded multiple have been devised. For example, for the multiple of Fig. 10(a), the probability of loss P_i as seen by a call entering subgroup number i, is approximately,

$$P_i = \sum_{r=0}^{y-1} \sum_{z=y}^{\infty} \theta_{x,i}(r) \cdot \psi(z - r) + \sum_{r=y}^{\infty} \theta_{x,i}(r) \qquad (9)$$

in which $\psi(z - r)$ is the probability of exactly $z - r$ overflow calls being present, or wanting to be present, on the alternate route from all the subgroups except the ith, and with no regard for the numbers of calls present in these subgroups. The $\theta_{x,i}(r) = f_i(x_i, r)$ term, of course, contemplates all paths in the particular originating call's subgroup being occupied, forcing the new call arriving in subgroup i to advance to the alternate route. This corresponds to the method of solving graded multiples developed by E. C. Molina[6] but has the advantage of overcoming the artificial "no holes in the multiple" assumption which he made. Similar calculating procedures have been suggested by Kosten.* These computational methods doubtless yield useful estimates of the resulting service, and for the limited numbers of multiple arrangements which might occur in within-office switching trains (particularly ones of a symmetrical variety) such procedures might be practicable. But it would be far too laborious to obtain the individual overflow distributions $\theta(n)$, and then convolute them for the large variety of loads and multiple arrangements expected to be met in toll alternate routing.

7.2. *Approximate Description of the Character of Overflow Traffic*

It was natural that various approximate procedures should be tried in the attempt to obtain solutions to the general loss formula sufficiently accurate for engineering and study purposes. The most obvious of these is to calculate the lower moments or semi-invariants of the loads overflowing the subgroups, and from them construct approximate fitting

* Kosten gives the above approximation (9), which he calls W_b^+, as an upper limit to the blocking. He also gives a lower limit, W_b^-, in which $z = y$ throughout (References 4, 5).

distributions for $\theta(n)$ and $\theta_x(n)$. Since each such overflow is independent of the others, they may be combined additively (or convoluted), to obtain the corresponding total distribution of calls appearing before the alternate route (or common group). It may further be possible to obtain an approximate fitting distribution to the sum-distribution of the overflow calls.

The ordinary moments about the 0 point of the subgroup overflow distribution, when m of the x paths are busy, are found by

$$\mu_i'(m) = \sum_{n=0}^{y} n^i f(m, n) \tag{10}$$

When an infinite number of y-paths is assumed, the resulting expressions for the mean and variance are found to be:*

Number of x-paths busy unspecified:†

$$\text{Mean} = \alpha = a \cdot E_{1,x}(a) \tag{11}$$

$$\text{Variance} = v = \alpha[1 - \alpha + a(x + 1 + \alpha - a)^{-1}] \tag{12}$$

*All x-paths occupied*₁

$$\text{Mean} = \alpha_x = a[x - a + 1 + aE_{1,x}(a)]^{-1} \tag{13}$$

$$\text{Variance} = v_x = \alpha_x[1 - \alpha_x + 2a(x + 2 + \alpha_x - a)^{-1}] \tag{14}$$

Equations (11) and (12) have been calculated for considerable ranges of offered load a and paths x. Figs. 12 and 13 are graphs of these results. For example when a load of 4 erlangs is submitted to 5 paths, the average overflow load is seen to be $\alpha = 0.80$ erlang, the same value, of course, as determined through a direct application of the Erlang E_1 formula. During the time that all x paths are busy, however, the overflow load will tend to exceed this general level as indicated by the value of $\alpha_x = 1.41$ erlangs calculated from (13). Similarly the variance of the overflow load will tend to increase when the x-paths are fully occupied,

* The derivation of these equations is given in Appendix I.
† The skewness factor may also be of interest:

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{2/3}}$$

$$= \frac{1}{v^{3/2}}\left[\frac{a}{x+1+\alpha-a}\left\{\frac{2}{x+2}\left(\frac{(x+\alpha-a)a^2}{(x-a)^2+2(x-a)+x+2+(x+2-a)\alpha}+a\right) + 3(1-\alpha)\right\} + \alpha(1-\alpha)(1-2\alpha)\right] \tag{15}$$

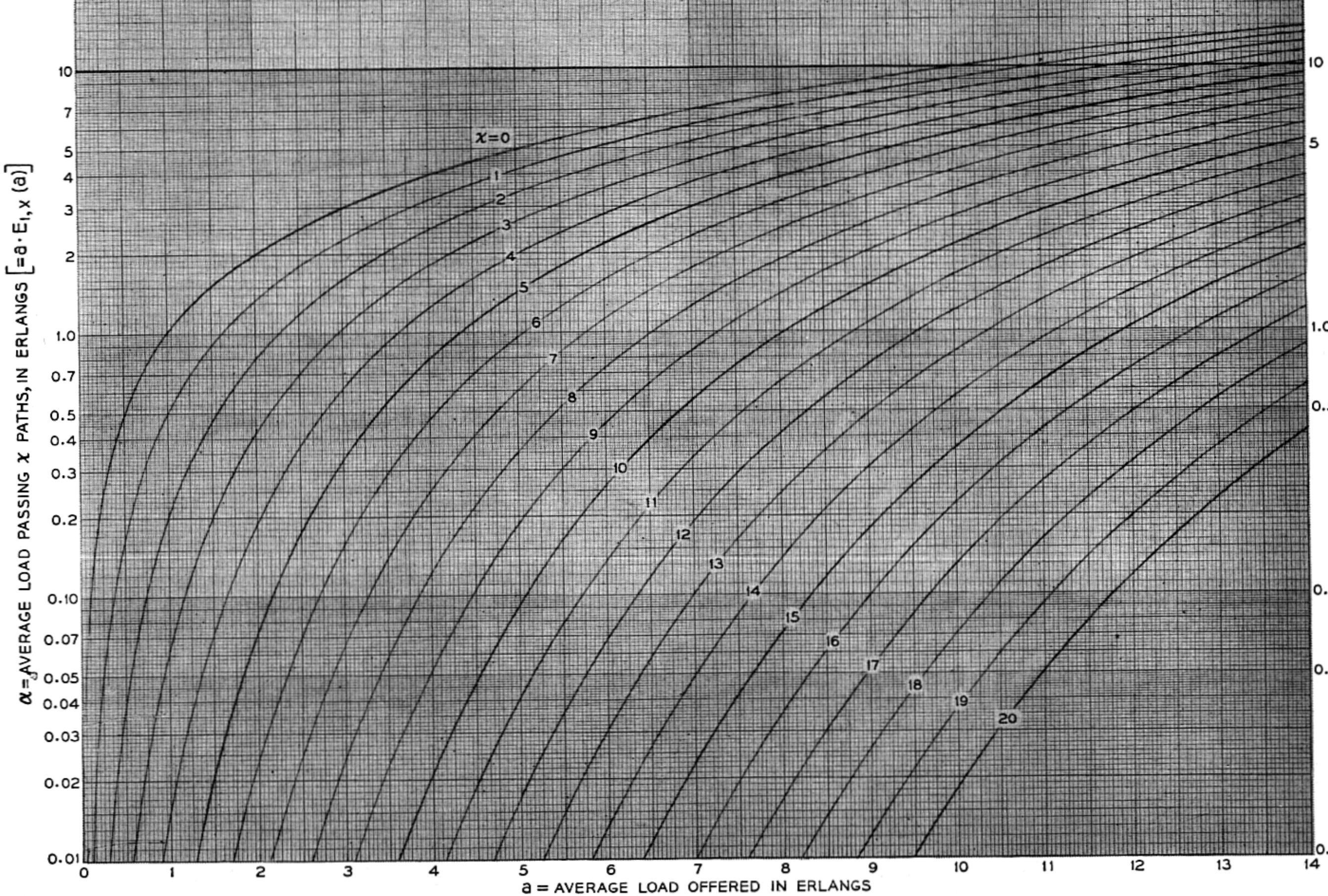

Fig. 12.1 — Average of overflow load, with 0 to 14 erlangs offered.

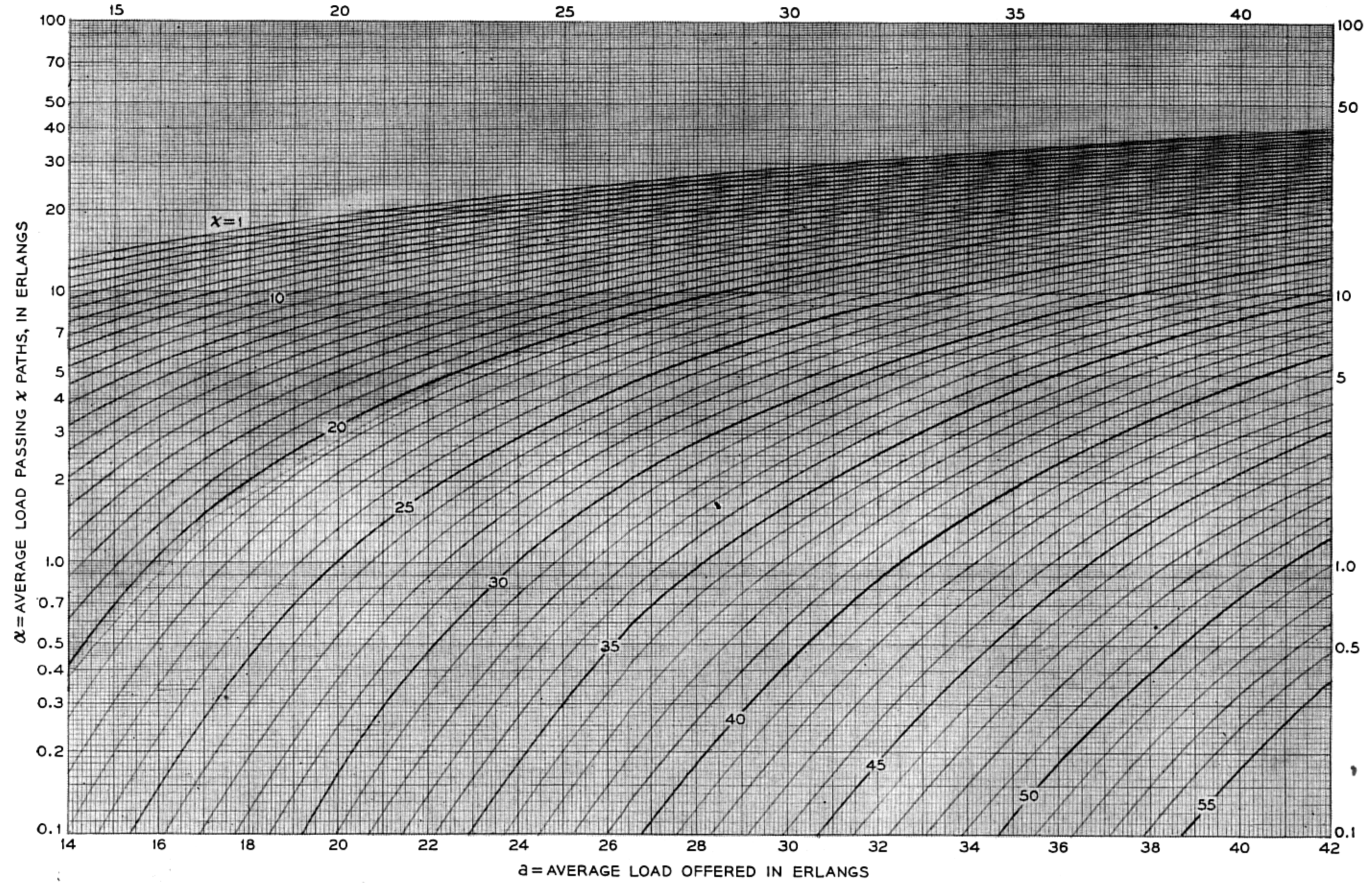

Fig. 12.2 — Average of overflow load, with 14 to 42 erlangs offered.

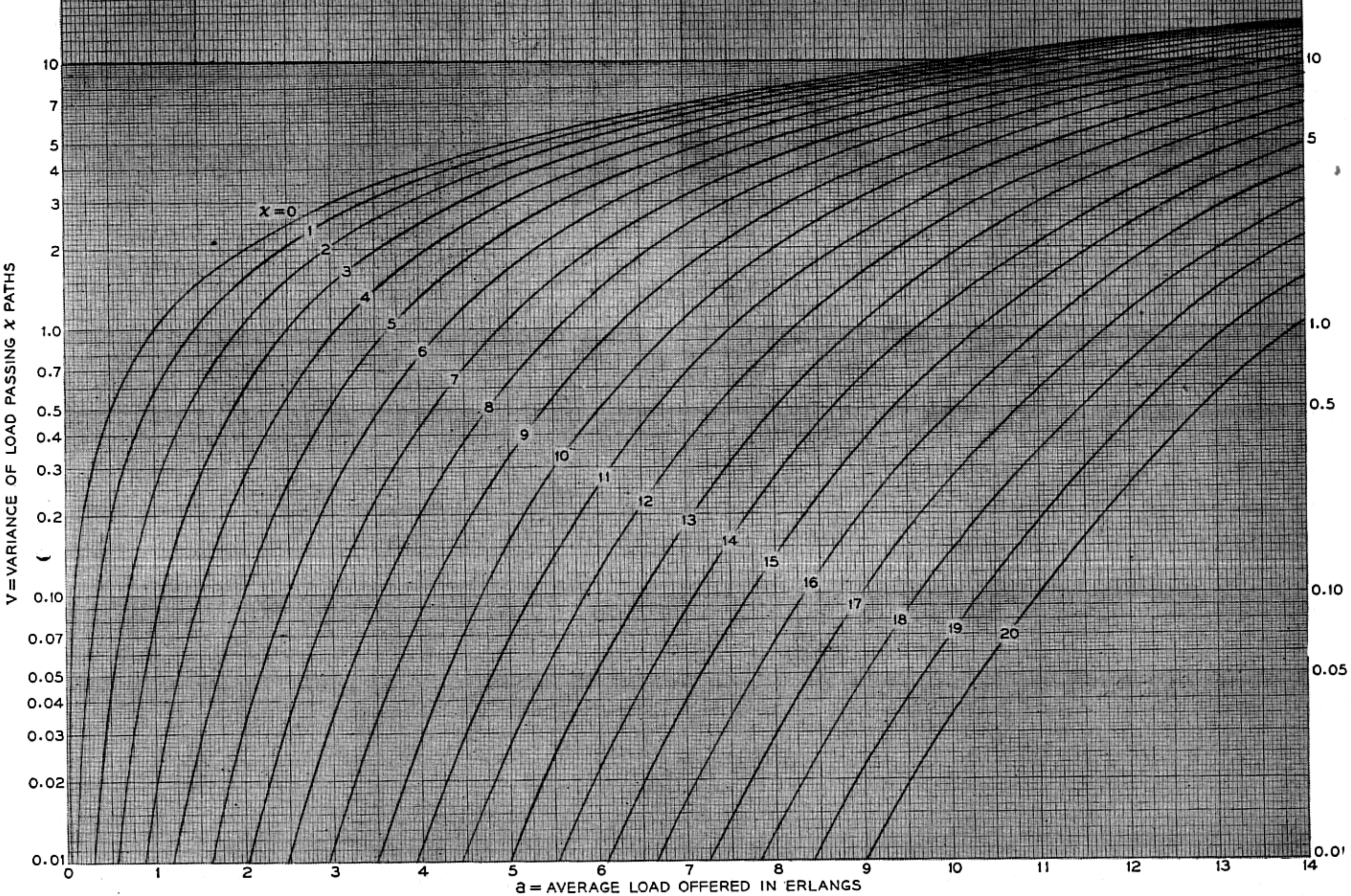

Fig. 13.1 — Variance of overflow load, with 0 to 14 erlangs offered.

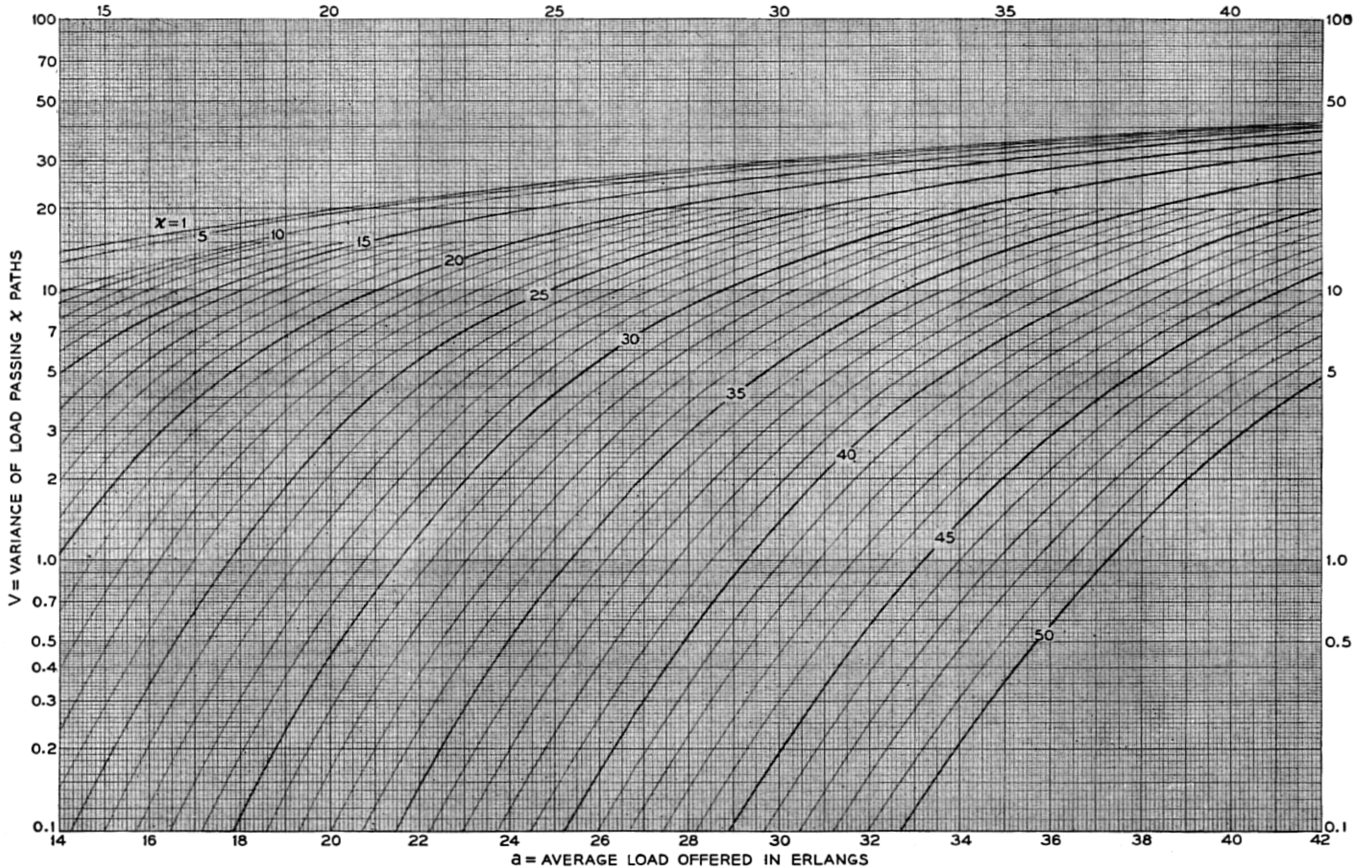

Fig. 13.2 — Variance of overflow load, with 14 to 42 erlangs offered.

as shown by $v = 1.30$, and $v_x = 1.95$. In all cases the variances v and v_x will exceed the variance of corresponding Poisson traffic (which would have variances of α and α_x respectively).

7.2.1. *A Probability Distribution for Overflow Traffic*

It would be of interest to be able, given the first several descriptive parameters of any traffic load (such as the mean and variance and skewness factors of the overflow from a group of trunks), to construct an approximate probability distribution $\theta(n)$ which would closely describe the true momentary distribution of simultaneous calls. Any proposed fitting distribution for the overflow from random traffic offered to x trunks, can, of course, be compared with

$$\theta(n) = \sum_{m=0}^{x} f(m, n)$$

determined from (7) or (8).

Suitable fitting curves should give probabilities for all possitive integral values of the variable (including zero), and have sufficient unspecified constants to accommodate the parameters selected for describing the distribution. Moreover, the higher moments of a fitting distribution should not diverge too radically from those of the true distribution; that is, the "natural shapes" of fitting and true distributions should be similar. Particularly desirable would be a fitting distribution form derived with some attention to the physical circumstances causing the ebb and flow of calls in an overflow situation. The following argument and derivation undertake to achieve these desiderata.*

A Poisson distribution of offered traffic is produced by a random arrival of calls. The assumption is made or implied that the probability of a new arrival in the next instant of time is quite independent of the number currently present in the system. When this randomness (and corresponding independence) are disturbed the resulting distribution will no longer be Poisson. The first important deviation from the Poisson would be expected to appear in a change from variance = mean, to variance ≠

* A two-parameter function which has the ability to fit quite well a wide variety of true overflow distributions, has the form

$$\xi(n) = K(n + 1)^b e^{-c(n+1)}$$

in which K is the normalizing constant. The distribution is displaced one unit from the usual discrete generalized exponential form, so that $\xi(0) \neq 0$. The expression, however, has little rationale for being selected a priori as a suitable fitting function.

mean. Corresponding changes in the higher moments would also be expected.

What would be the physical description of a cause system with a variance smaller or larger than the Poisson? If the variance is smaller, there must be forces at work which retard the call arrival rate as the number of calls recently offered exceeds a normal, or average, figure, and which increase the arrival rate when the number recently arrived falls below the normal level. Conversely, the variance will exceed the Poisson's should the tendencies of the forces be reversed.* This last is, in fact, a rough description of the incidence rates for calls overflowing a group of trunks.

Since holding times are attached to and extend from the call arrival instants, calls are enabled to project their influence into the future; that is, the presence of a considerable number of calls in a system at any instant reflects their having arrived in recent earlier time, and now can be used to modify the current rate of call arrival.

Let the probability of a call originating in a short interval of time dt be

$$P_{o,n} = [a + (n - a)\omega(n)]\, dt$$

where n = number of calls present in the system at time t,
a = base or average arrival rate of calls per unit time, and
$\omega(n)$ = an arbitrary function which regulates the modification in call origination rate as the number of calls rises above or falls below a.

Correspondingly, let the probability that one of n calls will end in the short interval of time dt be

$$P_{e,n} = n\, dt,$$

which will be satisfied in the case of exponential call holding times, with mean unity. Following the usual Erlang procedure, the general statistical equilibrium equation is

$$\begin{aligned} f(n) = f(n)[(1 - P_{o,n})(1 - P_{e,n})] + f(n - 1)P_{o,n-1}(1 - P_{e,n-1}) \\ + f(n + 1)(1 - P_{o,n+1})P_{e,n+1} \end{aligned} \tag{16}$$

which gives

$$(P_{o,n} + P_{e,n})f(n) = P_{o,n-1}f(n - 1) + P_{e,n+1}f(n + 1)$$

ignoring terms of order higher than the first in dt.

* The same thinking has been used by Vaulot[7] for decreasing the call arrival rate according to the number momentarily present; and by Lundquist[8] for both increasing and decreasing the arrival rate.

Or,

$$[a + (n - a)\omega(n) + n]f(n) \\ = [a + (n - a - 1)\omega(n - 1)]f(n - 1) + (n + 1)f(n + 1) \qquad (17)$$

The choice of $\omega(n)$ will determine the solution of (17). Most simply, $\omega(n) = k$, making the variation from the average call arrival rate directly proportional to the deviation in numbers of calls present from their average number. In this case, the solution for an unlimited trunk group becomes, with $a' = a(1 - k)$,

$$f(n) = \frac{\dfrac{a'(a' + k) \cdots [a' + (n - 1)k]}{n!}}{1 + a' + \dfrac{a'(a' + k)}{2!} + \dfrac{a'(a' + k)(a' + 2k)}{3!} + \cdots} \qquad (18)$$

which may also be written after setting $a'' = a'/k = a(1 - k)/k$, as

$$f(n) = \frac{\dfrac{a''(a'' + 1) \cdots [a'' + (n - 1)]k^n}{n!}}{(1 - k)^{-a''}} \qquad (19)$$

The generating function (g.f.) of (19) is

$$\sum_{n=0}^{\infty} f(n)T^n = \frac{(1 - kT)^{-a''}}{(1 - k)^{-a''}}$$

which is recognized as that for the negative binomial, as distinguished from the g.f.,

$$(q + pT)^N = \frac{\left(1 + \dfrac{p}{q}\, T\right)^N}{(1/q)^N}$$

for the positive binomial.

The first four descriptive parameters of $f(n)$ are:

Order	Moment about Mean	Descriptive Parameter	
1	$\mu_1 = 0$	Mean $= \bar{n} = a$	(20)
2	$\mu_2 =$ variance, $v = a/(1 - k)$	Std Devn, $\sigma = [a/(1 - k)]^{1/2}$	(21)
3	$\mu_3 = \dfrac{a(1 + k)}{(1 - k)^2}$	Skewness, $\sqrt{\beta_1} = \dfrac{\mu_3}{\sigma^3} = \dfrac{1 + k}{a^{1/2}(1 - k)^{1/2}}$	(22)
4	$\mu_4 = \dfrac{3a^2(1 - k) + a(k^2 + 4k + 1)}{(1 - k)^3}$	Kurtosis, $\beta_2 = \dfrac{\mu_4}{\sigma^4} = 3 + \dfrac{k^2 + 4k + 1}{a(1 - k)}$	(23)

Since only two constants, a and k, need specification in (18) or (19), the mean and variance are sufficient to fix the distribution. That is, with the mean $\bar{n}$ and variance v known,

$$a = \bar{n} \qquad \text{or} \qquad a' = \bar{n}(1 - k) = \bar{n}^2/v, \qquad \text{or} \qquad a'' = \bar{n}(1 - k)/k \tag{24}$$

$$k = 1 - a/v = 1 - \bar{n}/v. \tag{25}$$

The probability density distribution $f(n)$ is readily calculated from (19); the cumulative distribution $G(\geqq n)$ also may be found through use of the Incomplete Beta Function tables since

$$\begin{aligned} G(\geqq n) &= I_k(n - 1, a'') \\ &= I_k(n - 1, a(1 - k)/k) \end{aligned} \tag{26}$$

The goodness with which the negative binomial of (19) fits actual distributions of overflow calls requires some investigation. Perhaps a more elaborate expression for $\omega(n)$ than a constant k in (17) is required. Three comparisons appear possible: (1), comparison with a variety of $\theta_m(n)$ distributions with exactly m calls on the x trunks, or $\theta(n)$ with m unspecified, (obtained by solving the statistical equilibrium equations (7) for a divided group); (2), comparison with simulation or "throwdown" results; and (3), comparison with call distributions seen on actual trunk groups. These are most easily performed in the order listed.*

Comparison of Negative Binomial with True Overflow Distributions

Figs. 14 to 17 show various comparisons of the negative binomial distribution with true overflow distributions. Fig. 14 gives in cumulative form the cases of 5 erlangs offered to 1, 2, 5, and 10 trunks. The true

$$F(\geqq n) = \sum_{j=n}^{\infty} \theta(j)$$

distributions (shown as solid lines) are obtained by solving the difference equations (7) in the manner described in Section 7.1. The negative binomial distributions (shown dashed) are chosen to have the same mean and variance as the several $F(\geqq n)$ cases fitted. The dots shown on

* Comparison could also be made after equating means and variances respectively, between the higher moments of the overflow traffic beyond x trunks and the corresponding negative binomial moments: e.g., the skewness given by (15) can be compared with the negative binomial skewness of (22). The difficulty here is that one is unable to judge whether the disparity between the two distribution functions as described by differences in their higher parameters is significant or not for traffic engineering purposes.

the figure are for random (Poisson) traffic having the same mean values as the F distributions. The negative binomial provides excellent fits down to cumulated probabilities of 0.01, with a tendency thereafter to give somewhat larger values than the true ones. The Poisson agreement is good only for the overflow from a single trunk, as might have been anticipated, the divergence rapidly increasing thereafter.

Fig. 15 corresponds with the cases of Fig. 14 except that the true overflow $F_x(\geqq n)$ distributions for the conditional situation of all x-paths busy, are fitted. Again the negative binomial is seen to give a good agreement down to 0.01 probability, with somewhat too-high estimates for larger values of the simultaneous overflow calls n.

Fig. 16 shows additional comparisons of overflow and negative binomial distributions. As before, the agreement is quite satisfactory to 0.01 probability, the negative binomial thereafter tending to give somewhat high values.

On Fig. 17 are compared the individual $\theta(n)$ density distributions for several cases. The agreement of the negative binomial with the true distribution is seen to be uniformly good. The dots indicate the random (Poisson) individual term distribution corresponding to the $a = 9.6$ case;

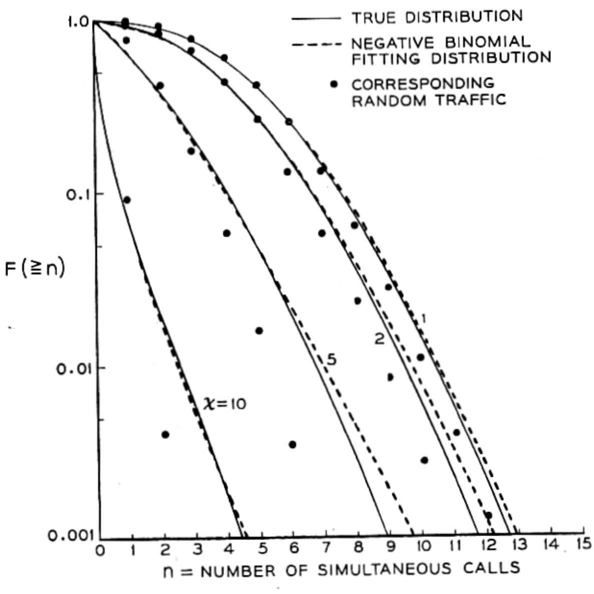

Fig. 14 — Probability distributions of overflow traffic with 5 erlangs offered to 1, 2, 5, and 10 trunks, fitted by negative binomial.

the agreement, of course, is poor since the non-randomness of the overflow here is marked, having an average of 1.88 and a variance of 3.84.

Comparison of Negative Binomial with Overflow Distributions Observed by Throwdowns and on Actual Trunk Groups

Fig. 18 shows a comparison of the negative binomial with the overflow distributions from four direct groups as seen in throwdown studies. The agreement over the range of group sizes from one to fifteen trunks is seen to be excellent. The assumption of randomness (Poisson) as shown by the dot values is clearly unsatisfactory for overflows beyond more than two or three trunks.

A number of switch counts made on the final group of an operating toll alternate routing system at Newark, New Jersey, during periods when few calls were lost, have also shown good agreement with the negative binomial distribution.

7.2.2. *A Probability Distribution for Combined Overflow Traffic Loads*

It has been shown in Section 7.2.1 that, at least for load ranges of wide interest, the negative binomial with but two parameters, chosen to agree

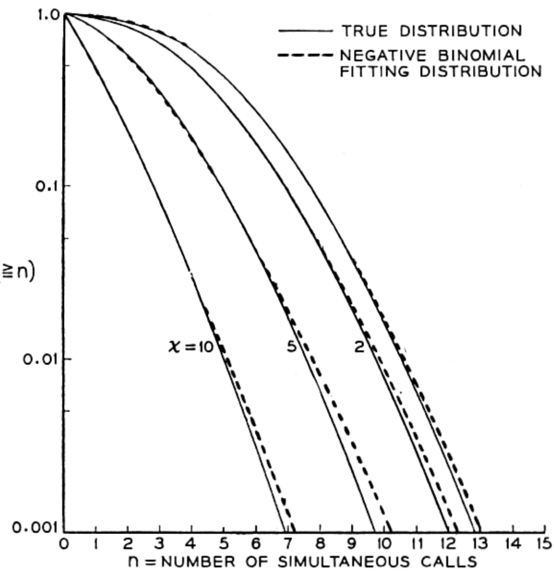

Fig. 15 — Probability distributions of overflow traffic with 5 erlangs offered to 1, 2, 5, and 10 trunks, when all trunks are busy; fitted by negative binomial.

with mean and variance, gives a satisfactory fit to the distribution of traffic overflowing a group of trunks. It is now possible, of course, to convolute the various overflows from any number of groups of varying sizes, to obtain a combined overflow distribution. This procedure, however, would be very clumsy and laborious since at each switching point in the toll alternate route system an entirely different layout of loads and high usage groups would require solution; it would be unfeasible for practical working.

We return again to the method of moments. Since the overflows of the several high usage groups will, in general, be independent of one another, the ith semi-invariants λ_i of the individual overflows can be combined to give the corresponding semi-invariants Λ_i of their total,

$$\Lambda_i = {}_1\lambda_i + {}_2\lambda_i + \cdots \tag{27}$$

Or, in terms of the overflow means and variances, the corresponding parameters of the combined loads are

$$\text{Average} = A' = \alpha_1 + \alpha_2 + \cdots \tag{28}$$

$$\text{Variance} = V' = v_1 + v_2 + \cdots \tag{29}$$

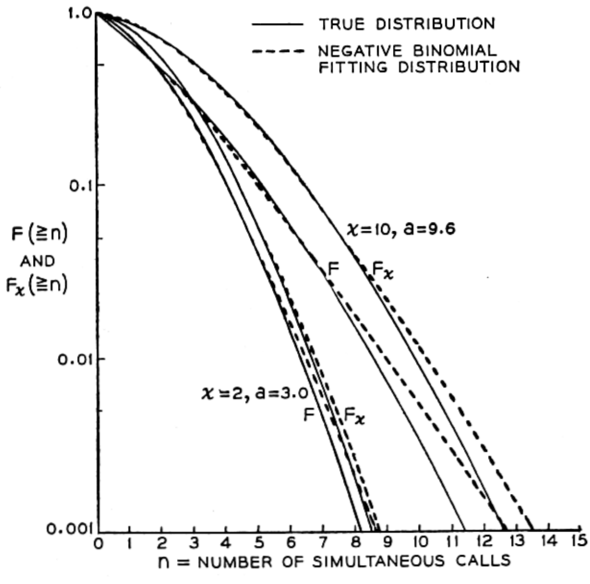

Fig. 16 — Probability distributions of overflow traffic: 3 erlangs offered to 2 trunks, and 9.6 erlangs offered to 10 trunks.

With the mean and variance of the combined overflows now determined, the negative binomial can again be employed to give an approximate description of the distribution of the simultaneous calls $\varphi(z)$ offered to the common, or alternate, group.

The acceptability of this procedure can be tested in various ways. One way is to examine whether the convolution of several negative binomials (representing overflows from individual groups) is sufficiently well fitted by another negative binomial with appropriate mean and variance, as found above.

It can easily be shown that the convolution of several negative binomials all with the same over-dispersion (variance-to-mean ratio) but not necessarily the same mean, is again a negative binomial. Shown in Table I are the distribution components and their parameters of two examples in which the over-dispersion parameters are not identical. The third and fourth semi-invariants of the fitted and fitting distributions, are seen to diverge considerably, as do the Pearsonian skewness and kurtosis factors. The test of acceptability for traffic fluctuation description comes in comparing the fitted and fitting distributions which are shown on Fig. 19. Here it is seen that, despite what might appear alarming dis-

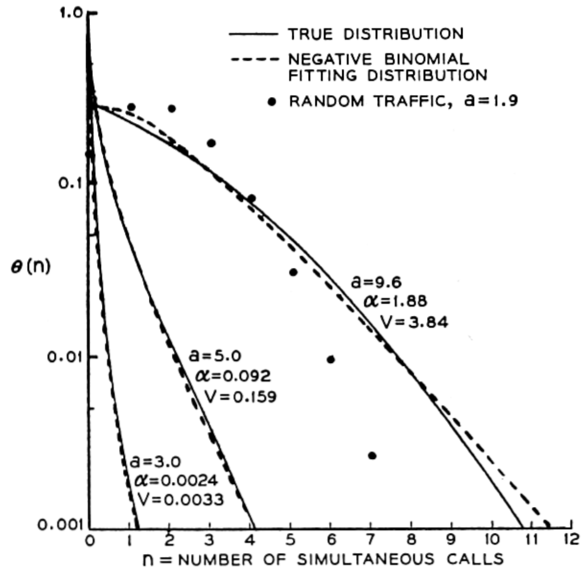

Fig. 17 — Probability density distributions of overflow traffic from 10 trunks, fitted by negative binomial.

parities in the higher semi-invariants, the agreement for practical traffic purposes is very good indeed.

Numerous throwdown checks confirm that the negative binomial employing the calculated sum-overflow mean and variance has a wide range over which the fit is quite satisfactory for traffic description purposes. Fig. 20 shows three such trunking arrangements selected from a considerable number which have been studied by the simulation method. Approximately 5,000, 3,500, and 580 calls were run through in the three examples, respectively. The overflow parameters obtained by experiment are seen to agree reasonably well with the theoretical ones from (28) and (29) when the numbers of calls processed is considered.

On Fig. 21 are shown, for the first arrangement of Fig. 20, distributions of simultaneous offered calls in each subgroup of trunks compared with the corresponding Poisson; the agreement is satisfactory as was to be expected. The sum distribution of the overflows from the eight subgroups is given at the foot of the figure. The superposed Poisson, of course, is a poor fit; the negative binomial, on the other hand, appears quite acceptable as a fitting curve.

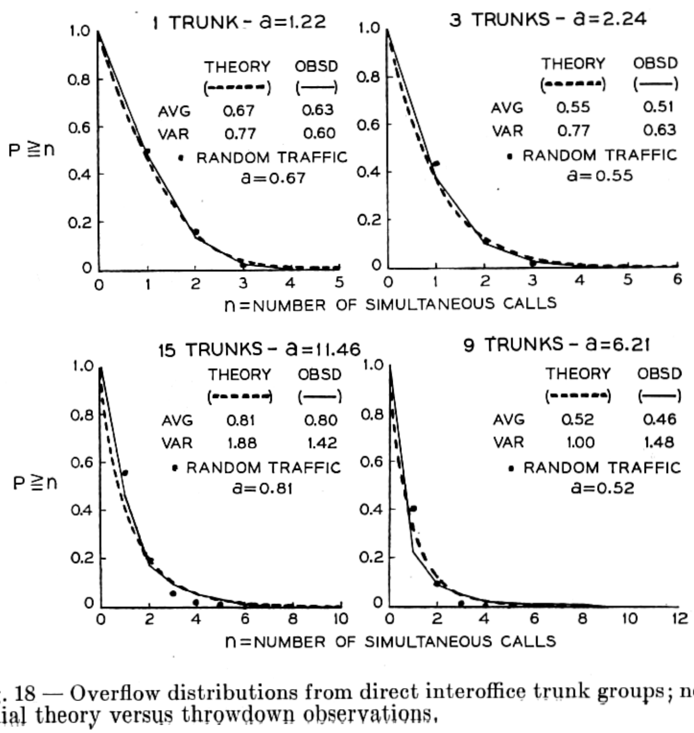

Fig. 18 — Overflow distributions from direct interoffice trunk groups; negative binomial theory versus throwdown observations.

TABLE I — COMPARISON OF PARAMETERS OF A FITTING NEGATIVE BINOMIAL TO THE CONVOLUTION OF THREE NEGATIVE BINOMIALS

Example No. 1			Example No. 2		
Component dist'n No.	Component parameters		Component dist'n No.	Component parameters	
	Mean	Variance		Mean	Variance
1	5	5	1	1	1
2	2	4	2	2	3
3	1	3	3	2	6
	8	12		5	10

Semi-Invariants Λ, Skewness $\sqrt{\beta_1}$, and Kurtosis β_2, of Sum Distributions

Parameter	Exact	Fitting	Parameter	Exact	Fitting
Λ_1	8	8	Λ_1	5	5
Λ_2	12	12	Λ_2	10	10
Λ_3	32	24	Λ_3	37	30
Λ_4	168	66	Λ_4	239.5	130
$\sqrt{\beta_1}$	0.770	0.577	$\sqrt{\beta_1}$	1.170	0.949
β_2	4.167	3.458	β_2	5.395	4.300

Fig. 22 shows the corresponding comparisons of the overflow loads in the other two trunk arrangements of Fig. 20. Again good agreement with the negative binomial is seen.

7.3. *Equivalent Random Theory for Prediction of Amount of Traffic Overflowing a Single Stage Alternate Route, and Its Character, with Lost Calls Cleared*

As discussed in Section 7.2, when random traffic is offered to a limited number of trunks x, the overflow traffic is well described (at least for traffic engineering purposes) by the two parameters, mean α and variance v. The result can readily be applied to a group divided (in one's mind) two or more times as in Fig. 23.

Employing the α and v curves of Figs. 12 and 13, and the appropriate numbers of trunks x_1, $x_1 + x_2$, and $x_1 + x_2 + x_3$, the pairs of descriptive parameters, α_1, v_1, α_2, v_2 and α_3, v_3 can be read at once. It is clear then that if at some point in a straight multiple a traffic with parameters α_1, v_1 is seen, and it is offered to x_2 paths, the overflow therefrom will have the characteristics α_2, v_2. To estimate the particular values of α_2 and v_2, one would first determine the values of the equivalent random

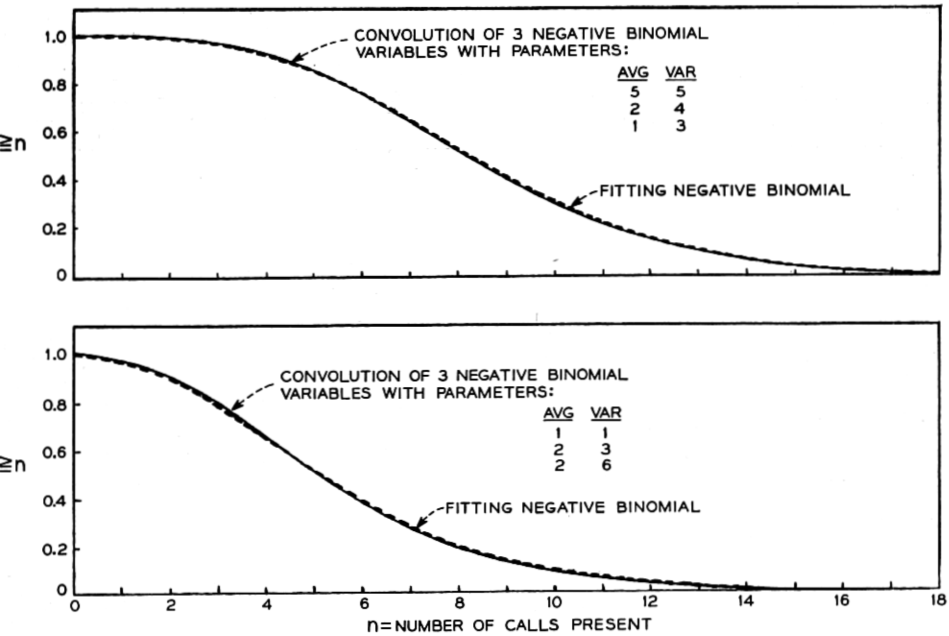

Fig. 19 — Fitting sums of negative binomial variables with a negative binomial.

traffic a and trunks x_1 which would have produced α_1 and v_1. Then proceeding in the forward direction, using a and $x_1 + x_2$, one consults the α and v charts to find α_2 and v_2. Thus, within the limitations of straight group traffic flow, the character (mean and variance) of any overflow load from x trunks can be predicted if the character (mean and variance) of the load submitted to them is known.

Curves could be constructed in the manner just described by which the overflow's α' and v' are estimated from a load, α and v, offered to x trunks. An illustrative fragment of such curves is shown in Appendix II, with an example of their application in the calculation of a straight trunk group loss by considering the successive overflows from each trunk as the offered loads to the next.

Enough, perhaps, has been shown in Section 7.2 of the generally excellent descriptions of a variety of non-random traffic loads obtainable by the use of only the two parameters α and v, to make one strongly suspect that most of the fluctuation information needed for traffic engineering purposes is contained in those two values. If this is, in fact, the case, we should then be able to predict the overflow α', v' from x trunks with an offered load α, v *which has arisen in any manner of overflow from earlier high usage groups*, as illustrated in Fig. 24.

This is found to be the case, as will be illustrated in several studies described in the balance of this section. In the determination of the characteristics of the overflow traffic α', v' in the cases of non-full-access groups, such as Figs. 24(b) and 24(c), the equivalent straight group is visualized [Fig. 24(a)], and the Equivalent Random load A and trunks S are found.* Using A, and $S + C$, to enter the α and v curves of Figs. 12 and 13, α' and v' are readily determined. To facilitate the reading of A and S, Fig. 25† and Fig. 26† (which latter enlarges the lower left corner of Fig. 25) have been drawn. Since, in general, α and v will not have come from a simple straight group, as in Fig. 24(a), it is not to be expected that S,

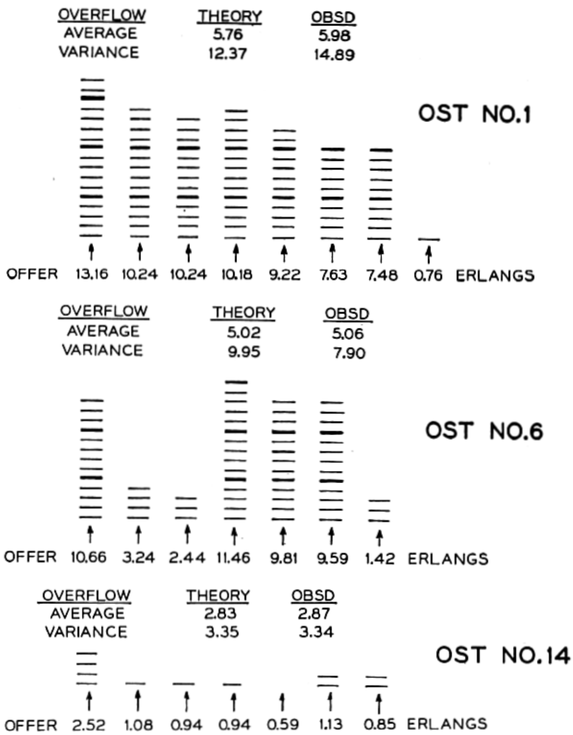

Fig. 20 — Comparison of joint-overflow parameters; theory versus throwdown.

* A somewhat similar method, commonly identified with the British Post Office, which uses one parameter, has been employed for solving symmetrical graded multiples (Ref. 9).

† Figs. 25 and 26 will be found in the envelope on the inside back cover.

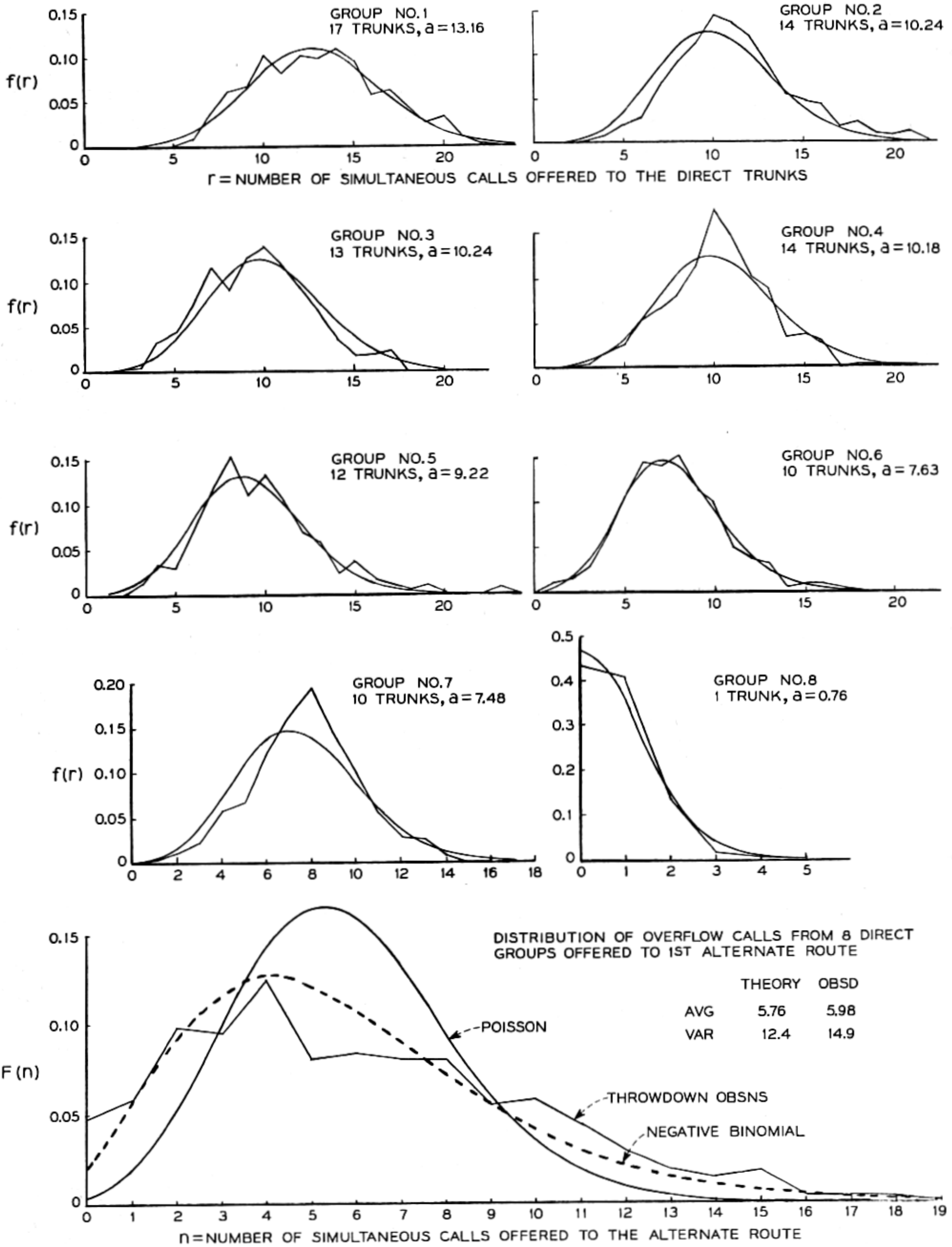

Fig. 21 — Comparison of theoretical and throwdown distributions of simultaneous calls offered to direct groups and to their first alternate route (OST No. 1).

read from Fig. 25, will be an integer. This causes no trouble and S should be carried along fractionally to the extent of the accuracy of result desired. Reading S to one-tenth of a trunk will usually be found sufficient for traffic engineering purposes.

Example 1: Suppose a simple graded multiple has three trunks in each of two subgroups, which overflow to C common trunks, where $C = 1$,

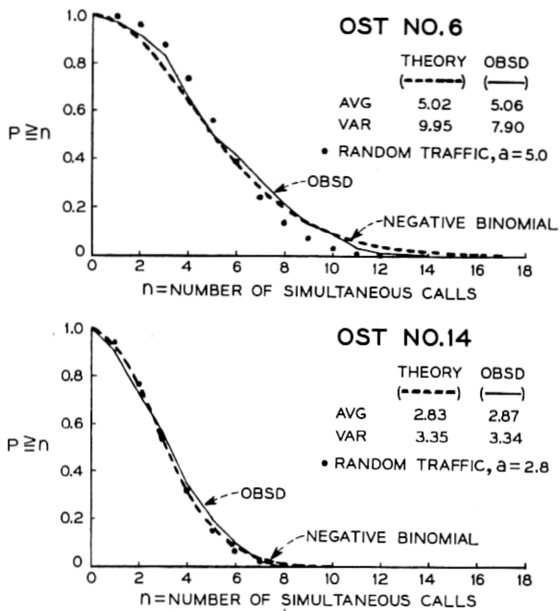

Fig. 22 — Combined overflow loads offered to alternate-route OST trunks from direct interoffice trunks; negative binomial theory vs throwdown observations.

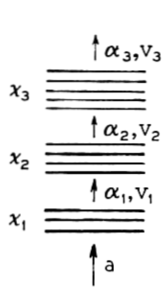

Fig. 23 — A full access group divided at several points to examine the traffic character at each point.

2 or 3. A load of a erlangs is submitted to each subgroup, a having the values 1, 2, 3, 4 or 5. What grade of service will be given?

Solution: The load overflowing each subgroup, when $a = 1$ for example, has the characteristics $\alpha = 0.0625$ and $v = 0.0790$. Then $A' = 2\alpha = 0.125$ and $V' = 2v = 0.158$. Reading on Fig. 26 gives the Equivalent Random values of $A = 1.04$ erlangs, $S = 2.55$ trunks. Reading on Fig. 12.1 with $C + S = 3.55$ when $C = 1$, and $A = 1.04$, we find $\alpha' = 0.0350$ and $\alpha'/(a_1 + a_2) = 0.0175$. We construct Table II in which loss values predicted by the Equivalent Random (ER) Theory are given in columns (3), (5) and (7). For comparison, the corresponding exact values given by Neovius* are shown in columns (2), (4) and (6). (Less exact loss

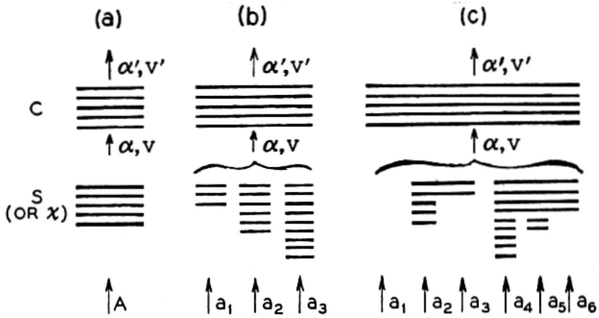

Fig. 24 — Various high usage trunk group arrangements producing the same total overflow α, v.

figures were given previously by Conny Palm[10]. The agreement is seen to be excellent for engineering needs for all values in the table.

Example 2: Suppose in Fig. 24(b) the random offered loads and paths are as given in Table III; we desire the proportion of overflow and the overflow load characteristics from an alternate route of 5 trunks.

Solution: The individual overflows α_1, v_1; α_2, v_2; and α_3, v_3 are read from Figs. 12 and 13 and recorded in columns (4) and (5) of the table. The α and v columns are totalled to obtain the sum-overflow average A' and variance V'. The Equivalent Random load A which, if submitted to S trunks would produce overflow A', V', is found from Fig. 26. Finally, with A submitted to $S + C$ trunks the characteristics α' and v', of the load overflowing the C trunks are found. The numerical values obtained

* Artificial Traffic Trials Using Digital Computers, a paper presented by G. Neovius at the First International Congress on the Application of the Theory of Probability on Telephone Engineering and Administration, Copenhagen, June, 1955.

TABLE II—CALCULATION OF LOSS IN A SIMPLE GRADED MULTIPLE
$g = 2, x_1 = x_2 = 3, \quad a_1 = a_2 = a = 1$ to 5, $\quad C = 1$ to 3

Load Submitted to each Subgroup in Erlangs a	Proportion of Each Subgroup Load which Overflows $= \alpha'/(a_1 + a_2)$					
	C = 1		C = 2		C = 3	
	True	ER	True	ER	True	ER
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.01737	0.0175	0.00396	0.0045	0.00077	0.00088
2	0.11548	0.115	0.05630	0.057	0.02438	0.024
3	0.24566	0.246	0.16399	0.163	0.10212	0.103
5	0.35935	0.363	0.27705	0.279	0.20535	0.210
5	0.44920	0.445	0.37336	0.370	0.30308	0.305

for this example are shown in the lower section of Table III. As before, of course, the "lost" calls are assumed cleared, and do not reappear in the system.

Example 3: A load of 18 erlangs is offered through four groups of 10-point selector switches to twenty-two trunks which have been designated as "high usage" paths in an alternate route plan. Which of the trunk arrangements shown in Fig. 27 is to be preferred, and to what extent?

Solution: By successive applications of the Equivalent Random method the overflow percentages for each of the three trunk arrangements are determined. The results are shown in column 2 of Table IV. The difference in percentage overflow between the three trunk plans is small; however, plan 2 is slightly superior followed by plans 3 and 1 in

TABLE III — CALCULATION OF OVERFLOWS FROM A SIMPLE ALTERNATE ROUTE TRUNK ARRANGEMENT

Subgroup Number	Offered Load in Erlangs a	Number of Trunks x	Overflow Loads	
			α	v
1	3.5	3	1.41	1.98
2	5.7	6	1.39	2.40
3	6.0	9	0.45	0.85
	15.2		3.25	5.23

Description of load offered to alternate route: $A' = 3.25$, $V' = 5.23$.
Equivalent straight multiple: $S = 5.8$ trunks, $A = 8.00$ erlangs (from Fig. 26).
Overflow from $C = 5$ alternate route trunks (enter Figs. 12 and 13 with $A = 8.0$ and $S + C = 10.8$: $\alpha' = 0.72$, $v' = 1.48$.
Proportion of load to commons which overflows = 0.72/3.25 = 0.22.
Proportion of offered load which overflows = 0.72/15.2 = 0.0475.

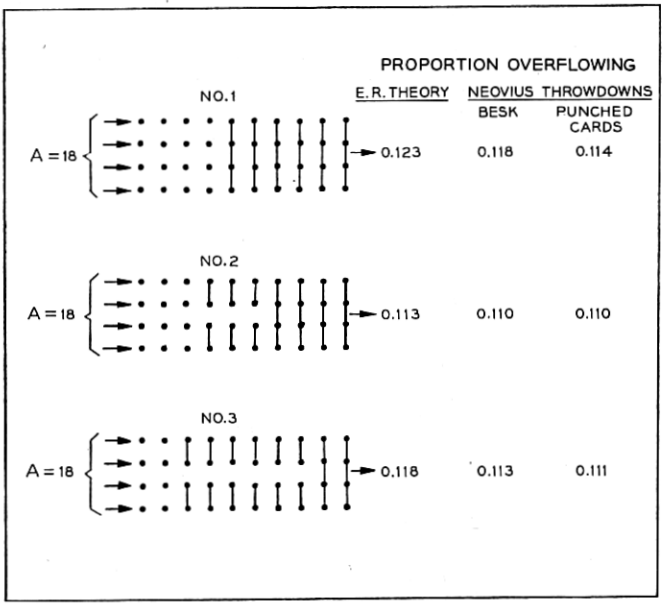

Fig. 27 — Comparison of losses on three graded arrangements of 22 trunks.

that order. The results of extensive simulations made by Neovius on the three trunk plans are available for comparison.* The values so obtained are seen to be very close to the ER theoretical ones; moreover the same order of preference among the three plans is indicated and with closely similar loss differentials between them.

7.3.1. *Throwdown Comparisons with Equivalent Random Theory on Simple Alternate Routing Arrangements with Lost Calls Cleared*

Results of manually run throwdowns on a considerable number of non-symmetrical single-stage alternate route arrangements are available. Some of these were shown in Fig. 20; they represent part of a projected multi-alternate route layout (to be described later) for outgoing calls from the local No. 1 crossbar Murray Hill–6 office in New York to all other offices in the metropolitan area. The paths hunted over initially are called direct trunks; they overflow calls to Office Selector Tandem (OST) groups, numbered from 1 to 17, which are located in widely dispersed central office buildings in the Greater New York area.

* Loc. cit.

TABLE IV—LOSS COMPARISON OF GRADED ARRANGEMENTS

Plan Number	Estimates of Percentage of Load Overflowing		
	ER Theory	Neovius Throwdowns	
		BESK Computer (262144 calls)	Punched Cards (10,000 calls)
(1)	(2)	(3)	(4)
1	12.3	11.81	11.4
2	11.3	10.98	11.0
3	11.8	11.25	11.1

TABLE V — COMPARISON OF THEORY AND THROWDOWNS FOR THE PARAMETERS OF LOADS OVERFLOWING THE COMMON TRUNKS IN SINGLE-STAGE GRADED MULTIPLES

OST (Alternate) Route Group		No. of Groups of Direct Trunks	Total No. of Trunks in Direct Groups	Total Load Offered to Direct Trunks		Total Overflow Load from OST			
						Theory		Throwdown	
Group no.	No. of trunks			Erlangs	Approximate No. of Calls (in 2.7 hours)	α'	v'	α'	v'
1	6	8	91	68.91	4950	2.00	5.50	2.36	6.52
2	3	3	45	37.49	2690	2.10	5.60	2.05	6.36
3	6	6	80	60.62	4355	1.50	4.00	1.30	5.67
4	3	6	52	38.49	2765	2.30	5.20	2.08	6.43
5	3	3	17	12.51	900	0.45	0.83	0.49	1.02
6	4	7	64	48.62	3490	2.50	5.90	2.36	4.88
7	8	12	78	57.42	4125	2.20	5.60	1.71	4.08
8	6	9	16	12.96	930	0.82	1.63	0.81	1.11
9	1	2	22	16.96	1220	1.30	2.60	1.02	1.73
10	5	6	10	9.52	685	0.78	1.40	1.05	2.07
11	8	13	16	16.43	1180	1.90	3.80	2.77	7.29
12	8	9	2	6.88	495	0.70	1.30	0.81	1.83
13	5	15	33	21.42	1540	1.75	3.30	1.16	2.01
14	2	7	11	8.05	580	1.46	2.20	1.63	2.14
15	9	15	8	11.97	860	1.60	3.25	1.55	4.12
16	11	22	34	27.46	1970	1.75	4.00	1.34	2.26
17	3	7	4	5.81	420	1.53	2.31	1.43	1.80
						26.64	58.42	25.92	61.32

In Table V are given certain descriptive data for the 17 OST trunk arrangements showing numbers of legs of direct trunks, total direct trunks, the offered erlangs and calls, and the mean and variance of the alternate routes' overflows, as obtained by the ER theory and by throwdowns.* The throwdown α' and v' values of the OST overflow

* Additional details of this simulation study are given in Section 7.4.

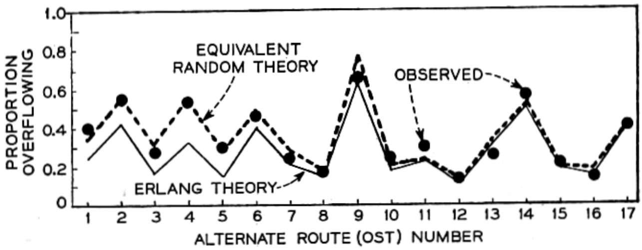

Fig. 28 — Comparison of theoretical and throwdown overflows from a number of first alternate routes.

were obtained by 36-second switch counts of those calls from each OST group which had come to rest on *subsequent* alternate routes.

On Fig. 28 is shown a summary of the observed and calculated proportions of "lost" to "offered" traffic at each OST alternate route group. As may be seen from the figure and the last four columns of Table V, the general agreement is quite good; the individual group variations are probably no more than to be expected in a simulation of this magnitude.

An assumption of randomness (which has sometimes been argued as returning when several overflows are combined) for the load offered to the OST's gives the Erlang E_1 loss curve on Fig. 28. This, as was to be expected, rather consistently understates the loss.

Since "switch-counts" were made on the calls overflowing each OST, the distributions of these overflows may be compared with those estimated by the Negative Binomial theory having the mean and variance predicted above for the overflow. Fig. 29 shows the individual and cumulative probability distributions of the overflow simultaneous calls from the first two OST alternate routes. As will be seen, the agreement is quite good even though this is traffic which has been twice "non-randomized." Comparison of the observed and calculated overflow means and variances in Table V indicates that similar agreement between observed and theoretical fitting distributions for most of the other OST's would be found.

7.3.2. *Comparison of Equivalent Random Theory with Field Results on Simple Alternate Routing Arrangements*

Data were made available to the author from certain measurements made in 1941 by his colleague C. Clos on the automatic alternate routing trunk arrangement in operation in the Murray Hill–2 central office in New York. Mr. Clos observed for one busy hour the load carried on several of its OST alternate route groups (similar to those shown in Table V for the Murray Hill–6 office, but not identical) by means of an electromechanical switch-counter having a six-second cycle. During each hour's observation, numbers of calls offered and overflowing were also recorded.

Although the loads offered to the corresponding direct trunks which overflowed to the OST group under observation were not simultaneously measured, such measurements had been made previously for several hours so that the relative contribution from each direct group was closely known. In this way the loads offered to each direct group which produced the total arriving before each OST group could be estimated with considerable assurance. From these direct group loads the character (mean and variance) of the traffic offered to and overflowing the OST's was predicted. The observed proportion of offered traffic which overflowed is shown on Fig. 30 along with the Equivalent Random theory prediction. The general agreement is again seen to be fairly good although with some tendency for the ER theory to predict higher than observed losses in the lower loss ranges; perhaps the disparity on in-

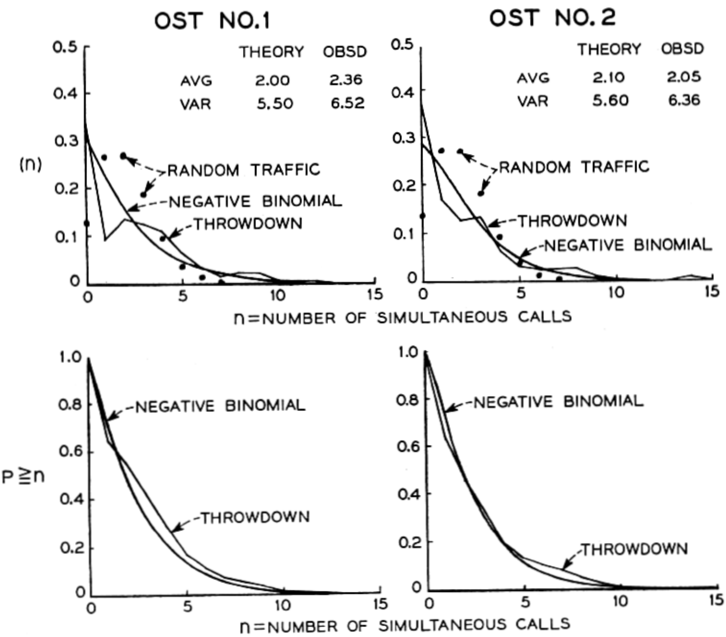

Fig. 29 — Distributions of loads overflowing from first alternate (OST) groups; negative binomial theory versus throwdown observations.

dividual OST groups is within the limits one might expect for data based on single-hour observations and for which the magnitudes of the direct group offered loads required some estimation. The assumption of random traffic offered to the OST gives, as anticipated, loss predictions (Erlang E_1) consistently below those observed.

More recently extensive field tests have been conducted on a working toll automatic alternate route system at Newark, New Jersey. High usage groups to seven distant large cities overflowed calls to the Newark-Pittsburgh alternate (final) route. Data describing the high usage groups and typical system busy hour loads are given in Table VI. (The loads, of course, varied considerably from day to day.) The size of the Pittsburgh route varied over the six weeks of the 1955 tests from 64 to 71 trunks. Altogether the system comprised some 255 intertoll trunks.

Observations were made at the Newark end of the groups by means of a Traffic Usage Recorder — making switch counts every 100 seconds — and by peg count and overflow registers. Register readings were photographically recorded by half-hourly, or more frequent, intervals. To

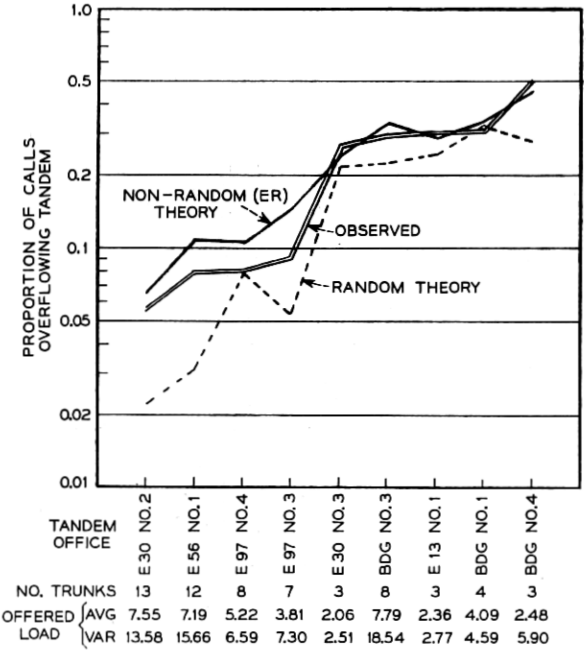

Fig. 30 — Observed tandem overflows in alternate route study at Murray Hill-2 (New York) 1940–1941.

TABLE VI—HIGH USAGE GROUPS AND TYPICAL SYSTEM BUSY HOUR LOADS

High Usage Group, Newark to:	Length of Direct Route (Air Miles)	Nominal Size of Group (Number of Trunks)	Typical Offered Load (erlangs)
Baltimore	170	18	19
Cincinnati	560	42	43
Cleveland	395	27	26
Dallas	1375	33	34
Detroit	470	37	36
Kansas City	1100	26	23
New Orleans	1170	5	4

compare theory with the observed overflow from the final route, estimates of the offered load A' and its variance V' are required. In the present case, the total load offered to the final route in each hour was estimated as

$$A' = \text{Average of Offered Load}$$

$$= \frac{\text{Peg Count of Calls Offered to Pittsburgh Group}}{(\text{Peg Count of Offered Calls}) - (\text{Peg Count of Overflow Calls})} \times \text{Average Load Carried by Pittsburgh Group}$$

The variance V' of the total load offered to the final route was estimated for each hour as

$$V' = \text{Variance of Offered Load}$$

$$= A' - \sum_{i=1}^{7} \alpha_i + \sum_{i=1}^{7} v_i$$

where α_i and v_i are, respectively, the average and variance of the load overflowing from the ith high usage group. (The expression, $A' - \sum_{i=1}^{7} \alpha_i$, is an estimate of the average — and, therefore of the variance — of the first-routed traffic offered directly to the final route. Thus the total variance, V', is taken as the sum of the direct and overflow components.) Using A', V' and the actual number, C, of final route trunks in service, the proportion of offered calls expected to overflow was calculated for the traffic and trunk conditions seen for 25 system busy hours from February 17 to April 1, 1955 on the Pittsburgh route. The results are displayed on Fig. 31, where certain traffic data on each hour are given in the lower part of the figure. The hours are ordered — for convenience in plotting and viewing — by ascending proportions of calls overflowing the group; observed results are shown by the double line

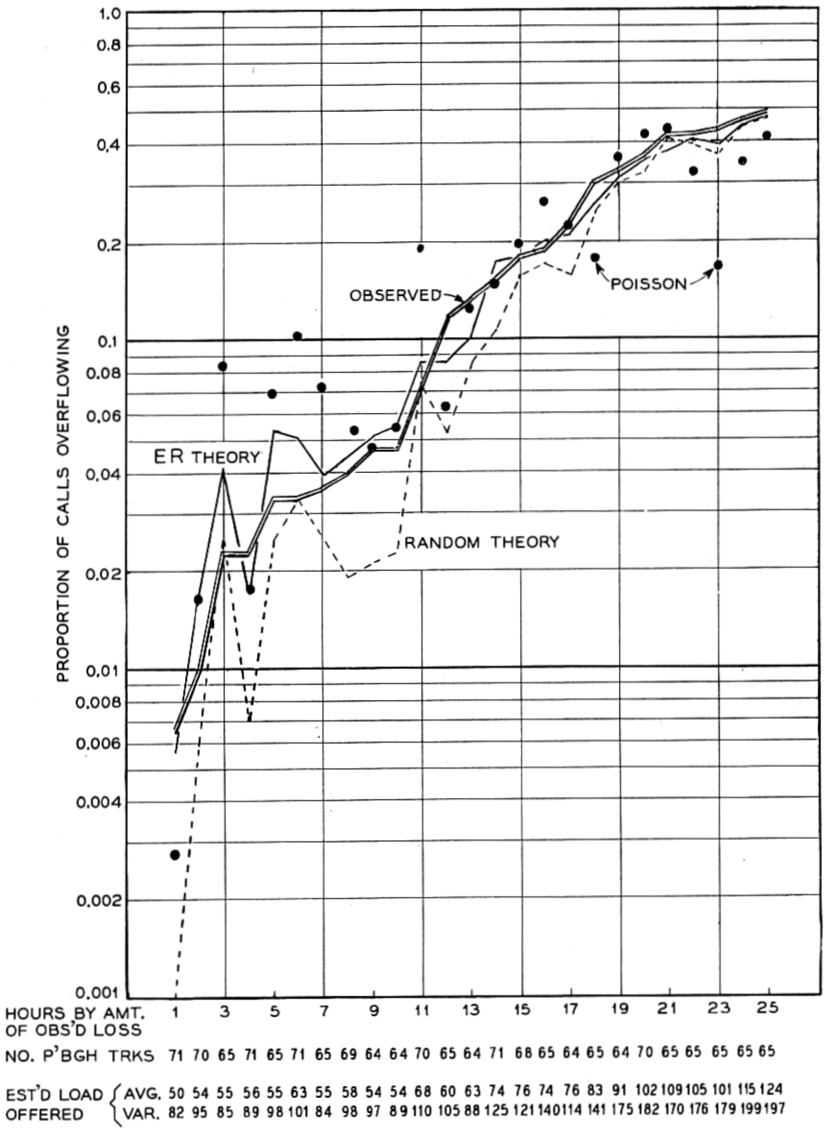

Fig. 31 — Final route (Newark-Pittsburgh) overflows in 1955 toll alternate route study.

curve. The superposed single line is the corresponding estimate by ER theory of the hour-to-hour call losses. As may be seen, theory and observation are in good agreement both point by point and on the average over the range of losses from 0.01 to 0.50. The dashed line shows the prediction of final route loss for each hour on the assumption that the offered traffic A' was random. Such an assumption gives consistently low estimates of the existing true loss.

As of interest, a series of heavy dots is included on Fig. 31. These are the result of calculating the Poisson Summation, $P(C,L)$, where L is the average load *carried* on, rather than offered to, the C trunks. It is interesting that just as in earlier studies in this paper on straight groups of intertoll trunks (for example as seen on Fig. 7), the Poisson Summation with load carried taken as the load offered parameter, gives loss values surprisingly close to those observed. Also, as before, this summation has a tendency to give too-great losses at light loadings of a group and too-small losses at the heavier loadings.

7.4 *Prediction of Traffic Passing Through a Multi-Stage Alternate Route Network*

In the contemplated American automatic toll switching plan, wide advantage is expected to be taken of the efficiency gains available in multi-alternate routing. Thus any procedure for traffic analysis and prediction needs to be adaptable for the more complex multi-stage arrangements as well as the simpler single-stage ones so far examined. Extension of the Equivalent Random theory to successive overflows is easily done since the characterizing parameters, average and variance, of the load overflowing a group of paths are always available.

Since few cases of more than single-stage automatic alternate routing are yet in operation in the American toll plant, it is not readily possible to check an extension of the theory with actual field data. Moreover collecting and analyzing observations on a large operating multi-alternate route system would be a comparatively formidable experiment.

However, in New York city's local interoffice trunking there is a very considerable development of multi-alternate routing made possible by the flexibility of the marker arrangements in the No. 1 crossbar switching system. None of these overflow arrangements has been observed as a whole, simultaneously and in detail. The Murray Hill–2 data in OST groups reviewed in Section 7.3.2 were among the partial studies which have been made.

In connection with studies made just prior to World War II on these

TABLE VII — SUM OF DIRECT GROUP OVERFLOW LOADS, OFFERED TO OST'S

	Theory	Observed
Average	86.06	87.12
Variance	129.5	127.4

local multi-alternate route systems, a throwdown was made in 1941 on a proposed trunk plan for the Murray Hill–6 office. The arrangement of trunks is shown on Fig. 32. Three successive alternate routes, Office Selector Tandems (OST), Crossbar Tandem (XBT), and Suburban Tandem (ST), are available to the large majority of the 123 direct trunk groups leading outward to 169 distant offices. (The remaining 46 parcels of traffic did not have direct trunks to distant offices but, as indicated on the diagram, offered their loads directly to a tandem group.) A total of 726 trunks is involved, carrying 475 erlangs of traffic.

A throwdown of 34,001 offered calls corresponding to 2.7 hours of traffic was run. Calls had approximate exponential holding times, averaging 135 seconds. Records were kept of numbers of calls and the load from the traffic parcels offered to each direct group, as they were carried or passed beyond the groups of paths to which they had access. Loads carried by each trunk in the system were also observed by means of a 36-second "switch-count." (The results on the 17 OST groups reported in Section 7.3.1 were part of this study.)

Comparisons of observation and theory which are of interest include the combined loads to and overflowing the 17 OST's. Observed versus calculated parameters (starting with theory from the original direct group submitted loads) are given in Table VII. The agreement is seen to be very good.

The corresponding comparison of total load from all the OST's is given in Table VIII. Again the agreement is highly satisfactory.

Not all of the overflow from the OST's was offered to the 22 crossbar tandem trunks; for economic reasons certain parcels by-passed XBT and were sent directly to Suburban Tandem.* This posed the problem of breaking off certain portions of the overflow from the OST's, to be added again to the overflow from XBT. An estimate was needed of the contribution made by each parcel of direct group traffic to any OST's overflow. These were taken as proportional to the loads *offered* the OST by each direct group (this assumes that each parcel suffers the same over-

* In the toll alternate route system by-passing of this sort will not occur.

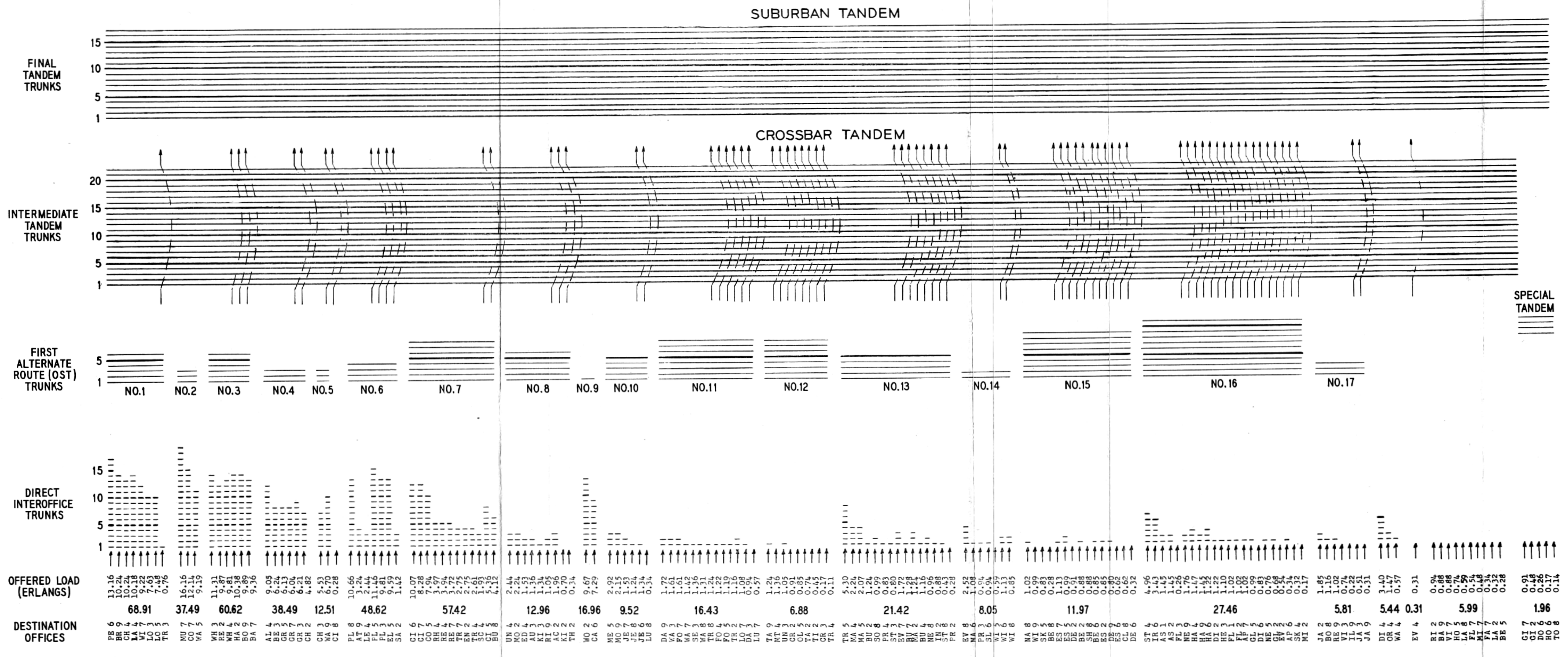

Fig. 32. — Multi-alternate route trunking arrangement at Murray Hill—6 (New York) local No. 1 crossbar office.

flow probability). The variance of this overflow portion by-passing XBT was estimated by assigning to it the same variance-to-average ratio as was found for the total load overflowing the OST. Subtracting the means and variances so estimated for all items by-passing XBT, left an approximate load for XBT from each OST. Combining these corrected overflows gave mean and variance values for offered load to XBT. Observed values

TABLE VIII — SUM OF LOADS OVERFLOWING OST'S

	Theory	Observed
Average	26.64	25.92
Variance	58.42	61.32

TABLE IX — LOAD OFFERED TO CROSSBAR TANDEM

	Theory	Observed
Average	25.18	25.51
Variance	47.67	56.10

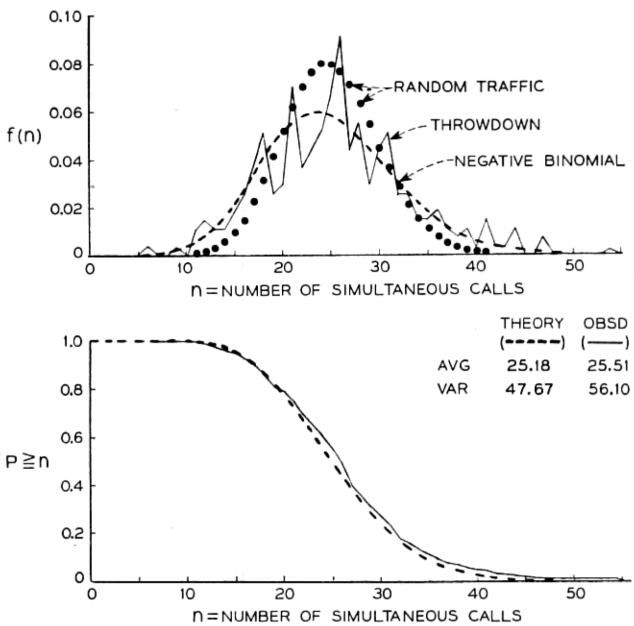

Fig. 33 — Distribution of load offered to crossbar tandem trunks; negative binomial theory versus throwdown observations.

TABLE X — LOAD OVERFLOWING CROSSBAR TANDEM

	Theory	Observed
Average	6.55	6.47
Variance	23.80	33.48

and those calculated (in the above manner) are given in Table IX. Fig. 33 shows the distribution of XBT offered loads, observed and calculated. The agreement is very satisfactory. The random traffic (Poisson) distribution, is of course, considerably too narrow.

In a manner exactly similar to previous cases, the Equivalent Random load method was applied to the XBT group to obtain estimated parameters of the traffic overflowing. Comparison of observation and theory at this point is given in Table X.

Fig. 34 shows the corresponding observed and calculated distributions

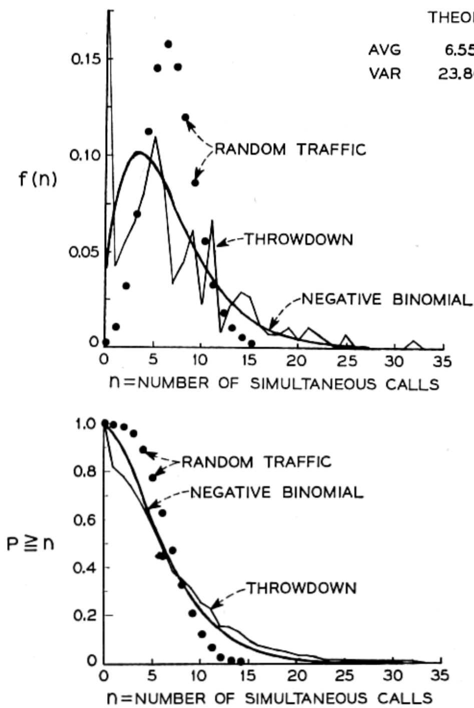

Fig. 34 — Distribution of calls from crossbar tandem trunks; negative binomial theory versus throwdown observations.

of simultaneous calls. The agreement again is reasonably good, in spite of the considerable disparity in variances.

The overflow from XBT and the load which by-passed it, as well as some other miscellaneous parcels of traffic, were now combined for final offer to the Suburban Tandem group of 17 trunks. The comparison of parameters here is again available in Table XI. On Fig. 35 are shown the observed and calculated distributions of simultaneous calls for the load offered to the ST trunks. The agreement is once again seen to be very satisfactory.

We now estimate the loss from the ST trunks for comparison with the actual *proportion of calls* which failed to find an idle path, and finally

TABLE XI — LOAD OFFERED TO SUBURBAN TANDEM

	Theory	Observed
Average	15.38	14.52
Variance	42.06	48.53

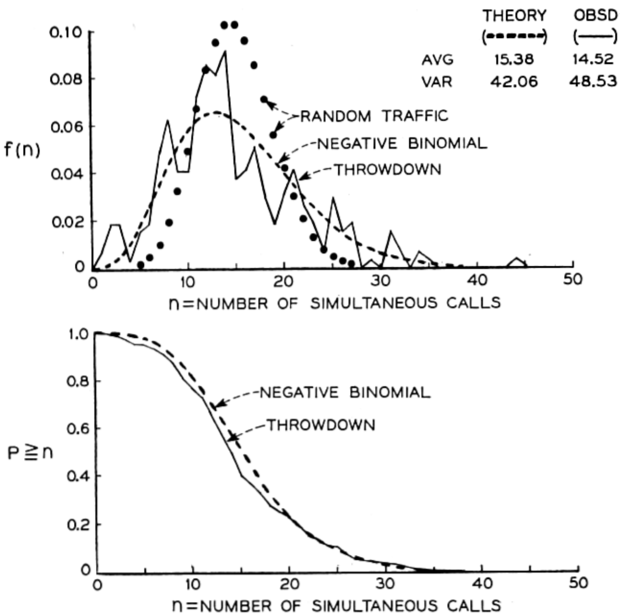

Fig. 35 — Distribution of load offered to suburban tandem trunks; negative binomial theory versus throwdown observations.

TABLE XII — GRADE OF SERVICE ON ST GROUP

	Theory	Observation	Observation	
Load submitted (erlangs)	15.38	14.52	Number of calls submitted	1057
Load overflowing (erlangs)	3.20	2.63	Number of calls overflowing	200
Proportion load overflowing	0.209	0.181	Proportion of calls overflowing	0.189

TABLE XIII — GRADE OF SERVICE ON THE SYSTEM

	Theory	Observed
Total load submitted	475 erlangs	34,001 calls
Total load overflowing	3.20 erlangs	200 calls
Proportion of load not served	0.00674	0.00588

compare the proportions of all traffic offered the system which failed to find a trunk immediately. See Tables XII and XIII.

After these several and varied combinations of offered and overflowed loads to a system of one direct and three alternate routes it is seen that the final prediction of amount of load finally lost beyond the ST trunks is gratifyingly close to that actually observed in the throwdown. The prediction of the system grade of service is, of course, correspondingly good.

It is interesting in this connection to examine also the proportions overflowing the ST group when summarized by parcels contributed from the several OST groups. The individual losses are shown on Fig. 36; they appear well in line with the variation one would expect from group to group with the moderate numbers of calls which progressed this far through the multiple.

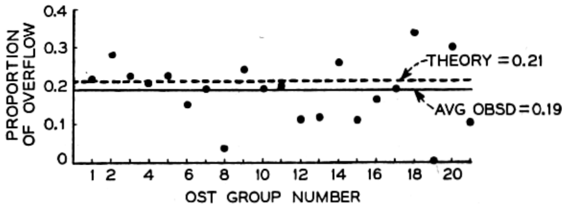

Fig. 36 — Overflow calls on third alternate (ST) route.

7.4.1 *Correlation of Loss with Peakedness of Components of Non-Random Offered Traffic*

Common sense suggests that if several non-random parcels of traffic are combined, and their joint proportion of overflow from a trunk group is P, the parcels which contain the more peaked traffic should experience overflow proportions larger than P, and the smoother traffic an overflow proportion smaller than P. It is by no means clear however, *a priori*, the extent to which this would occur. One might conjecture that if any one parcel's contribution to the total combined load is small, its loss would be caused principally by the aggregate of calls from the other parcels, and consequently its own loss would be at about the general average loss P, and hence not very much determined by its own peakedness. The Murray Hill-6 throwdown results may be examined in this respect. The mean and variance of each OST-parcel of traffic, for example, arriving at the final ST route was recorded, together with, as noted before, its own proportion of overflow from the ST trunks. The variance/mean over-dispersion ratio, used as a measure of peakedness, is plotted for each parcel of traffic against its proportion of loss on Fig. 37. There is an undoubted, but only moderate, increase in proportion of overflow with increased peakedness in the offered loads.

It is quite possible, however, that by recognizing the differences between the service given various parcels of traffic, significant savings in final route trunks can be effected for certain combinations of loads and trunking arrangements. Of particular interest is the service given to a parcel of random traffic offered directly to the final route when compared

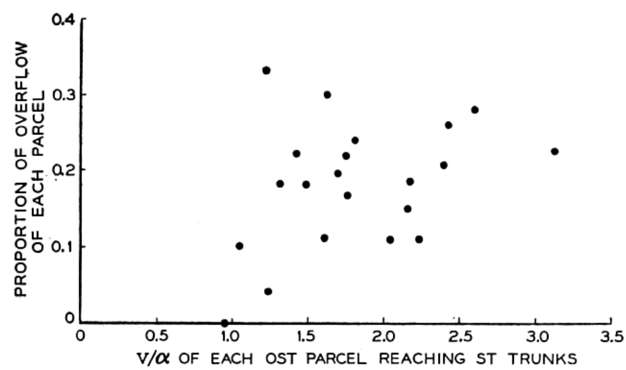

Fig. 37 — Effect of peakedness on overflow of a parcel of traffic reaching an alternate route.

with that received by non-random parcels overflowing to it from high usage groups.

7.5 *Expected Loss on First Routed Traffic Offered to Final Route*

The congestion experienced by the first-routed traffic offered to the final group in a complex alternate route arrangement [such as the right hand parcels in Figs. 10(c) and (d)] will be the same as encountered in a series of random tests of the final route by an independent observer, that is, it will be the proportion of time that all of the final trunks are busy. As noted before, the distribution of simultaneous calls n (and hence the congestion) on the C final trunks produced by some specific arrangement of offered load and high usage trunks can be closely simulated by that due to a single Equivalent Random load offered to a straight group of $S + C$ trunks. Then the proportion of time that the C trunks are busy in such an equivalent system provides an estimate of the corresponding time in the real system; and this proportion should be approximately the desired grade of service given the first routed traffic.

Brockmeyer[11] has given an expression (his equation 36) for the proportion of time, R_1, in a simple $S + C$ system with random offer A, and "lost calls cleared," that all C trunks are busy, independent of the condition of the S-trunks:

$$\begin{aligned} R_1 &= f(S,C,A) \\ &= E_{1,S+C}(A) \frac{\sigma_{C+1}(S)}{\sigma_C(S)} \end{aligned} \qquad (30)$$

where

$$\sigma_C(S) = \sum_{m=0}^{S} \binom{C - 1 + m}{m} \frac{A^{S-m}}{(S - m)!}$$

However, $\sigma_C(S)$ is usually calculated more readily step-by-step using the formula

$$\sigma_C(S) = \sigma_C(S - 1) + \sigma_{C-1}(S),$$

starting with

$$\sigma_C(O) = 1 \qquad \text{and} \qquad \sigma_O(S) = A^S/S!$$

The average load carried on the C paths is clearly

$$L_C = A[E_{1,S}(A) - E_{1,S+C}(A)], \qquad (31)$$

and the variance of the carried load can be shown to be*

$$V_C = AL_C \frac{\sigma_1(S)}{\sigma_2(S)} - ACE_{1,S+C}(A) + L_C - L_C^2 \qquad (32)$$

On Fig. 38, R_1 values are shown in solid line curves for several combinations of A and C over a small range of S trunks. The corresponding losses R_2 for *all* traffic offered the final group, where $R_2 = \alpha'/A'$, are shown as broken curves on the same figure. The R_2 values are always above R_1, agreeing with the common sense conclusion that a random component of traffic will receive better service than more peaked non-random components.

However, there are evidently considerable areas where the loss difference between the two R's will not be large. In the loss range of principal interest, 0.01 to 0.10, there is less proportionate difference between the R's as the $A = C$ paired values increase on Fig. 38. For example, at $R_2 = 0.05$, and $A = C = 10$, $R_2/R_1 = 0.050/0.034 = 1.47$; while for $A = C = 30$, $R_2/\mathrm{R}_1 = 0.050/0.044 = 1.13$. Similarly for $A = 2C$, the R_2/R_1 ratios are given in Table XIV. Again the rapid decrease in the R_2/R_1 ratio is notable as A and C increase.

F. I. Tånge of the Swedish Telephone Administration has performed elaborate simulation studies on a variety of semi-symmetrical alternate route arrangements, to test the disparity between the R_1 and R_2 types of losses on the final route.† For example if g high-usage groups of 8 paths each, jointly overflow 2.0 erlangs to a final route which also serves 2.0 erlangs of first routed traffic, Tånge found the differences in losses between the two 2-erlang parcels, $R_{\text{high usage (h.u.)}} - R_1$, shown in column 9 of Table XV. The corresponding ER calculations are performed in columns 2 to 8, the last of which is comparable with the throwdown values of column 9. The agreement is not unreasonable considering the sensitiveness of determining the difference between two small probabilities of loss. A quite similar agreement was found for a variety of other loads and trunk arrangements.

* In terms of the first two factorial moments of n: V_C is given by

$$V_C = M_{(2)} + M_{(1)} - M_{(1)}^2, \qquad \text{where } M_{(1)} = L_C$$

General expressions $M_{(i)}$ for the factorial moments of n are derived in an unpublished memorandum by J. Riordan.

† Optimal Use of Both-Way Circuits in Cases of Unlimited Availability, a paper by F. I. Tånge, presented at the First International Congress on the Application of the Theory of Probability in Telephone Engineering and Administration, June 1955, Copenhagen.

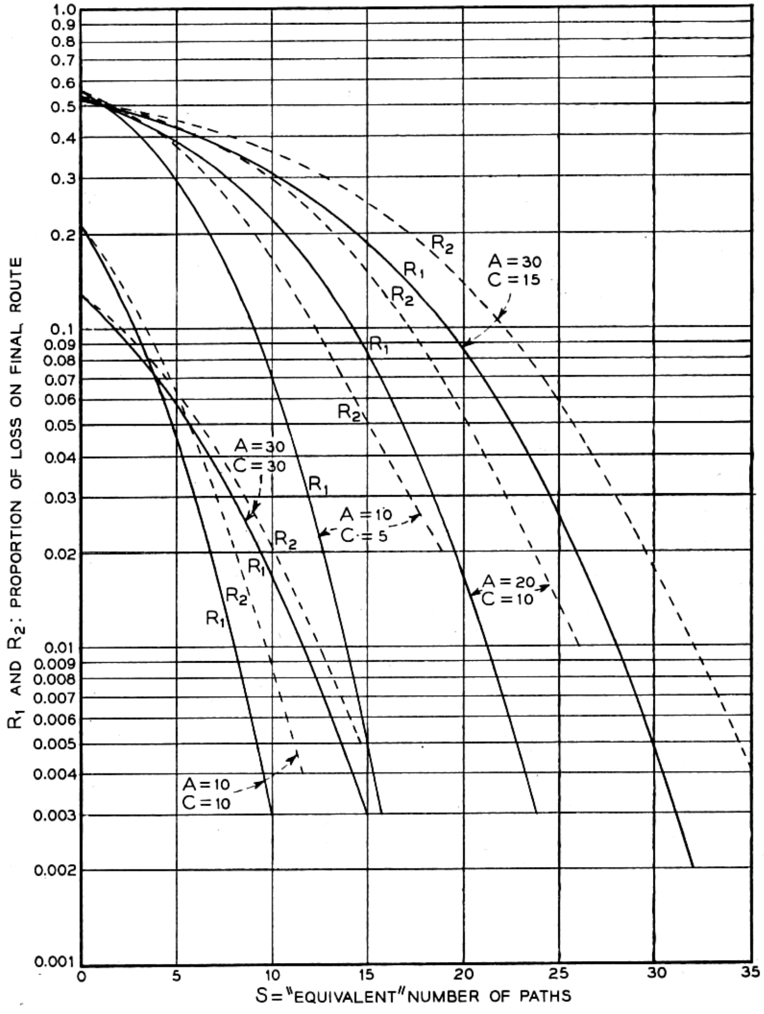

Fig. 38 — Comparison of R_1 and R_2 losses under various load and trunk conditions.

TABLE XIV—THE R_2/R_1 RATIOS FOR $A = 2C$

A	C	R_2/R_1 when $R_2 = 0.05$
10	5	10.6
20	10	3.25
30	15	2.44

TABLE XV—COMPARISON OF E.R. THEORY AND THROWDOWNS ON DISPARITY OF LOSS BETWEEN HIGH USAGE OVERFLOW AND RANDOM OFFER TO A FINAL GROUP

(8 trunks in each high usage group; 9 final trunks serving 2.0 erlangs high usage overflow and 2.0 erlangs first routed traffic.)

Number of Groups of 8 High Usage Trunks	ER Theory ($A' = 4.0$)							Tånge Throwdown $R_{h.u.} - R_1$
	V'	A	S	$R_2 = \alpha'/A'$	R_1	$R_{h.u.} = 2R_2 - R_1$	$R_{h.u.} - R_1 = 2(R_2 - R_1)$	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	5.77	7.51	4.17	0.0375	0.0251	0.0499	0.0248	0.0180
2	5.80	7.50	4.25	0.0383	0.0255	0.0511	0.0256	0.0247
3	5.74	7.44	4.08	0.0369	0.0248	0.0490	0.0242	0.0286
4	5.68	7.30	3.91	0.0362	0.0247	0.0477	0.0230	0.0276
5	5.64	7.20	3.80	0.0355	0.0242	0.0468	0.0226	0.0245
6	5.58	7.06	3.64	0.0350	0.0240	0.0460	0.0220	0.0221
7	5.55	7.00	3.56	0.0345	0.0238	0.0452	0.0204	0.0202
8	5.51	6.91	3.45	0.0335	0.0236	0.0434	0.0198	0.0188
9	5.47	6.81	3.34	0.0325	0.0231	0.0419	0.0188	0.0177
10	5.45	6.76	3.29	0.0312	0.0225	0.0399	0.0174	0.0166

Limited data are available showing the disparity of R_1 and R_2 in actual operation in a range of load and trunk values well beyond those for which R_1 values have been calculated. Special peg count and overflow registers were installed for a time on the final route during the 1955 Newark alternate route tests. These gave separate readings for the calls from high usage groups, and for the first routed Newark to Pittsburgh calls. Comparative losses for 17 hours of operation over a wide range of loadings are shown on Fig. 39. The numbers at each pair of points give the per cent of final route offered traffic which was first routed (random). In general, approximately equal amounts of the two types of traffic were offered.

In 6 of the hours almost identical loss ratios were observed, in 7 hours the overflow-from-high-usage calls showed higher losses, and in 4 hours lower losses, than the corresponding first routed calls. The non-random calls clearly enjoyed practically as good service as the random calls. This result is not in disagreement with what one might expect from theory. To compare directly with the Newark-Pittsburgh case we should need curves on Fig. 38 expanded to correspond to A', V' values of (50, 85) to (120, 200). Examining the mid-range case of $C = 65$, $A' = 70$, $V' = 120$, we find $A \doteq 123$, $S \doteq 54$. Here A is approximately $2C$; extrapolating the $A = 2C$ curves of Fig. 38 to these much higher values of A and C suggests that R_2/R_1 would be but little different from unity.

It is clear from the above theory, throwdowns, and actual observation that there are certain areas where the service differences given first routed and high usage trunk overflow parcels of traffic are significant. In Section 8, where practical engineering methods are discussed, curves are presented which permit recognition of this fact in the determination of final trunk requirements.

7.6 *Load on Each Trunk, Particularly the Last Trunk, in a Non-Slipped Alternate Route*

In the engineering of alternate route systems it is necessary to determine the point at which to limit a high usage group of trunks and send the overflow traffic via an alternate route. This is an economic problem whose solution requires an estimate of the load which will be carried on

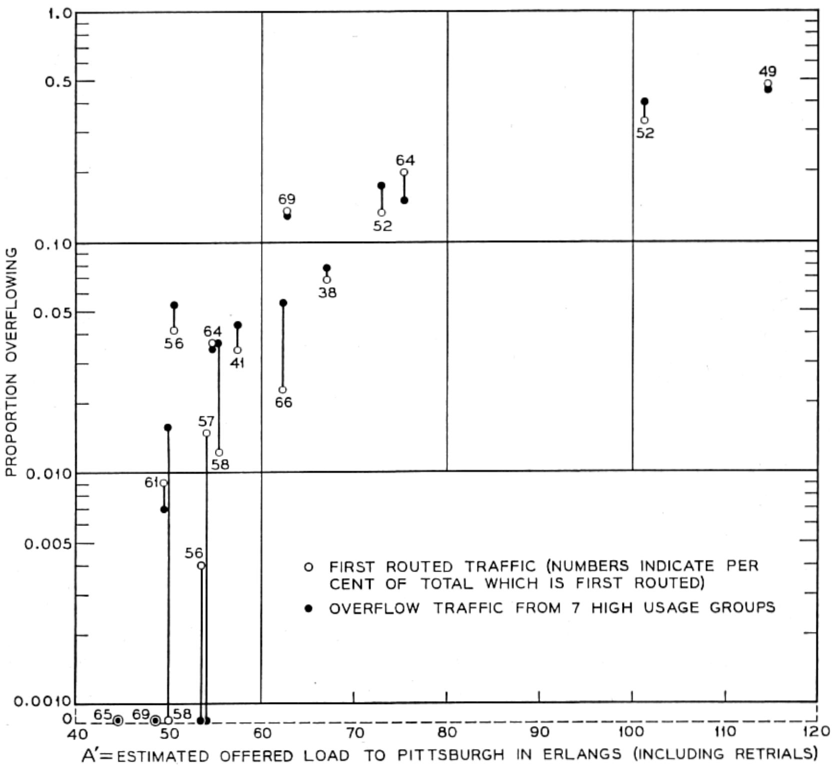

FIG. 39 — Comparison of losses on final route (Newark to Pittsburgh) for high usage overflow and first routed traffic.

the last trunk of a straight high usage group of any specified size, carrying either first or higher choice traffic or a mixture thereof.*

The Equivalent Random theory readily supplies estimates of the loads carried by any trunk in an alternate routing network. After having found the Equivalent Random load A offered to $S + C$ trunks which corresponds to the given parameters of the traffic offered to the C trunks, it is a simple matter to calculate the expected load ℓ on any one of the C trunks if they are not slipped or reversed. The load on the ith trunk in a simple straight multiple (or the $S + j$th in a divided multiple of S lower and C upper trunks), is

$$\ell_i = l_{S+j} = A[E_{1,S+j-1}(A) - E_{1,S+j}(A)] \tag{33}$$

where $E_{1,n}(A)$ is the Erlang loss formula. A moderate range of values of ℓ_i versus load A is given on Figure 40.†

Using this method, selected comparisons of theoretical versus observed loads carried on particular trunks at various points in the Murray-Hill–6 throwdown are shown in Fig. 41; these include the loads on each of the trunks of the first two OST groups of Fig. 32, and on the second and third alternate routes, crossbar and suburban tandem, respectively. The agreement is seen to be fairly good, although at the tail end of the latter two groups the observed values drop away somewhat from the theoretical ones. There seems no explanation for this beyond the possibility that the throwdown load samples here are becoming small and might by chance have deviated this far from the true values (or the arbitrary breakdown of OST overflows into parcels offered to and bypassing XBT may well have introduced errors of sufficient amount to account for this disparity). As is well known, (33) gives good estimates of the loads carried by each trunk in a high usage group to which random (Poisson) traffic is offered; this relationship has long been used for the purpose in Bell System trunk engineering.

8. PRACTICAL METHODS FOR ALTERNATE ROUTE ENGINEERING

To reduce to practical use the theory so far presented for analysis of alternate route systems, working curves are needed incorporating the

* The proper selection point will be where the circuit annual charge per erlang of traffic carried on the last trunk, is just equal to the annual charge per erlang of traffic carried by the longer (usually) alternate route enlarged to handle the overflow traffic.

† A comprehensive table of ℓ_i is given by A. Jensen as Table IV in his book "Moe's Principle," Copenhagen, 1950; coverage is for $\ell \geqq 0.001$ erlang, $i = 1(1)140$; $A = 0.1(0.1)10, 10(1)50, 50(4)100$. Note that $n + 1$, in Jensen's notation, equals i here.

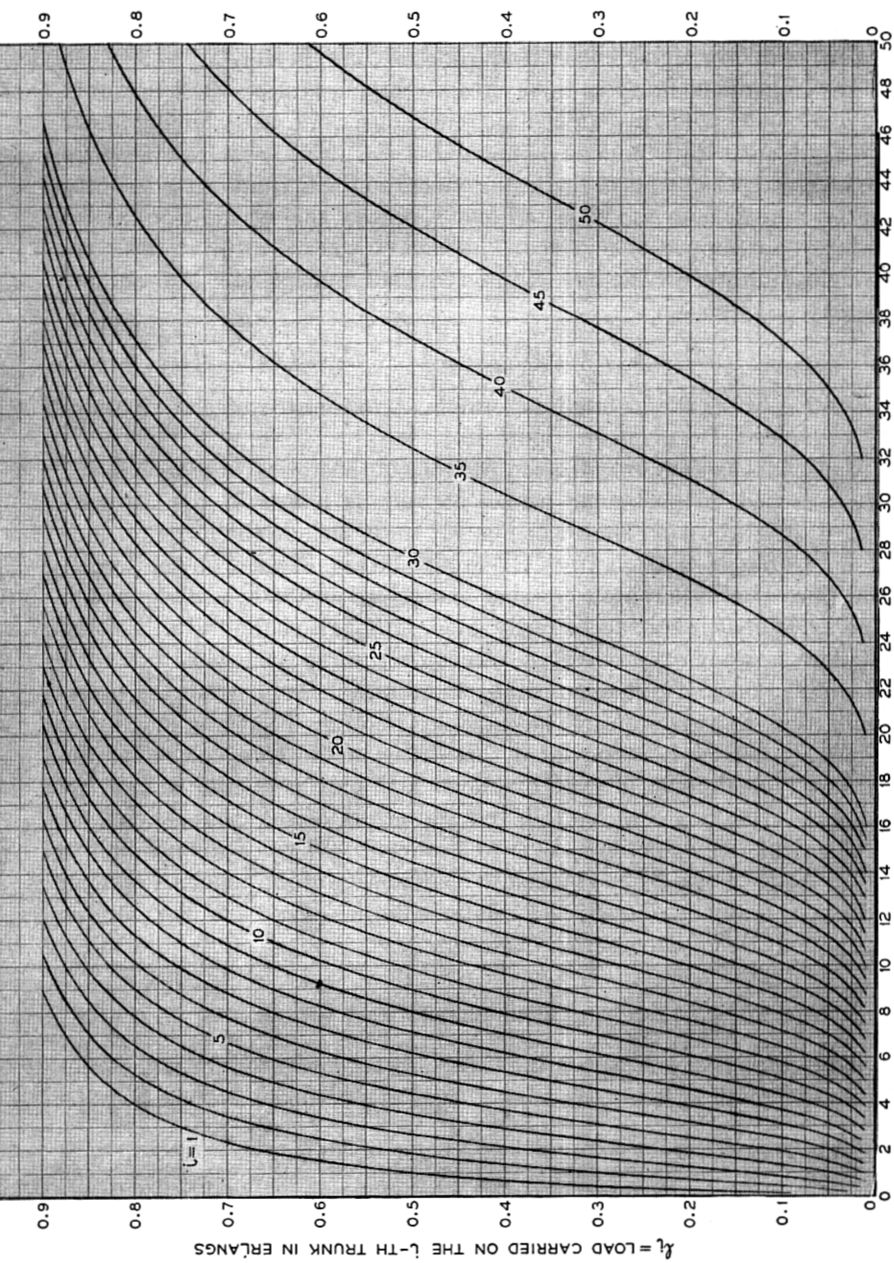

Fig. 40 — Load on each trunk of a straight multiple.

pertinent load-loss relationships. The methods so far discussed, and proposed for use, will be briefly reviewed.

The dimensioning of each high usage group of trunks is expected to be performed in the manner currently in use, as described in Section 7.6. The critical figure in this method is the load carried on the last high usage trunk, and is chosen so as to yield an economic division of the offered load between high usage and alternate route trunks. Fig. 40 is one form of load-on-each-trunk presentation suitable for choosing economic high usage group size once the permitted load on the last trunk is established.

The character (average α and variance v) of the traffic overflowing each high usage group is easily found from Figs. 12 and 13 (or equivalent

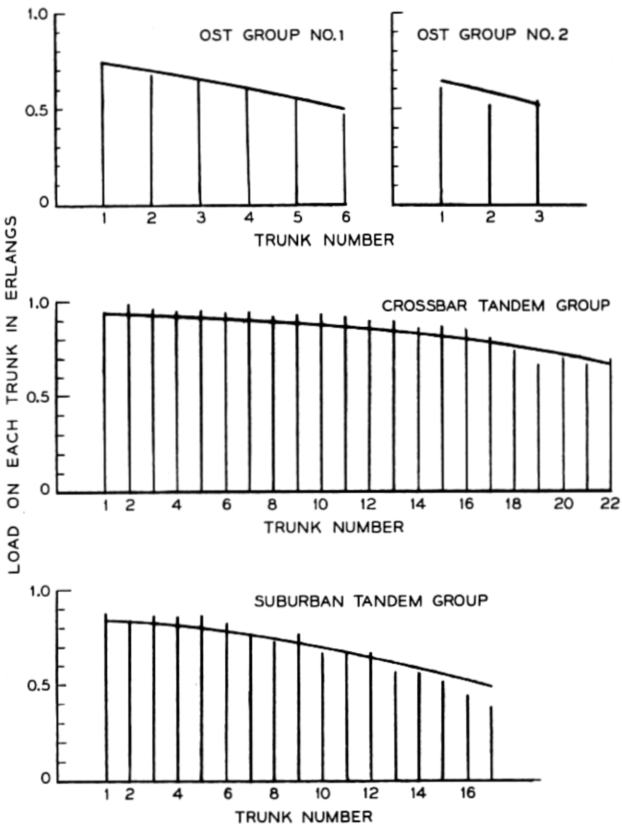

Fig. 41 — Comparison of load carried by each alternate route trunk; theory versus throwdowns.

tables). The respective sums of the overflow α's and v's, give A' and V' by (28) and (29); they provide the necessary statistical description of traffic offered to the alternate route.

According to the Equivalent Random method for estimating the alternate route trunks required to provide a specified grade of service to the overflow traffic A', one next determines a random load A which when submitted to S trunks will yield an overflow with the same character (A', V') as that derived from the complex system's high usage groups. An alternate route of C trunks beyond these S trunks is then imagined. The erlang overflow α', with random offer A, to $S + C$ trunks is found from standard E_1-formula tables or curves (such as Fig. 12).

The ratio $R_2 = \alpha'/A'$ is a first estimate of the grade of service given to each parcel of traffic offered to the alternate route. As discussed in Section 7.5, this service estimate, under certain conditions of load and trunk arrangement, may be significantly pessimistic when applied to a first routed parcel of traffic offered directly to the alternate route. An improved estimate of the overflow probability for such first routed traffic was found to be R_1 as given by (30).

8.1 *Determination of Final Group Size with First Routed Traffic Offered Directly to the Final Group*

When first routed traffic is offered directly to the final group, its service R_1 will nearly always be poorer than the *overall* service given to those other traffic parcels enjoying high usage groups. The first routed traffic's service will then be controlling in determining the final group size. Since R_1 is a function of S, C and A in the Equivalent Random solution (30), and there is a one-to-one correspondence of pairs of A and S values with A' and V' values, engineering charts can be constructed at selected service levels R_1 which show the final route trunks C required, for any given values of A' and V'. Figs. 42 to 45 show this relation at service levels of $R_1 = 0.01, 0.03, 0.05$ and 0.10, respectively.*

* On Fig. 42 (and also Figs. 46–49) the low numbered curves assume, at first sight, surprising shapes, indicating that a load with given average and variance would require fewer trunks if the average were *increased*. This arises from the sensitivity of the tails of the distribution of offered calls, to the V'/A' peakedness ratio which, of course, decreases with increases in A'. For example, with $C = 4$ trunks and fixed $V' = 0.52$, the loss rapidly decreases with increasing A':

A'	V'/A'	A	S	α'	α'/A'
0.28	1.86	6.1	10.	0.0155	0.055
0.33	1.58	3.0	5.0	0.0081	0.025
0.40	1.30	1.42	2.03	0.0036	0.009
0.52	1.00	0.52	0	0.0008	0.002

These four R_1 levels would appear to cover the most used engineering range. For example, if the traffic offered to the final route (including the first routed traffic) has parameters $A' = 12$ and $V' = 20$, reading on Fig. 43 indicates that to give $P = 0.03$ "lost calls cleared" service to the first routed traffic, $C = 19$ final route trunks should be provided. (For random traffic ($V' = A' = 12$), 17.8 trunks would be required.)

Other charts, of course, might be constructed from which R_1 could be read for specific values of A', V' and C. They would become voluminous, however, if a wide range of all three variables were required.

8.2 *Provision of Trunks Individual to First Routed Traffic to Equalize Service*

If the difference between the service R_1 given the first routed parcel of traffic and the service given all of the other parcels, is material, it may be desirable to take measures to diminish these inequities. This may readily be accomplished by setting aside a number of the otherwise full access final route trunks, for exclusive and first choice use of the first routed traffic. High usage groups are now provided for all parcels of traffic. The alternate route then services their combined overflow. The overall grade of service given the ith parcel of offered traffic in a single stage alternate route system will then be approximately

$$P_i \doteq E_{1,x_i}(a_i)R_2 = E_{1,x_i}(a_i)\frac{\alpha'}{A'}{}^{*} \tag{34}$$

Thus the service will tend to be uniform among the offered parcels when all send substantially identical proportions of their offered loads to the alternate route. And the natural provision of "individual" trunks for the exclusive use of the first routed traffic would be such that the same proportion should overflow as occurs in the associated high usage groups.

This procedure cannot be followed literally since high usage group size is fixed by economic considerations rather than any predetermined overflow value. The resultant overflow proportions will commonly vary over a considerable range. In this circumstance it would appear reasonable to estimate the objective overflow proportion to be used in establishing the individual group for the first routed traffic, as some weighted average $\bar{b}$ of the overflow proportions of the several high usage groups. Thus with weights g and overflow proportions b,

$$\bar{b} = \frac{g_1 b_1 + g_2 b_2 + \cdots}{g_1 + g_2 + \cdots} \tag{35}$$

* Although not exact, this equation can probably be accepted for most engineering purposes where high usage trunks are provided for each parcel of traffic.

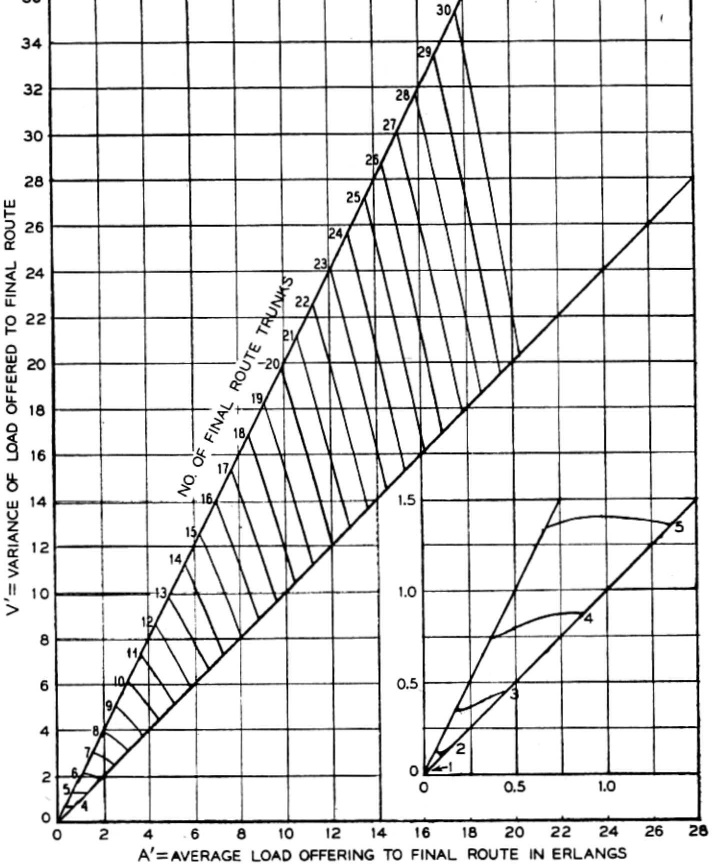

Fig. 42 — Provision of final route trunks to give first routed traffic service of $R_1 = 0.01$.

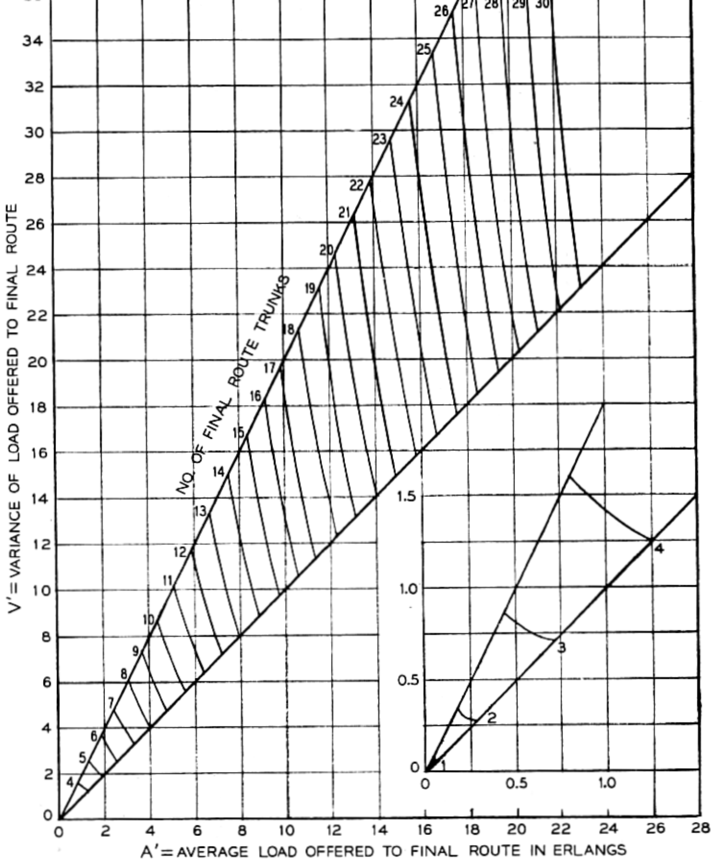

Fig. 43 — Provision of final route to trunks give first routed traffic service of $R_1 = 0.03$.

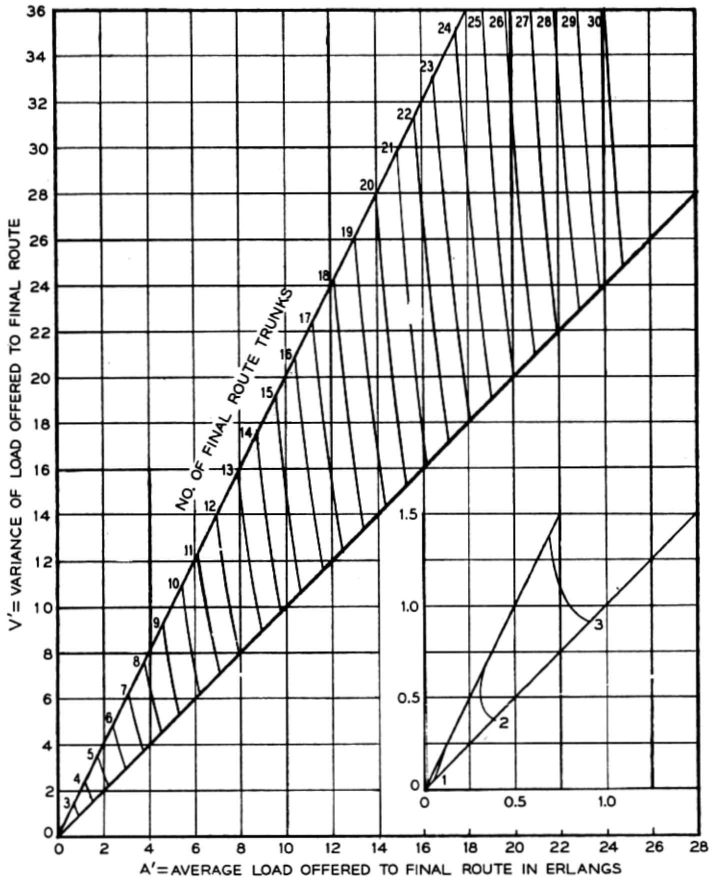

Fig. 44 — Provision of final route trunks to give first routed traffic service of $R_1 = 0.05$

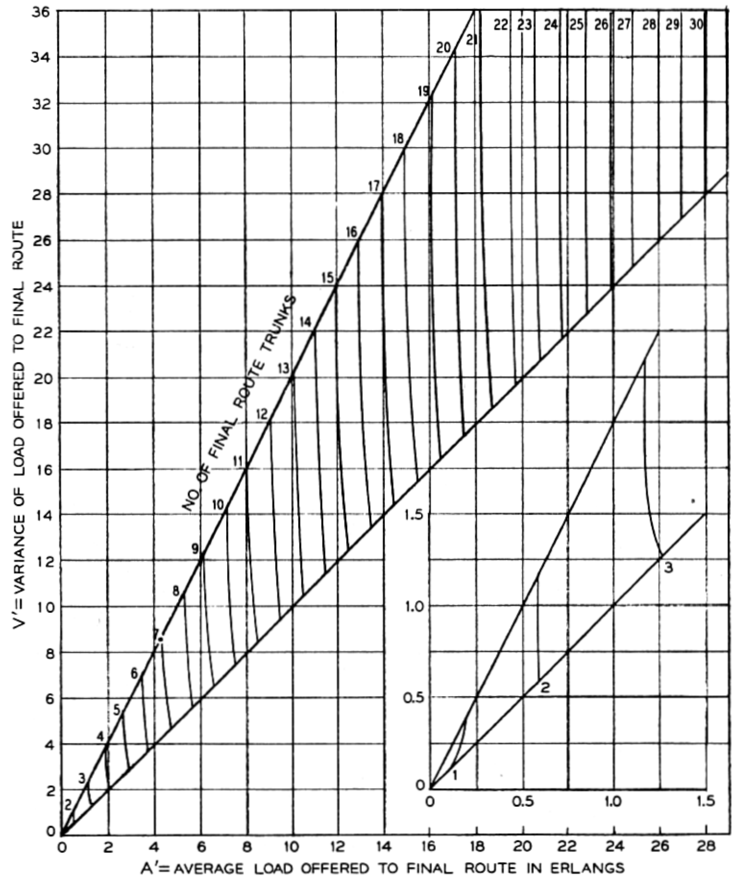

Fig. 45 — Provision of final route trunks to give first routed traffic service of $R_1 = 0.10$.

A choice of all weights g equal to unity will often be satisfactory for the present purpose. The desired high usage group size for the first routed traffic is then found from standard E_1-tables showing trunks x required, as a function of offered traffic a and proportion overflow $\bar{b}$.

Since the different parcels of traffic have varying proportions b of their loads overflowing to the final route, by equation (34) the parcel with the largest proportion will determine the permitted value of R_2. Thus

$$R_2 = P/b_{\max} \tag{36}$$

where P is the specified poorest overall service (say 0.03) for any parcel. It may be noted that on occasion some one parcel, perhaps a small one, may provide an outstandingly large $b_{\max}$ value, which will tend to give a considerably better than required service to all the major traffic parcels. Some compromise with a literal application of a fixed poorest service criterion may be indicated in such cases.

An alternative and somewhat simpler procedure here is to use an average value $\bar{b}$ in (36) instead of $b_{\max}$, with a compensating modification of P, so that substantially the same R_2 is obtained as before. The allowance in P will be influenced by the choice of weights g in (35). It will commonly be found in practice that overflow proportions to final groups for large parcels of traffic are lower than for small parcels. Choosing all weights, as unity, as opposed to weighting by traffic volumes for example, tends to insert a small element of service protection for those traffic parcels (often the smaller ones) with the higher prportionate high usage group overflows.

Having determined R_2, a ready means is needed for estimating the required number of final route trunks, Curves for this purpose are provided on Figs. 46 to 49, within whose range, $R_2 = 0.01$ to 0.10, it will usually be sufficiently accurate to interpolate for trunk engineering purposes. These R_2-curves exactly parallel the R_1-curves for use when first routed traffic is offered directly to the final group without benefit of individual high usage trunks. If R_2 is well outside the charted range a run-through of the ER calculations may be required.

8.3 *Area in Which Significant Savings in Final Route Trunks are Realized by Allowing for the Preferred Service Given a First Routed Traffic Parcel*

Considerable effort has been expended by alternate route research workers in various countries to discover and evaluate those areas where first routed (random) traffic offered to a final route enjoys a substantial service advantage over competing parcels of traffic which have overflowed from high usage groups. A comparison of Figs. 42 to 45, (which indicate trunk provision for meeting a first routed traffic criterion R_1) with Figs. 46 to 49 (which indicate trunk provision for meeting a composite-load-offered-to-the-final-route criterion R_2) gives a means for deciding under what conditions in practice it is important to distinguish between the two criteria. Fig. 50 shows the borders of areas, defined in terms of A' and V', the characterizing parameters of the total load offered to the final route, where a 2 and 5 per cent overprovision of final trunks would occur using R_2 for R_1 as the loss measure for first routed traffic. Thus in the alternate route examples displayed in Table XV, where $x = 8$, $g = 2$ to 10, $A' = 4.0$ and V' varies from 5.80 to 5.45, Fig. 50 shows that by failing to allow for the preferred position of the 2 erlang first routed parcel, we should at $R = 0.02$ engineered loss, provide a little over 5 per cent more final trunks than necessary. (Actually 10.2 and 9.9 versus 9.6 and 9.4 trunks for $g = 2$ and 10, respectively.)

The curves of Fig. 50 for final route loads larger than a few erlangs, are almost straight lines. At an objective engineering base of $R = 0.03$, for example, the 2 and 5 per cent trunk overprovision areas through using R_2 instead of R_1 are outlined closely by:

$$\text{2 per cent overprovision occurs at } V'/(A' - 1) \doteq 1.4$$
$$\text{5 per cent overprovision occurs at } V'/(A' - 1) \doteq 1.8.$$

Thus in the range of loads covered by Fig. 50, one might conclude that useful and determinable savings in final trunks can be achieved by use of the specialized R_1-curves instead of the more general R_2-curves, when the ratio $V'/(A' - 1)$ exceeds some figure in the 1.4 to 1.8 range, say 1.6. (In the examples just cited the $V'/(A' - 1)$ ratio is approximately 1.9.)

8.4. *Character of Traffic Carried on Non-Final Routes*

Telephone traffic which is carried by a non-final route will ordinarily be subjected to a peak clipping process which will depress the variance of the carried portion below that of the offered load. If this traffic terminates at the distant end of the route, its character, while conceivably affecting the toll and local switching trains in that office, will not require further consideration for intertoll trunk engineering. If, however, some or all of the route's load is to be carried on toll facilities to a more distant point (the common situation), the character of such parcels of traffic will be of interest in providing suitable subsequent paths. For this purpose it will be desirable to have etimates of the mean and variance of these carried parcels.

When a random traffic of "a" erlangs is offered to a group of "c" paths

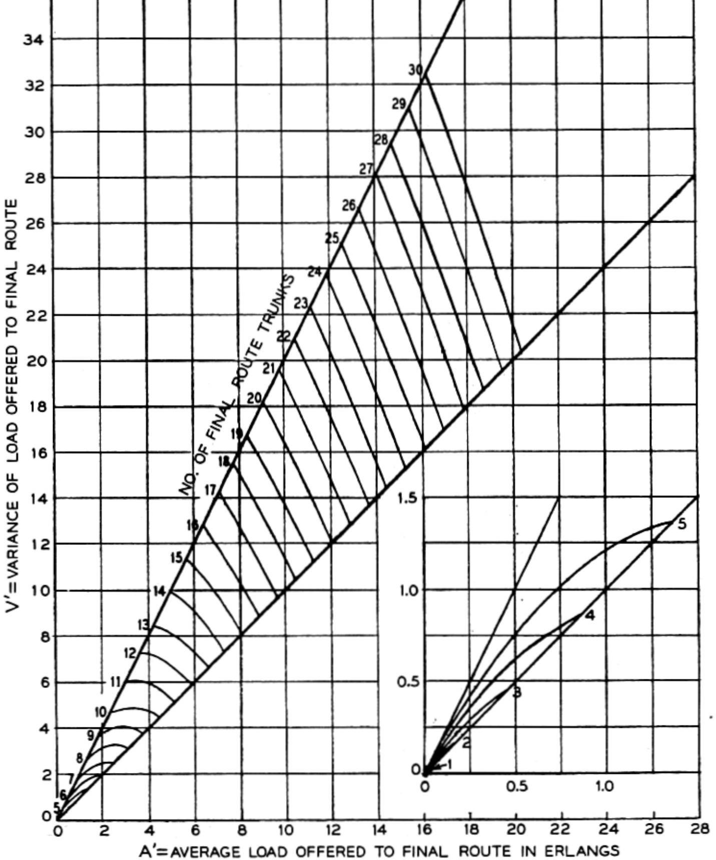

Fig. 46 — Provision of final route trunks to give combined offered load a service of $R_2 = 0.01$.

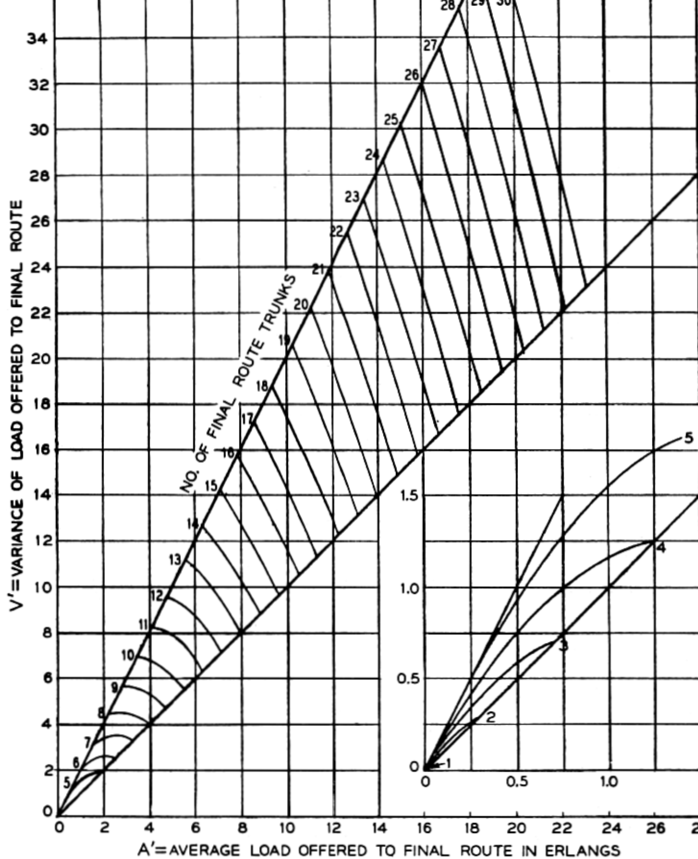

Fig. 47 — Provision of final route trunks to give combined offered load a service of $R_2 = 0.03$.

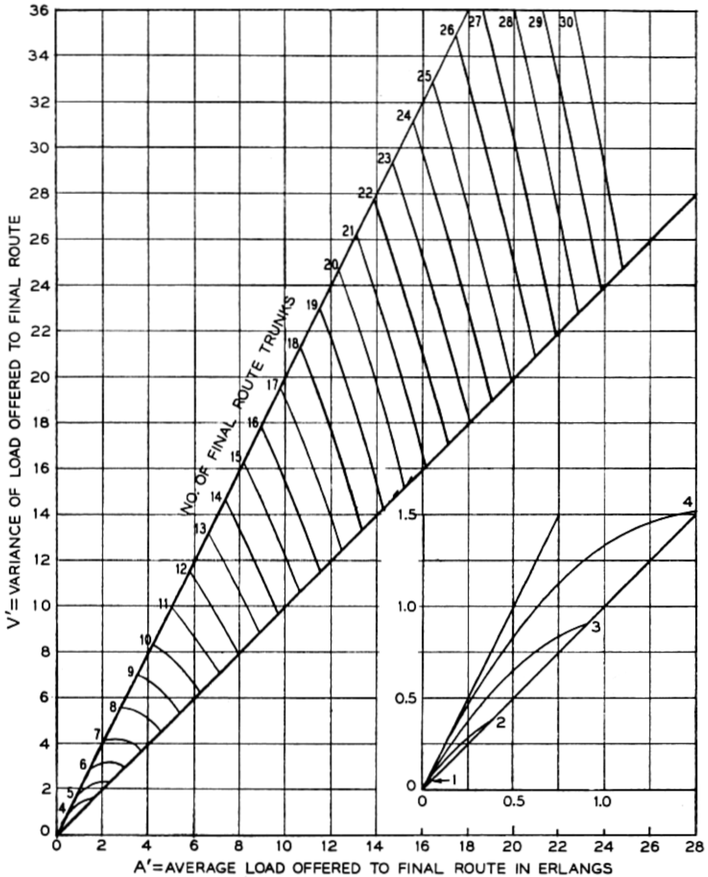

Fig. 48 — Provision of final route trunks to give combined offered load a service of $B_a = 0.05$.

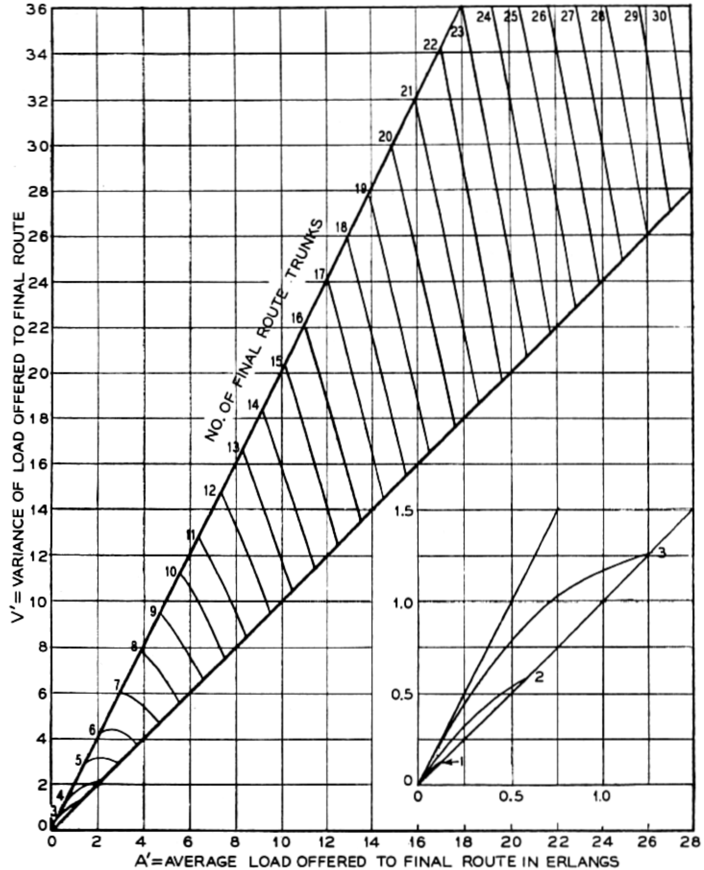

Fig. 49 — Provision of final route trunks to give combined offered load a service of $B_a = 0.10$.

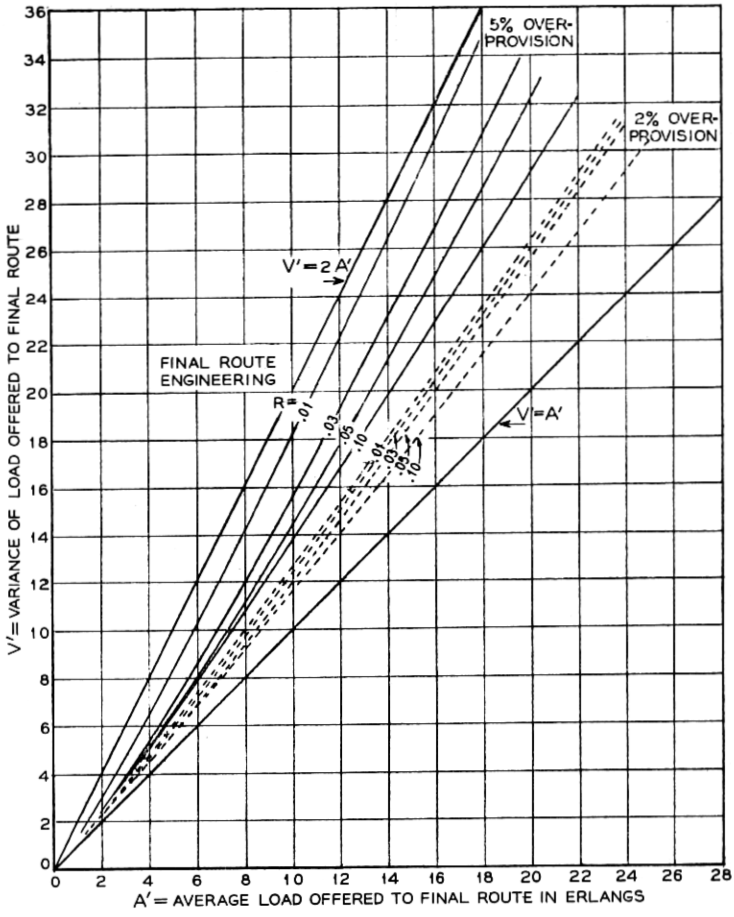

Fig. 50 — Overprovision of final route trunks when R_2 is used instead of R_1 as service to first routed traffic.

and overflowing calls do not return, the variance of the carried load is

$$V_{cd} = a[1 - E_{1,c}(a)]\,[1 + aE_{1,c}(a) - aE_{1,c-1}(a)]^* \qquad (37)$$

and the ratio of variance to average of the carried load is

$$\begin{aligned} \frac{V_{cd}}{L} &= 1 - a\,[E_{1,c-1}(a) - E_{1,c}(a)]^* \\ &= 1 - \left(\frac{c}{L} - 1\right)(a - L)^* \qquad (38) \\ &= 1 - \ell_c \end{aligned}$$

* These particular forms are due to P. J. Burke.

From (38) it is easy to see that

$$V_{cd} = L(1 - \ell_c)$$
$$= \text{(Load carried by the group)}(1 - \text{load on last trunk}) \quad (39)$$

This is a convenient relationship since for high usage trunk study work, both the loads carried (in erlangs) on the group and on the last trunk will ordinarily be at hand.

If the high usage group's load is to be split in various directions at the distant point for re-offer to other groups, it would appear not unreasonable to assign a variance to each portion so as to maintain the ratio expressed in equation (38). That is, if a carried load L is divided into parts $\lambda_1, \lambda_2 \cdots$ where $L = \lambda_1 + \lambda_2 \cdots$, then the associated variances $\gamma_1, \gamma_2 \ldots$ would be

$$\gamma_1 = \lambda_1 (1 - \ell_c)$$
$$\gamma_2 = \lambda_2 (1 - \ell_c) \quad (40)$$
$$\cdots\cdots\cdots\cdots$$

If, however, the load offered to the group is non-random (e.g., the group is an intermediate route in a multi-alternate route system), the procedure is not quite so simple as in the random case just discussed. Equation (32) expresses the variance V_c of the carried load on a group of C paths whose offered traffic consists of the overflow from a first group of S paths to which a random load of A erlangs has been offered. V_c could of course be expressed in terms of A', V' and C, and curves or tables constructed for working purposes. However, such are not available, and in any case might be unwieldy for practical use.

A simple alternative procedure can be used which yields a conservative (too large) estimate of carried load variance. With random load offered to a divided two stage multiple of x paths followed by y paths, a positive correlation exists between the numbers m and n of calls present simultaneously on the x and y paths, respectively. Then the variance V_{m+n} of the $m + n$ distribution is greater than the sum of the individual variances of m and n,

$$V_{m+n} \gtrless V_m + V_n$$

or

$$V_m \leq V_{m+n} - V_n \quad (41)$$

Now n can be chosen arbitrarily, and if made very large, V_{m+n} becomes the offered load variance, and V_n the overflow load variance. Both of these are usually (or can be made) available. Their difference then, according to (41) gives an upper limit to V_m, the desired carried load

TABLE XVI—APPROXIMATE DETERMINATION OF THE VARIANCE OF CARRIED LOADS;
x lower paths, 8 upper paths; offer to upper paths = 3 erlangs

Lower Paths, x					Upper Paths, y			
No. Lower Paths x	Random offered load A (= V)	Variance of overflow V_n	Estimated variance of carried load $V - V_n$	True variance of carried load Eq (37)	Variance of offer V' (= V_n) (Col 3)	Variance of overflow V''	Estimated variance of cd load $V' - V''$	True variance of cd load (Brock-meyer)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
0	3.00	3.00	0	0	3.00	0.035	2.97	2.853
3	5.399	4.05	1.35	0.60	4.05	0.121	3.93	3.664
6	7.856	4.98	2.88	1.418	4.95	0.236	4.74	4.175
12	12.882	6.22	6.66	3.538	6.22	0.520	5.70	4.790

variance. Corresponding reasoning yields the same conclusion when the offered load before the x paths is non-random.

A numerical example by Brockmeyer[11] while clearly insufficient to establish the degree of the inequality (41), indicates something as to the discrepancy introduced by this approximate procedure. Comparison with the true values is shown in Table XVI.

In the case of random offer to the 0, 3, 6, 12 "lower paths," the approximate method of equation (41) overestimates the variance of the carried load by nearly two to one (columns 4 and 5 of Table XVI). The exact procedure of (37) is then clearly desirable when it is applicable, that is when random traffic is being offered. For the 8 upper paths to which non-random load is offered (the non-randomness is suggested by comparing the variance of column 6 in Table XVI with the average offered load of 3 erlangs), the approximate formula (41) gives a not too extravagant overestimate of the true carried load variance. Until curves or tables are computed from equation (32), it would appear useful to follow the above procedure for estimating the carried load variance when non-random load is offered.

8.5. *Solution of a Typical Toll Multi-Alternate Route Trunking Arrangement: Bloomsburg, Pa.*

In Fig. 9 a typical, moderately complex, toll alternate route layout was illustrated. It is centered on the toll office at Bloomsburg, Pa. The loads to be carried between Bloomsburg and the ten surrounding cities are indicated in CCS (hundred call seconds per hour of traffic; 36 CCS = 1 erlang). The numbers of direct high usage trunks shown are assumed to have been determined by an economic study; we are asked to find the number of trunks which should be installed on the Bloomsburg-Harrisburg route, so that the last trunk will carry approximately 18 *CCS* (0.50 erlang). Following this determination, (*a*) the number of final trunks from Bloomsburg to Scranton is desired so that the poorest service given to any of the original parcels of traffic will be no more than 3 calls in 100 meeting *NC*. Also (*b*) the modified Bloomsburg-Scranton trunk arrangement is to be determined when a high usage group is provided for the first routed traffic.

Solution (a): First Routed Traffic Offered Directly to Final Group

The offered loads in *CCS* to each distant point are shown in column (2) of Table XVII; the corresponding erlang values are in column (3). Consulting Figs. 12 and 13, the direct group overflow load parameters, average and variance, are read and entered in columns (5) and (6) respectively for the four groups overflowing to Harrisburg, and in columns (7) and (8) for the four groups directly overflowing to Scranton. The variance for the direct Bloomsburg-Harrisburg traffic equals its average; likewise for the direct Bloomsburg-Scranton traffic. They are so entered in the table. The parameters of the total load on the Harrisburg group are found by totalling, giving $A' = 11.19$, and $V' = 19.90$.

The required size C_1 of the Harrisburg group is now determined by the Equivalent Random theory. Entering Fig. 25 with A' and V' just determined, the ER values of trunks and load found are $S_1 = 13.55$, and $A_1 = 23.75$. C_1 is to be selected so that on a straight group of $S_1 + C_1$ trunks with offered load A, the last trunk will carry 0.50 erlang. Reading from Fig. 40, the load carried by the 26th trunk approximates this figure. Hence $C_1 = 26 - S_1 = 12.45$ trunks; or choose 12 trunks.

The overflow load's mean and variance from the Harrisburg group with 12 trunks, is now read from Figs. 12 and 13, entering with load $A_1 = 23.75$ and $C_1 + S_1 = 25.55$ trunks. The overflow values ($\alpha' = 2.50$ and $v' = 7.50$) are entered in columns (7) and (8) of the table. The total offered load to Scranton is now obtained by totalling columns (7) and (8), giving $A'' = 16.27$ and $V'' = 25.60$.

We desire now to know the number of trunks C_2 for the Scranton group which will provide *NC* 3 per cent of the time to the poorest service parcel of traffic, i.e., the first routed Bloomsburg-Scranton parcel. The $R_1 = 0.03$ and $R_2 = 0.03$ solutions are available, the former of course being more closely applicable. A check reference to Fig. 50 shows a difference of approximately 4 per cent in trunk provision would result from the two methods. Entering Figs. 43 and 47 with $A'' = 16.27$ and

TABLE XVII — ILLUSTRATIVE CALCULATION OF ALTERNATE ROUTE TRUNKS AT BLOOMSBURG, PA.

Distant Office	Load Between Bloomsburg and Distant Office		No. Trunks Between Bloomsburg and Distant Office x	Characteristics of Load to Harrisburg Group		Characteristics of Load to Scranton Group		Approximate Proportion of Original Offer Going To Final Route
	CCS	Erlangs		α (Fig. 12)	v (Fig. 13)			
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
								$\frac{\text{Col. 5}}{\text{Col. 3}} \times \frac{2.50}{11.19}$
Pottsville	26	0.72	1	0.30	0.35	2.50 (Pottsville–Harrisburg)	7.50 (Pottsville–Harrisburg)	0.093
Shamokin	540	15.0	15	2.70	6.35			0.040
Sunbury	691	19.19	20	2.66	7.00			0.031
Williamsport	160	4.44	5	1.06	1.73			0.053
Harrisburg	161	4.47	C_1	4.47	4.47			0.223
				$A' = 11.19$	$V' = 19.90$			
								Col 7 ÷ Col 3
Frackville	123	3.42	5			0.50	0.81	0.146
Hazleton	836	23.22	28			1.28	3.90	0.055
Wilkes-Barre	228	6.33	8			0.89	1.68	0.141
Philadelphia	154	4.28	5			0.96	1.57	0.224
Scranton	365	10.14	C_2			10.14	10.14	$\bar{b} = 0.112$
						$A'' = 16.27$	$V'' = 25.60$	= unweighted average

$A_1 = 23.75$ $C_1 = 12$ $S_1 = 13.55$

Final route trunks required = 24.1. (Read from Fig. 43 using $A''' = 16.57$, $V''' = 25.90$ for 3 per cent first routed retrials.)

$V'' = 25.60$, we obtain the trunk requirements:

R_1 Method........ 23.8 trunks
R_2 Method........ 24.8 trunks

Thus the more precise method of solution here yields a reduction of 1.0 in 25 trunks, a saving of 4 per cent, as had been predicted.

The above calculation is on a Lost Calls Cleared basis. Since the overflow direct traffic calls will return to this group to obtain service, to assure their receiving no more than 3 per cent NC, the provision of the final route would theoretically need to be slightly more liberal. An estimate of the allowance required here may be made by adding the expected erlangs loss Δ for the direct traffic (most of the final route overflow calls which come from high usage routes will be carried by their respective groups on the next retrial) to both the A'' and V'' values previously obtained, and recalculating the trunks required from that point onward. (In fact this could have been included in the initial computation.) Thus:

$$\begin{aligned} \Delta &= 0.03 \times 10.14 = 0.30 \text{ erlang} \\ A''' &= 16.27 + 0.30 = 16.57 \text{ erlangs} \\ V''' &= 25.60 + 0.30 = 25.90 \text{ erlangs} \end{aligned}$$

Again consulting Figs. 43 and 47 gives the corresponding final trunk values

R_1 Method........ 24.1 trunks
R_2 Method........ 25.1 trunks

Of the above four figures for the number of trunks in the Scranton route, the R_1-Method with retrials, i.e., 24.1 trunks, would appear to give the best estimate of the required trunks to give 0.03 service to the poorest service parcel.

Solution (b): With High Usage Group Provided for First Routed Traffic

Following the procedure outlined in Section 8.2, we obtain an average of the proportions overflowing to the final route for all offered load parcels. The individual parcel overflow proportion estimates are shown in the last column of Table XVII; their unweighted average is 0.112. With a first routed offer to Scranton of 10.14 erlangs, a provision of 12 high usage trunks will result in an overflow of $\alpha = 1.26$ erlangs, or a *proportion* of 0.125 which is the value most closely attainable to the objective 0.112. With 12 trunks the overflow variance is found to be 2.80.

Replacing 10.14 in columns 7 and 8 of Table XVII with 1.26 and 2.80, respectively, gives new estimates characterizing the offer to the final route, $A'' = 7.39$ and $V'' = 18.26$. We now proceed to insure that the poorest service parcel obtains 0.03 service. This occurs on the Philadelphia and Harrisburg groups, which overflow to the final group approximately 0.224 of their original offered loads. The final group must then, according to equation (34) be engineered for

$$R_2 = 0.03/0.224 = 0.134 \text{ service.}$$

This value lies above the highest R_2 engineering chart (Fig. 49, $R_2 = 0.10$), so an ER calculation is indicated.

The Equivalent Random average is 28.6 erlangs, and $S = 23.5$ trunks. We determine the total trunks $S + R$ which, with 28.6 erlangs offered, will overflow $0.134(7.39) = 0.99$ erlang. From Fig. 12.2, 35.6 trunks are required. Then the final route provision should be $C = 35.6 - 23.5 = 12.1$ trunks; and a total of $12 + 12.1$ or 24.1 Scranton trunks is indicated.

Simplified Alternative Solution: In Section 8.2 a simplified approximate procedure was described using a modified probability P' for the average overall service for all parcels of traffic, instead of P for the poorest service parcel. Suppose $P' = 0.01$ is chosen as being acceptable. Then

$$R_2 = \frac{P'}{\bar{b}} = \frac{0.01}{0.112} = 0.089$$

Interpolating between the $R_2 = 0.05$ and 0.10 curves (Figs. 48 and 49) gives with $A'' = 7.39$ and $V'' = 18.26$, $C = 13.4$, the number of final trunks required. Again the same result could have been obtained by making the suitable ER computation. It may be noted that if P' had been chosen as 0.015 (one-half of P), R_2 would have become 0.134, exactly the same value found in the poorest-service-parcel method. The final trunk provision, of course, would have again been 12.1 trunks.

Disscussion

In the first solution above, 24.1 full access final trunks from Bloomsburg to Scranton were required. The service on the first routed traffic was 0.03; however, the service enjoyed by the offered traffic as a whole was markedly better than 0.03. The corresponding ER calculation shows ($A = 28.3$, $S + C = 12.3 + 24.1$) a total overflow of $\alpha'' = 0.72$ erlangs, or an overall service of $0.72/91.21 = 0.008$.

In the second solution, 12 high usage and 12.1 common final, or a total of 24.1, trunks were again required, to give 0.03 service to the poorest service parcels of offered load. The overall service here, however, was 0.99/91.21 = 0.011. Thus, with the same number of paths provided, in the second solution (high usage arrangement) the overall call loss was 40 pes cent larger than in the first solution.* However, it may well be desirable to accept such an average service penalty since by providing high usage trunks for the first routed traffic, the latter's service cannot be degraded nearly so readily should heavy overloads occur momentarily in the other parcels of traffic.

9. CONCLUSION

As direct distance dialing increases, it will be necessary to provide intertoll paths so that substantially no-delay service is given at all times. To do this economically, automatic multi-alternate routing will replace the present single route operation. Traffic engineering of these complicated trunking arrangements will be more difficult than with simple intertoll groups.

One of the new problems is to describe adequately the non-random character of overflow traffic. In the present paper this is proposed to be done by employing both mean and variance values to describe each parcel of traffic, instead of only the mean as used heretofore. Numerous comparisons are made with simulation results which indicate that adequate predictive reliability is obtained by this method for most traffic engineering and administrative purposes. Working curves are provided by which trunking arrangements of considerable complexity can readily be solved.

A second problem requiring further review is the day-to-day variation among the primary loads and their effect on the alternate route system's grade of service. A thorough study of these variations will permit a reevaluation of the service criteria which have tentatively been adopted. A closely allied problem is that of providing the necessary kind and amounts of traffic measuring devices at suitable points in the toll alternate route systems. Requisite to the solution of both of these problems is an understanding of traffic flow character in a complex overflow-type

* The actual loss difference may be slightly greater than estimated here since in the first solution (complete access final trunks), an allowance was included for return attempts to the final route by first routed calls meeting an 0.03 loss, while in the second solution (high usage group for first routed traffic) no return attempts to the final route were considered. These would presumably be small since only 1 per cent of all calls would overflow and most of these upon retrial would be handled on their respective high usage groups.

of trunking plan, and a method for estimating quantitatively the essential fluctuation parameters at each point in such a system. The present paper has undertaken to shed some light on the former, and to provide an approximate yet sufficiently accurate method by which the latter can be accomplished. It may be expected then that these studies, as they are developed, will provide the basis for assuring an adequate direct distance dialing service at all times with a minimum investment in intertoll trunk facilities.

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The extensive calculations and chart constructions are principally the work of Miss C. A. Lennon.

REFERENCES

1. Rappleye, S. C., A Study of the Delays Encountered by Toll Operators in Obtaining an Idle Trunk, B.S.T.J., **25,** p. 539, Oct., 1946.
2. Kosten, L., Over de Invloed van Herhaalde Oproepen in de Theorie der Blokkeringskausen, De Ingenieur, **59,** p. E123, Nov. 21, 1947.
3. Clos, C., An Aspect of the Dialing Behavior of Subscribers and Its Effect on the Trunk Plant, B.S.T.J., **27,** p. 424, July, 1948.
4. Kosten, L., Uber Sperrungswahrscheinlichkeiten bei Staffelschaltungen, E.N.T., **14,** p. 5, Jan., 1937.
5. Kosten, L., Over Blokkeerings-en Wachtproblemen, Thesis, Delft, 1942.
6. Molina, E. C., Appendix to: Interconnection of Telephone Systems — Graded Multiples (R. I. Wilkinson), B.S.T.J., **10,** p. 531, Oct., 1931.
7. Vaulot, A. E., Application du Calcul des Probabilités à l'Exploitation Téléphonique, Revue Gen. de l'Electricité, **16,** p. 411, Sept. 13, 1924.
8. Lundquist, K., General Theory for Telephone Traffic, Ericsson Technics, **9,** p. 111, 1953.
9. Berkeley, G. S., Traffic and Trunking Principles in Automatic Telephony, 2nd revised edition, 1949, Ernest Benn, Ltd., London, Chapter V.
10. Palm, C., Calcul Exact de la Perte dans les Groupes de Circuits Échelonnés, Ericsson Technics, **3,** p. 41, 1936.
11. Brockmeyer, E., The Simple Overflow Problem in the Theory of Telephone Traffic, Teleteknik, **5,** p. 361, December, 1954.

ABRIDGED BIBLIOGRAPHY OF ARTICLES ON TOLL ALTERNATE ROUTING

Clark, A. B., and Osborne, H. S., Automatic Switching for Nationwide Telephone Service, A.I.E.E., Trans., **71,** Part I, p. 245, 1952. (Also B.S.T.J., **31,** p. 823, Sept., 1952.)
Pilliod, J. J., Fundamental Plans for Toll Telephone Plant, A.I.E.E. Trans., **71,** Part I, p. 248, 1952. (Also B.S.T.J., **31,** p. 832, Sept., 1952.)
Nunn, W. H., Nationwide Numbering Plan, A.I.E.E. Trans., **71,** Part I, p. 257, 1952. (Also B.S.T.J., **31,** p. 851, Sept., 1952.)
Clark, A. B., The Development of Telephony in the United States, A.I.E.E. Trans., **71,** Part I, p. 348, 1952.
Shipley, F. F., Automatic Toll Switching Systems, A.I.E.E. Trans., **71,** Part I, p. 261, 1952. (Also B.S.T.J., **31,** p. 860, Sept., 1952.)
Myers, O., The 4A Crossbar Toll System for Nationwide Dialing, Bell Lab. Record, **31,** p. 369, Oct., 1953.
Clos, C., Automatic Alternate Routing of Telephone Traffic, Bell Lab. Record, **32,** p. 51, Feb., 1954.
Truitt, C. J., Traffic Engineering Techniques for Determining Trunk Requirements in Alternate Routing Trunk Networks, B.S.T.J., **33,** p. 277, March, 1954.
Molnar, I., Some Recent Advances in the Economy of Routing Calls in Nationwide Dialing, A.E. Tech. Jl., **4,** p. 1, Dec., 1954.
Jacobitti, E., Automatic Alternate Routing in the 4A Crossbar System, Bell Lab. Record, **33,** p. 141, April, 1955.

APPENDIX I*

DERIVATION OF MOMENTS OF OVERFLOW TRAFFIC

This appendix gives a derivation of certain factorial moments of the equilibrium probabilities of congestion in a divided full-access multiple used as a basis for the calculations in the text. These moments were derived independently in unpublished memoranda (1941) by E. C. Molina (the first four) and by H. Nyquist; curiously, the method of derivation here, which uses factorial moment generating functions, employs auxiliary relations from both Molina and Nyquist. Although these factorial moments may be obtained at a glance from the probability expressions given by Kosten in 1937, if it is remembered that

$$p(x) = \sum_{k=0}^{\infty} (-1)^{k-x} \binom{k}{x} \frac{M_{(k)}}{k!}, \tag{1.1}$$

where $p(x)$ is a discrete probability and $M_{(k)}$ is the kth factorial moment of its distribution, Kosten does not so identify the moments and it may be interesting to have a direct derivation.

Starting from the equilibrium formulas of the text for $f(m, n)$, the probability of m trunks busy in the specific group of x trunks, and n in

* Prepared by J. Riordan.

the (unlimited) common group, namely

$$
\begin{aligned}
(a + m + n)f(m, n) &- (m + 1)f(m + 1, n) \\
&- (n + 1)f(m, n + 1) - af(m - 1, n) = 0 \\
(a + x + n)f(x, n) &- af(x, n - 1) \\
&- (n + 1)f(x, n + 1) - af(x - 1, n) = 0
\end{aligned}
\tag{1.2}
$$

and

$$f(m, n) = 0, \qquad m < 0 \qquad \text{or} \qquad n < 0 \qquad \text{or} \qquad m > x,$$

factorial moment generating function recurrences may be found and solved.

With m fixed, factorial moments of n are defined by

$$M_{(k)}(m) = \sum_{n=0}^{\infty} (n)_k f(m, n) \tag{1.3}$$

or alternatively by the factorial moment exponential generating function

$$M(m, t) = \sum_{k=0}^{\infty} M_{(k)}(m)t^k/k! = \sum_{n=0}^{\infty} (1 + t)^n f(m, n) \tag{1.4}$$

In (1.3), $(n)_k = n(n - 1) \cdots (n - k + 1)$ is the usual notation for a falling factorial.

Using (1.4) in equations (1.2), and for brevity $D = d/dt$, it is found that

$$
\begin{aligned}
a + m + tD)M(m, t) &- (m + 1)M(m + 1, t) \\
&- aM(m - 1, t) = 0 \\
(x - at + tD)M(x, t) &- aM(x - 1, t) = 0
\end{aligned}
\tag{1.5}
$$

which correspond (by equating powers of t) to the factorial moment recurrences

$$
\begin{aligned}
(a + m + k)M_{(k)}(m) &- (m + 1)M_{(k)}(m + 1) \\
&- aM_{(k)}(m - 1) = 0 \\
(x + k)M_{(k)}(x) - akM_{(k-1)}(x) &- aM_{(k)}(x - 1) = 0
\end{aligned}
\tag{1.6}
$$

Notice that the first of (1.6) is a recurrence in m, which suggests (following Molina) introducing a new generating function defined by

$$G_k(u) = \sum M_{(k)}(m)u^m \tag{1.7}$$

Using this in (1.5), it is found that

$$\left[(a + k - au + (u - 1)\frac{d}{du}\right] G_k(u) = 0 \tag{1.8}$$

Hence

$$\frac{1}{G_k(u)} \frac{dG_k(u)}{du} = a + \frac{k}{1 - u} \tag{1.9}$$

and, by easy integrations,

$$G_k(u) = ce^{au} (1 - u)^{-k}, \tag{1.10}$$

with c an arbitrary constant, which is clearly identical with $G_k(0) = M_{(k)}(0)$.

Expansion of the right-hand side of (1.10) shows that

$$M_{(k)}(m) = M_{(k)}(0) \sum_{j=0}^{m} \binom{k + j - 1}{j} \frac{a^{m-j}}{(m - j)!} = M_{(k)}(0)\sigma_k(m), \tag{1.11}$$

if

$$\sigma_0(m) = a^m/m! \qquad \text{and,} \quad \sigma_k(m) = \sum_{j=0}^{m} \binom{k + j - 1}{j} \frac{a^{m-j}}{(m - j)!} \tag{1.12}$$

The notation $\sigma_k(m)$ is copied from Nyquist; the functions are closely related to the $\varphi_x^{(n)}$ used by Kosten; indeed $\sigma_k(m) = e^a \varphi_m^{(k)}$. They have the generating function

$$g_k(u) = \sum_{m=0}^{\infty} \sigma_k(m)u^m = e^{au}(1 - u)^{-k} \tag{1.13}$$

from which a number of recurrences are found readily. Thus

$$g_k(u) = (1 - u)g_{k+1}(u)$$

$$u \frac{dg_k(u)}{du} = aug_k(u) + kug_{k+1}(u)$$

$$= -ag_{k-1}(u) + (a - k)g_k(u) + kg_{k-1}(u)$$

(the last by use of the first) imply

$$\sigma_k(m) = \sigma_{k+1}(m) - \sigma_{k+1}(m - 1)$$

$$m\sigma_k(m) = a\sigma_k(m - 1) + k\sigma_{k+1}(m - 1)$$

$$= - a\sigma_{k-1}(m) + (a - k)\sigma_k(m) + k\sigma_{k+1}(m)$$

The first of these leads to

$$\sigma_k(0) + \sigma_k(1) + \cdots + \sigma_k(x) = \sigma_{k+1}(x) \tag{1.14}$$

and the last is useful in the form

$$k\sigma_{k+1}(m) = (m + k - a)\sigma_k(m) + a\sigma_{k-1}(m) \tag{1.15}$$

Also, the first along with $\sigma_0(m) = a^m/m!$ leads to a simple calculation procedure, as Kosten has noticed.

By (1.11) the factorial moments are now completely determined except for $M_{(k)}(0)$. To determine the latter, the second of (1.6) and the normalizing equation

$$\sum_{m=0}^{x} M_0(m) = 1 \tag{1.16}$$

are available.

Thus from the second of (1.6)

$$[(x + k)\sigma_k(x) - a\sigma_k(x - 1)]M_{(k)}(0) = ak\sigma_{k-1}(x)M_{(k-1)}(0) \tag{1.17}$$

Also

$$\begin{aligned}(x + k)\sigma_k(x) - a\sigma_k(x - 1) & \\ &= (x + k - a)\sigma_k(x) + a[\sigma_k(x) - \sigma_k(x - 1)] \\ &= (x + k - a)\sigma_k(x) + a\sigma_{k-1}(x) \\ &= k\sigma_{k+1}(x),\end{aligned}$$

the last step by (1.15). Hence

$$M_{(k)}(0) = a\frac{\sigma_{k-1}(x)}{\sigma_{k+1}(x)} M_{(k-1)}(0) \tag{1.18}$$

and by iteration

$$M_{(k)}(0) = a^k \frac{\sigma_1(x)\sigma_0(x)}{\sigma_{k+1}(x)\sigma_k(x)} M_0(0) \tag{1.19}$$

From (1.11) and (1.16), and in the last step (1.14),

$$\sum_{m=0}^{x} M_0(m) = \sum_{m=0}^{x} M_0(0)\sigma_0(m) = M_0(0)\sigma_1(x) = 1 \tag{1.20}$$

Hence finally

$$\begin{aligned}M_{(k)}(m) &= M_{(k)}(0)\sigma_k(m) \\ &= a^k \frac{\sigma_0(x)\sigma_k(m)}{\sigma_{k+1}(x)\sigma_k(x)}\end{aligned} \tag{1.21}$$

and
$$M_{(k)} = \sum_{m=0}^{x} M_{(k)}(m) = a^k \sigma_0(x)/\sigma_k(x) \tag{1.22}$$

Ordinary moments are found from the factorial moments by linear relations; thus if m_k is the kth ordinary moment (about the origin)

$$m_0 = M_{(0)} \qquad m_1 = M_{(1)} \qquad m_2 = M_{(2)} + M_{(1)}$$
$$m_3 = M_{(3)} + 3M_{(2)} + M_{(1)}$$

Thus

$$m_0(m) = \sigma_0(m)/\sigma_1(x)$$
$$m_1(m) = a\sigma_1(m)\sigma_0(x)/\sigma_1(x)\sigma_2(x)$$
$$m_2(m) = a^2\sigma_2(m)\sigma_0(x)/\sigma_2(x)\sigma_3(x) + a\sigma_1(m)\sigma_0(x)/\sigma_1(x)\sigma_2(x)$$

and, in particular, using notation of the text

$$m_0(x) = \sigma_0(x)/\sigma_1(x) = E_{1,x}(a)$$

$$\alpha_x = \frac{m_1(x)}{m_0(x)} = a\,\frac{\sigma_1(x)}{\sigma_2(x)} = \frac{a}{x - a + 1 + aE_{1,x}(a)} \tag{1.23}$$

$$\begin{aligned} v_x &= \frac{m_2(x)}{m_0(x)} - \alpha_x^2 = \frac{a^2\sigma_1(x)}{\sigma_3(x)} + \alpha_x - \alpha_x^2 \\ &= \alpha_x[1 - \alpha_x + 2a(x + 2 + \alpha_x - a)^{-1}] \end{aligned} \tag{1.24}$$

Finally the sum moments: $m_k = \sum_0^x m_k(m)$ are

$$\begin{aligned} m_0 &= 1 \\ m_1 &= \alpha = a\sigma_0(x)/\sigma_1(x) = aE_{1,x}(a) \end{aligned} \tag{1.25}$$

$$\begin{aligned} m_2 &= a^2\sigma_0(x)/\sigma_2(x) + m_1 = m_1[a(x + 1 + m_1 - a)^{-1} + 1] \\ v &= m_2 - m_1^2 = m_1[1 - m_1 + a(x + 1 + m_1 - a)^{-1}] \end{aligned} \tag{1.26}$$

In these, $E_{1,x}(a) = \sigma_0(x)/\sigma_1(x)$ is the familiar Erlang loss function.

Appendix II — Character of Overflow Load When Non-Random Traffic is Offered to a Group of Trunks

It has long been recognized that it would be useful to have a method by which the character of the overflow traffic could be determined when non-random traffic is offered to a group of trunks. Excellent agreement has been found in both throwdown and field observation over ranges of considerable interest with the "equivalent random" method of describ-

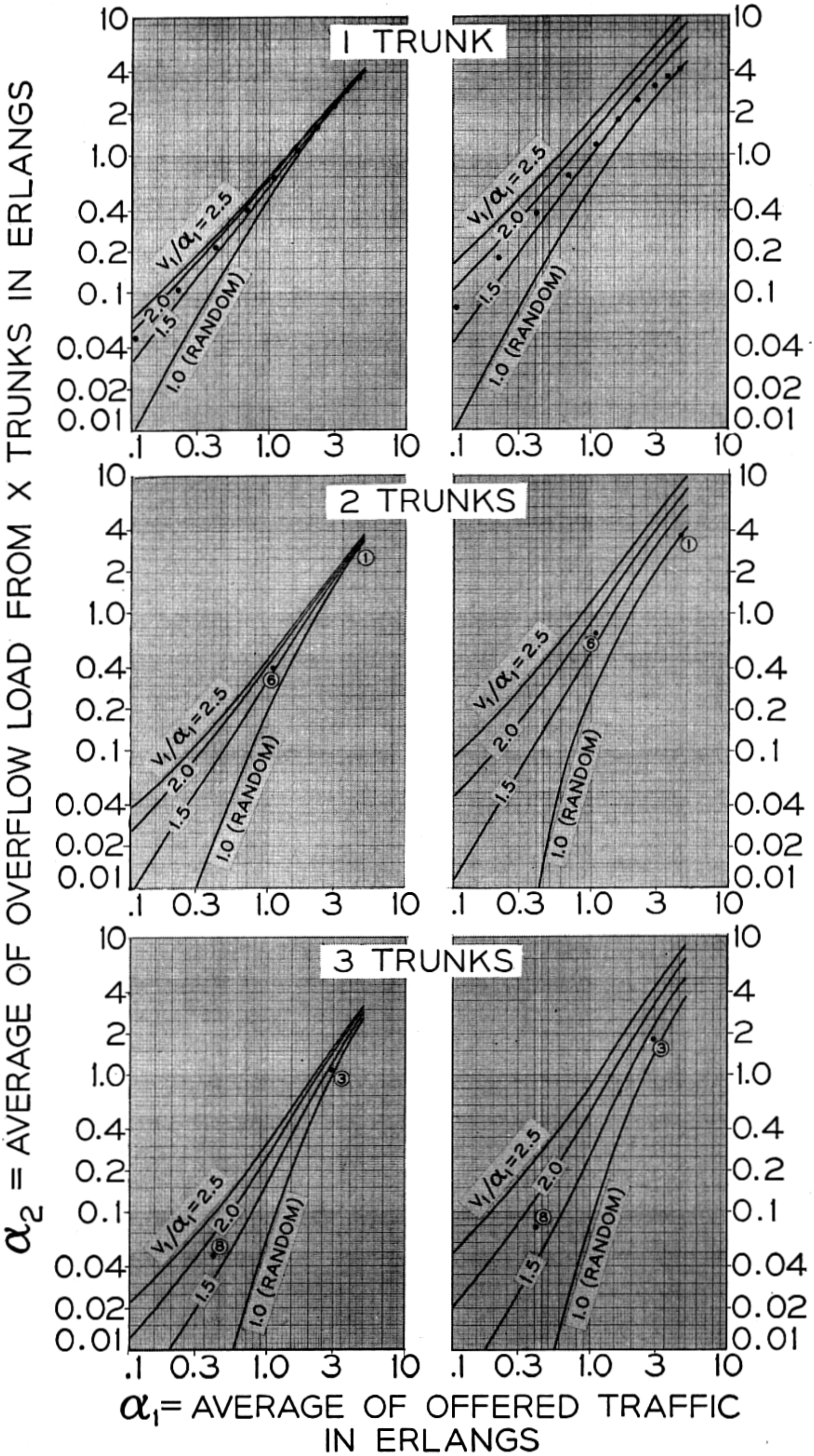
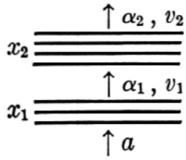


Fig. 51 — Mean and variance of overflow load when non-random traffic is offered to a group of trunks.

ing the character of non-random traffic. An approximate solution of the problem is offered based on this method.

Suppose a random traffic a is offered to a straight multiple which is divided into a lower x_1 portion and an upper x_2 portion, as follows:



From Nyquist's and Molina's work we know the mean and variance of the two overflows to be:

$$\alpha_1 = a \cdot E_{1,x_1}(a) = a \frac{\dfrac{a^{x_1}}{x_1!}}{1 + a + \dfrac{a^2}{2!} + \cdots + \dfrac{a^{x_1}}{x_1!}}$$

$$v_1 = \alpha_1 \left[1 - \alpha_1 + \frac{a}{x_1 - a + \alpha_1 + 1} \right]$$

$$\alpha_2 = a \cdot E_{1,x_1+x_2}(a)$$

$$v_2 = \alpha_2 \left[1 - \alpha_2 + \frac{a}{x_1 + x_2 - a + \alpha_2 + 1} \right]$$

Since α_1 and v_1 completely determine a and x_1, and these in turn, with x_2, determine α_2 and v_2, we may express α_2 and v_2 in terms of only α_1, v_1, and x_2. The overflow characteristics (α_2 and v_2), are then given for a non-random load (α_1 and v_1) offered to x trunks as was desired.

Fig. 51 of this Appendix has been constructed by the Equivalent Random method. The charts show the expected values of α_2 and v_2 when α_1, v_1 (or v_1/α_1), and x_2, are given. The range of α_1 is only 0 to 5 erlangs, and v/α is given only from the Poisson unity relation to a peakedness value of 2.5. Extended and more definitive curves or tables could readily, of course, be constructed.

The use of the curves can perhaps best be illustrated by the solution of a familiar example.

Example: A load of 4.5 erlangs is submitted to 10 trunks; on the "lost calls cleared" basis; what is the average load passing to overflow?

Solution: Compute the load characteristics from the first trunk when 4.5 erlangs of random traffic are submitted to it. These values are found to be $\alpha_1 = 3.68$, $v_1 = 4.15$. Now using α_1 and v_1 (or $v_1/\alpha_1 = 4.15/3.68 = 1.13$) as the offered load to the second trunk, read on the chart the parameters of the overflow from the second trunk, and so on. The successive overflow values are given in Table XVIII.

The proportion of load overflowing the group is then 0.0472/4.50 = 0.0105, which agrees, of course, with the Erlang $E_{1,10}(4.5)$ value. The successive overflow values are shown on the chart by the row of dots along the α_2 and v_2 1-trunk curves.

Instead of considering successive single-trunk overflows as in the example above, other numbers of trunks may be chosen and their overflows determined. For example suppose the 10 trunks are subdivided into 2 + 3 + 2 + 3 trunks. The loads overflowing these groups are given in Table XIX.

Again the overflow is 0.0472 erlang, or a proportion lost of 0.0105, which is, as it should be, the same as found in the previous example. The values read in this example are indicated by the row of dots marked 1, 3, 6, 8 on the 2-trunk and 3-trunk curves.

The above procedure and curves should be of use in obtaining an estimate of the character of the overflow traffic when a non-random load is offered to a group of paths.

Table XVIII — Successive Non-Random Overflows

Trunk Number i	Characteristics of Load Offered to Trunk No. i (same as overflow from previous trunk)		
	Average	Variance	Ratio of variance to average
1	4.50	4.50	1.00 (Random)
2	3.68	4.15	1.13
3	2.92	3.68	1.26
4	2.22	3.11	1.40
5	1.61	2.46	1.53
6	1.09	1.80	1.64
7	0.694	1.19	1.72
8	0.406	0.709	1.75
9	0.217	0.377	1.74
10	0.106	0.180	1.70
Overflow	0.0472	0.077	1.64

Table XIX — Sucessive Non-Random Overflows

Trunk Number i	No. Trunks in Next Bundle	Offered Load Characteristics (same as overflow from previous trunk)		
		Average	Variance	Ratio of variance to average
1	2	4.50	4.50	1.00 (Random)
3	3	2.92	3.68	1.26
6	2	1.09	1.80	1.64
8	3	0.406	0.709	1.75
Overflow		0.0472	0.077	1.64